The Economics of Time as it is Embedded in the Prices of Options

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Outline

- General remarks on additive processes and option pricing.

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- Observations on the elasticity of variance.

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- Separating TC and SS for the positive and negative moves.

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- For an underlying Brownian motion the two approaches are equivalent. In general they are different.

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- The limit laws were characterized by Lévy (1937) and Khintchine (1938).
- Separating out the processes for the positive and negative moves and selecting the maximal entropy unit time distribution on the half line given by the gamma distribution one arrives at space scaling a bilateral gamma distribution at unit time.

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- It measures the percentage increase in variance for a percentage increase in the time to maturity.
- We recognize that an additional month at one month is not comparable to an additional month at five years or sixty months.
- The function $\gamma(t)$ may be approximated from data on option prices with maturities in a neighborhood of a given maturity t.

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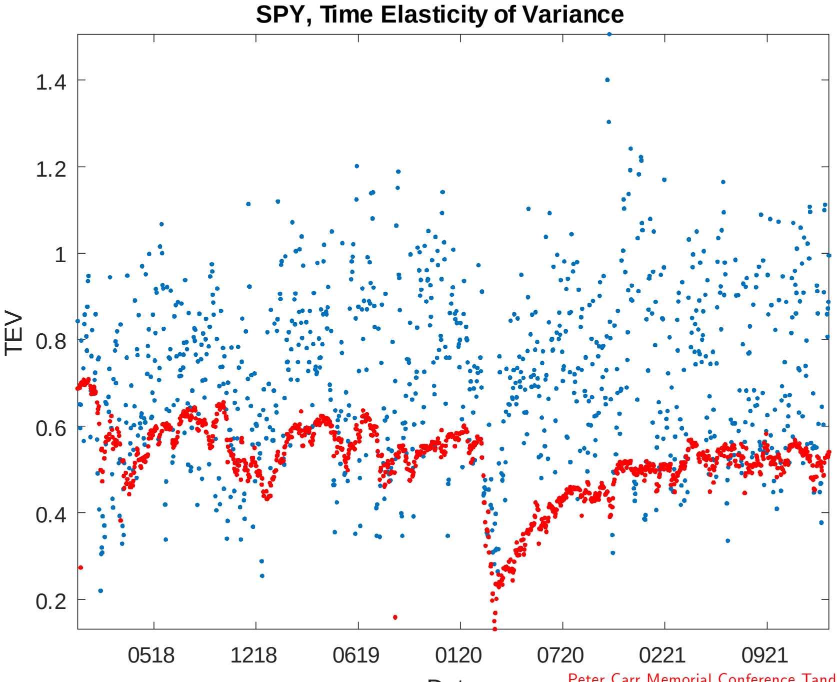
- The comparison with $1/2$ implies that at the money implied volatilities drop to zero in normal times as maturity drops to zero.
- In stressed times implied volatilities rise towards infinity as we approach zero from above.


SPY, Time Elasticity of Variance
TEV
1.4
1.2
1
0.8
0.6
0.4
0.2
0518
1218
0619
0120
0720
0221
0921

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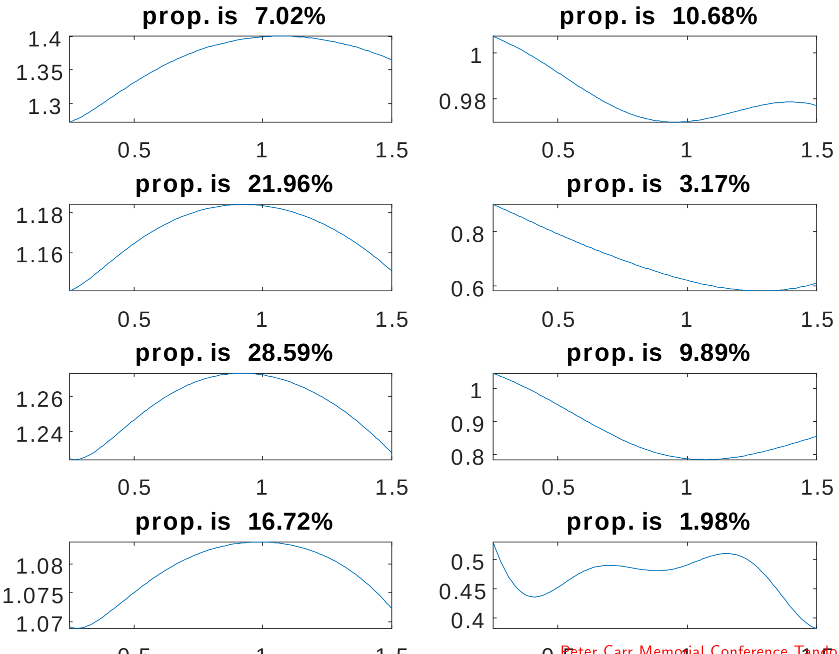
- The shapes we build into the models allow for at most one interior maximum or minimum but not two such points.


prop. is 7.02%
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prop. is 10.68%
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prop. is 21.96%
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prop. is 1.98%
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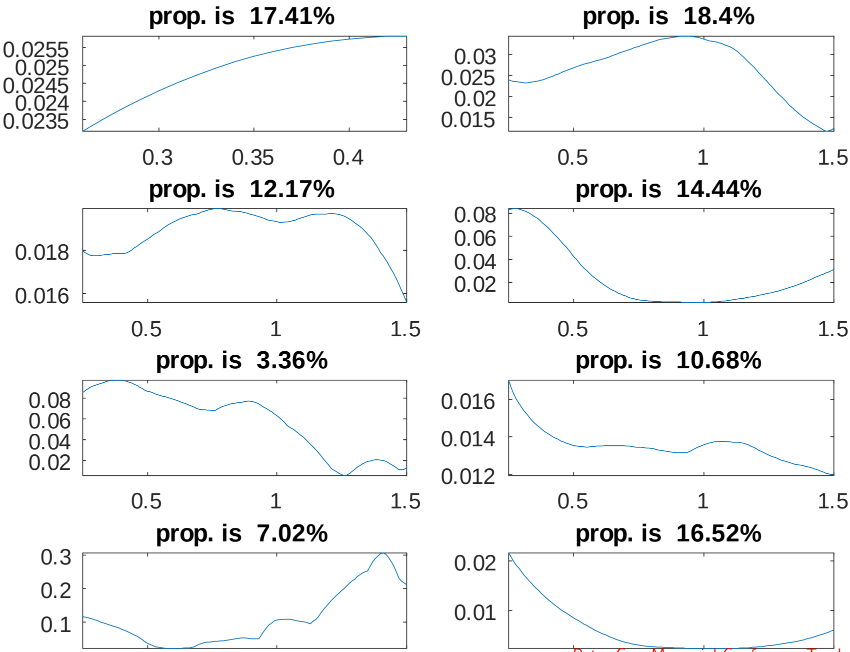
- We observe that the time changing elasticities are significant larger than the corresponding space scaling elasticities.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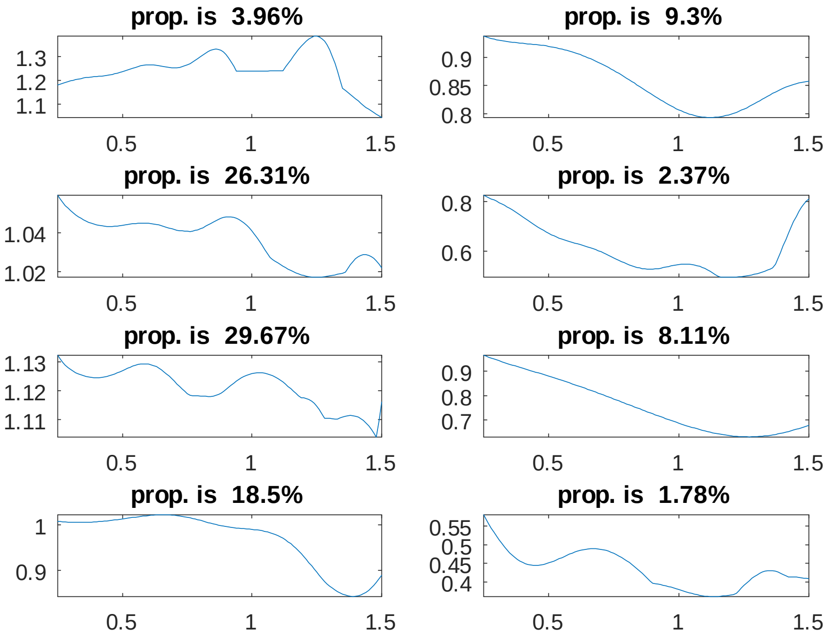
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prop. is 1.78%
0.55
0.5
0.45
0.4
0.5
1
1.5

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- We observe from equation that space scaling makes contributions at small levels of t and larger levels of x.

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- With X the law at unit time we have
$$X(t) - X = A(\ln t) + B(\ln t)$$

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- Then

$$\begin{aligned} A(v) + B(v) &= \exp\left(\int_0^v \gamma(e^s)ds\right) W(v) \\ dW &= -\gamma(e^v)W(v)dv + dZ \\ dZ &= dY\left(\frac{e^u f(e^u)g'(e^u)}{g(e^u)}\right) + dL\left(f'(e^u)e^u\right) \end{aligned}$$

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- Hence the process X is a scaled version of a solution to an OU equation reflecting perpetual motion via the effects on current motion of past shocks.

Relative incremental variance contributions

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- where γ_g, γ_f are the time elasticities of g and f.

Modeling Time Elasticities of Variance

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- This gives us two models for the time elasticity of variance that we designate exponential/gamma time elasticity of variance. The specific functions are

$$\begin{aligned} \gamma_e(t) &= b + (a-b)e^{-\gamma t} \\ \gamma_g(t) &= a + \frac{(b-a)c^{\gamma}}{\Gamma(\gamma)} \int_0^t s^{\gamma-1} e^{-cs} ds \end{aligned}$$

The Time Change Functions

- The specific time change functions $f_e(t)$ and $f_g(t)$ are given by

$$
\begin{aligned}
f_e(t) &= t^b \exp((b-a)\,(expint(\gamma t) - expint(\gamma)))\\
f_g(t) &= t^a \exp\left((a-b)\left[\begin{array}{c} \frac{c^\gamma\ {}_2F_2(\gamma,\gamma;\gamma+1,\gamma+1;-c)}{\gamma^2\Gamma(\gamma)} \\ -\frac{(ct)^\gamma\ {}_2F_2(\gamma,\gamma;\gamma+1,\gamma+1;-ct)}{\gamma^2\Gamma(\gamma)} \end{array}\right]\right)
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- The Lévy density is given by

$$k_{BG}(x) = c_n \frac{\exp\left(-\frac{|x|}{b_n}\right)}{|x|} \mathbf{1}_{x<0} + c_p \frac{\exp\left(-\frac{x}{b_p}\right)}{x} \mathbf{1}_{x>0}.$$

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d &= b_p c_p - b_n c_n \\
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s &= 2\frac{b_p^3 c_p - b_n^3 c_n}{v^3} \\
k &= 3 + 6\frac{b_p^4 c_p + b_n^4 c_n}{v^4}.
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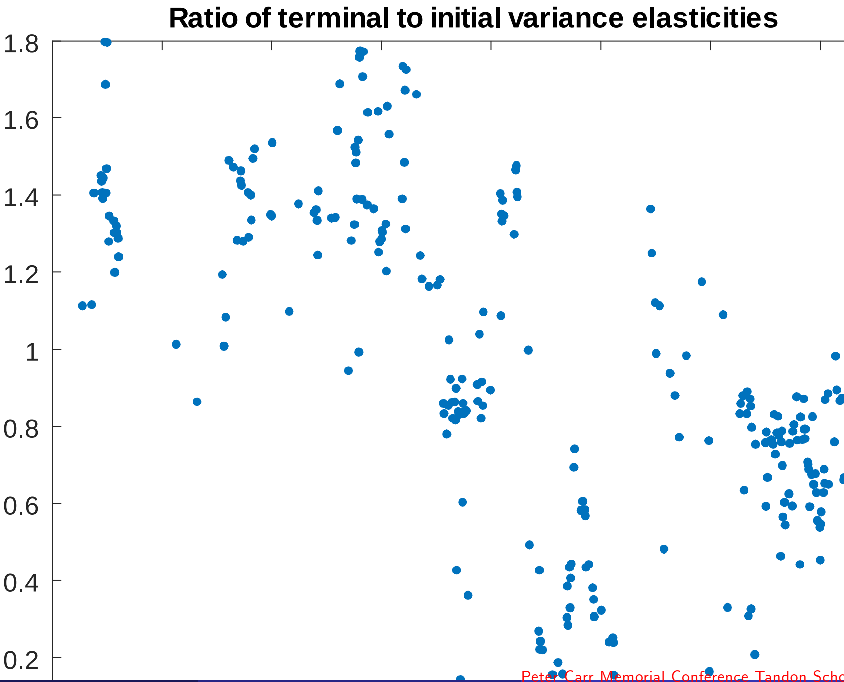
Ratio of terminal to initial variance elasticities
1.8
1.6
1.4
1.2
1
0.8
0.6
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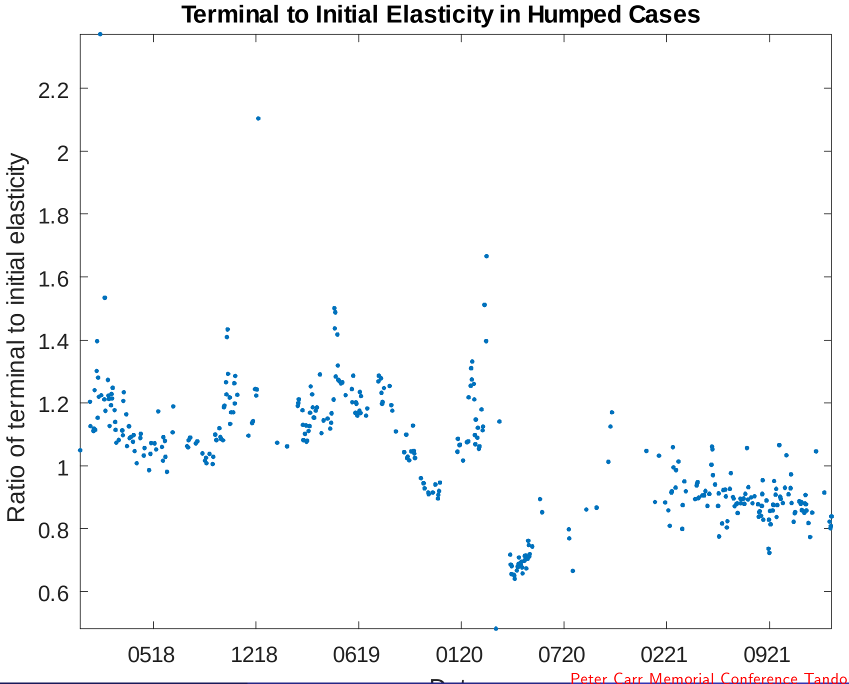
Terminal to Initial Elasticity in Humped Cases
Ratio of terminal to initial elasticity
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0518
1218
0619
0120
0720
0221
0921

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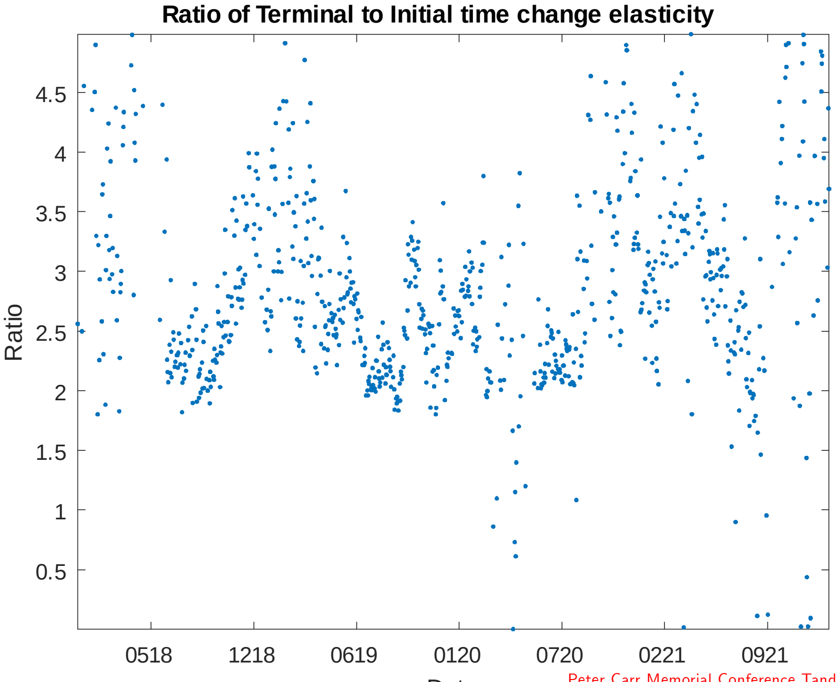
- First we note that the time change elasticity is mostly increasing as is evidenced by the ratio of terminal to initial elasticities presented in Figure


Ratio of Terminal to Initial time change elasticity
Ratio
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0518
1218
0619
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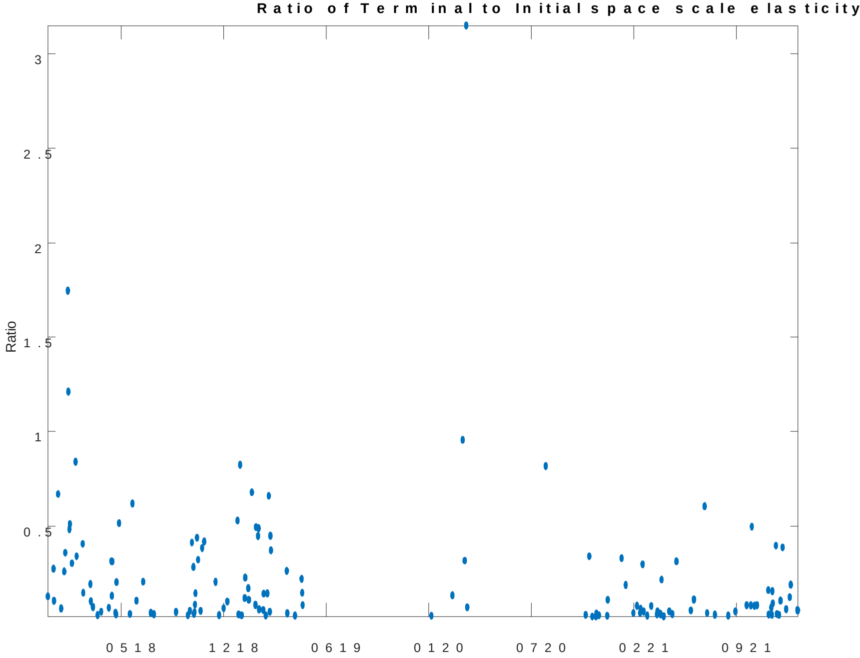
- Figure presents the ration of terminal to initial space scale elasticities where the terminal elasticity is above 0.01.


Ratio of Terminal to Initial space scale elasticity
Ratio
3
2.5
2
1.5
1
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0518
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- In 103 of the 203 cases both humps were between a quarter and a year and a half.

- For these cases the initial time change elasticity is essentially zero and below 0.001 in 77 of the 103 cases. Hence the time change elasticity rises from zero to the terminal elasticity that had a median value of 0.5425 with an interquartile range from 0.3979 to 0.8672.

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- The richer term structure of the gamma based case makes effective contributions in the modeling of the time elasticities of variance.

Result on splitting the positive and negative sides at the short end

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- We observe that the up variance elasticities are substantially lower than the down elasticities.

Table 1
Positive and Negative Move
Variance Elasticities

Percentile	Positive	Negative
25	0.1828	0.9808
50	0.3308	1.1592
75	0.4826	1.3314

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- With regard to the space scaling elasticities of variance we have backwardation on the positive side just 30 percent of the time while on the negative side there is backwardation 79 percent of the time.

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Conclusion II

- It is observed that space scaling generally contributes to the modeling of front end options while the back end switches towards the contributions of time changing.
- The space scaled and time changed densities are shown to be consistent with additive processes made up by aggregating over time the space scaled innovations of the sum of two underlying time changed Lévy processes taken at log time.
- The space scaled process is also a space scaled solution to a time varying OU equation driven by a time changed Lévy process, taken at log time, where the OU mean reversion rates are the time elasticities.
- The two processes are termed the space scaled and time change components and their relative contributions, space to time are determined to be twice the ratio of their elasticities of variance.