



Generalized Power-Gaussian Copula & CMS Spread Option Smiles

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Acknowledgements

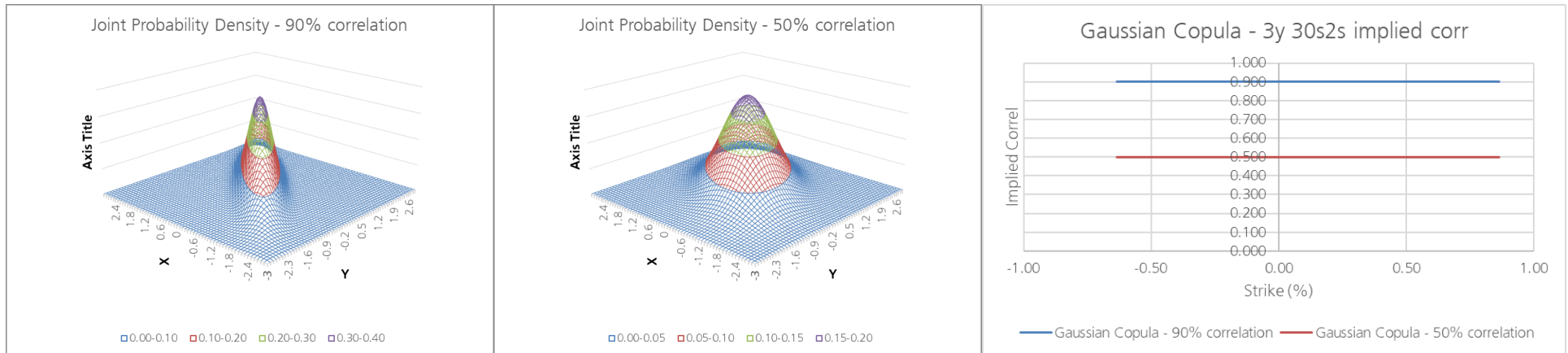
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Introduction: CMS Spread Options & Copulas

- CMS spread options are options on the spread between two swap rates observed at expiry and settled as a % of notional.
- $V(t) = P(t, T) E^T [(X(T) - Y(T) - K)^+ | F_t]$
- To evaluate this, we require the joint distribution of X and Y in the T-forward measure.
- Marginal-distributions in the T-forward measure
 - The marginal distributions of X & Y are known in their respective swap-Annuity measures via standard swaption pricing models such as SABR, calibrated to European Swaptions.
 - These can be transformed to the T-forward measure using standard techniques, for example:
 - a) Represent the Radon-Nikodym derivative as a function of the swap rate
 - b) Static replication of the payoff as a continuum of swaptions (as devised by Carr, Madan, 1998)
 - $E^T [f(S) | F_t] = \frac{A(t)}{P(t, T)} E^A \left[\frac{P(T, T)}{A(T)} f(S) | F_t \right] = \frac{A(t)}{P(t, T)} E^A [g(S) f(S) | F_t]$
- Copula function
 - A function $C: [0,1]^2 \rightarrow [0,1]$ is defined as a (2-dimensional) Copula if:
 - $C(u, 0) = C(0, v) = 0 \forall u, v \in [0,1]$
 - $C(u, 1) = u; C(1, v) = v \forall u, v \in [0,1]$
 - $(C(u_2, v_2) - C(u_2, v_1)) - (C(u_1, v_2) - C(u_1, v_1)) \geq 0 \forall 0 \leq u_1 \leq u_2 \leq 1 \text{ and } 0 \leq v_1 \leq v_2 \leq 1$
- Joint-distribution
 - Now, given marginal CDFs of X, Y denoted by F_X, F_Y , a the joint-distribution of X, Y can be specified as:
 $F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$
- $V(t) = \int \int (x - y)^+ \frac{\partial F_{X,Y}}{\partial x \partial y}(x, y) dx dy$ which can be reduced to a single integral using conditioning.

Gaussian Copula & Implied Correlations

- The Gaussian Copula, parameterized using a correlation ρ , is defined as:
 $C_{GC}^{\rho}(u, v) = \Phi^{\rho}(\Phi^{-1}(u), \Phi^{-1}(v))$ where Φ^{ρ} is a cumulative joint Normal distribution with correlation ρ , and Φ is a cumulative standard normal distribution
- We define implied correlation ρ_{imp} as the correlation required in a Gaussian copula to recover a given price using a Gaussian Copula model to jointly distribute the CMS indices.
- The implied correlations of prices generated using a Gaussian Copula C_{GC}^{ρ} , plotted as a function of strike will be flat with level ρ , by definition.

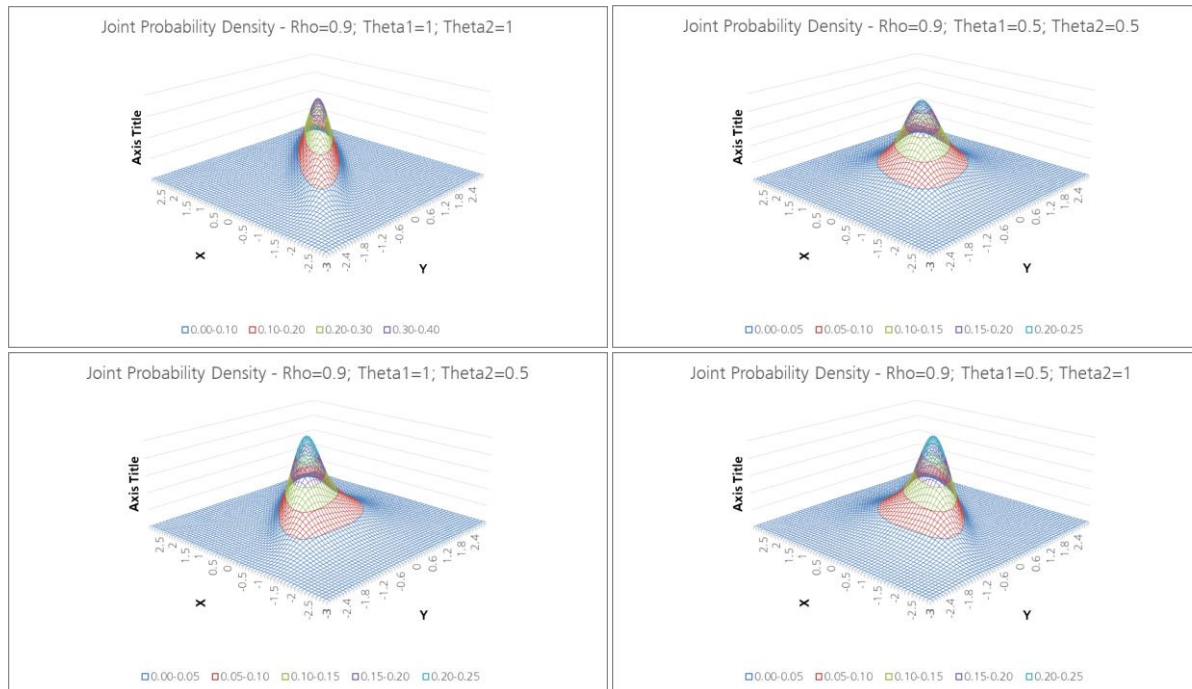


The joint-density charts above are illustrative, and assume Standard Normal marginal distributions.

Power-Gaussian Copula & Implied Correlation Smiles

- The Power-Gaussian Copula, introduced in Andersen & Piterbarg [1], parameterized using ρ, θ_1, θ_2 , is defined as follows:

$$C_{PGC}^{\rho, \theta_1, \theta_2}(u, v) = u^{1-\theta_1} v^{1-\theta_2} \Phi^\rho(\Phi^{-1}(u^{\theta_1}), \Phi^{-1}(v^{\theta_2}))$$
 where Φ^ρ is a cumulative joint Normal distribution with correlation ρ , and Φ is a cumulative standard normal distribution.
- Implied correlation ρ_{imp} is defined as before - as the correlation required in a Gaussian copula to recover the price derived from this Power-Gaussian Copula model.
- As one would expect, the Power-Gaussian allows us to generate an implied correlation smile.
- We characterize and rationalize the domain of smile shapes that this copula can generate.

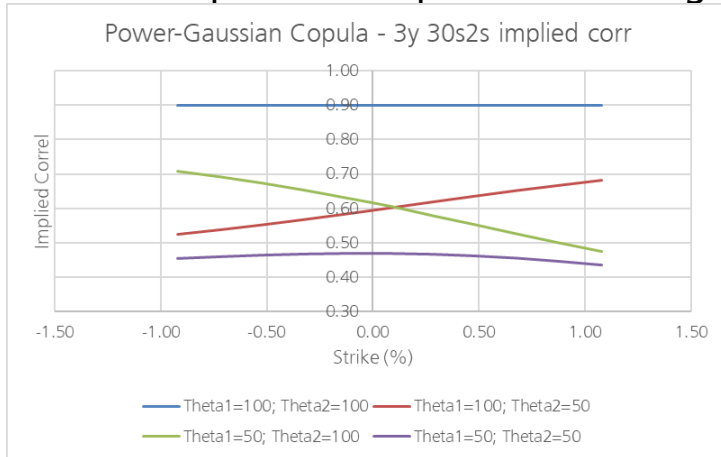


- We use $\rho = 0.9$ in each of these examples.
- $\theta_1 = \theta_2 = 1, \rho = 0.9$ is equivalent to a Gaussian Copula so we have a symmetric (w.r.t X & Y) density concentrated along the diagonal X=Y.
- $\theta_1 = \theta_2 = 0.5$ maintains symmetry, introduces a strong decorrelation and fat tails.
- $\theta_1 = 1; \theta_2 = 0.5$ implies "negative" skew on X-Y.
- $\theta_1 = 0.5; \theta_2 = 1$ implies "positive" skew on X-Y.
- This is consistent with the smile plots we see in the next slide.

The joint-density charts above are illustrative, and assume Standard Normal marginal distributions.

Power-Gaussian Copula & Implied Correlation Smiles (contd.)

- The four examples of PGC parameter sets generate the following implied correlation smiles:



- $\theta_1 = 1; \theta_2 = 0.5$: The -ve skew in X-Y cheapens high-strike options on X-Y, and hence the positive slope.
- $\theta_1 = 0.5; \theta_2 = 1$: A similar argument explains the case when parameters are flipped.
- $\theta_1 = 0.5; \theta_2 = 0.5$: The fat tails explain a richening of the wings, and consequently the concave implied correlations.
- This chart relies on marginal CMS distributions derived from calibrated SABR models.

- The skew injected by lowering θ_1 or θ_2 can be mathematically justified, and is an artifact of PGC.

- Reducing θ_1 or θ_2 richens options / decorrelates X & Y across strikes, the reason being two-fold.

$$C_{PGC}^{\rho, \theta_1, \theta_2}(u, v) = C_{GC}^0(u^{1-\theta_1}, v^{1-\theta_2}) C_{GC}^{\rho}(u^{\theta_1}, v^{\theta_2}), \text{ a product of Gaussian copulas.}$$

1) Reduction of θ_1 & θ_2 : $\lim_{\theta_1, \theta_2 \rightarrow 0} C_{PGC}^{\rho, \theta_1, \theta_2}(u, v) = C_{GC}^0(u, v) C_{GC}^{\rho}(1, 1)$, which would make X & Y independent.

2) Divergence of θ_1 & θ_2 : $\lim_{\theta_1 \rightarrow 1; \theta_2 \rightarrow 0} C_{PGC}^{\rho, \theta_1, \theta_2}(u, v) = C_{GC}^0(1, v) C_{GC}^{\rho}(u, 1) = uv$ which describes independence.

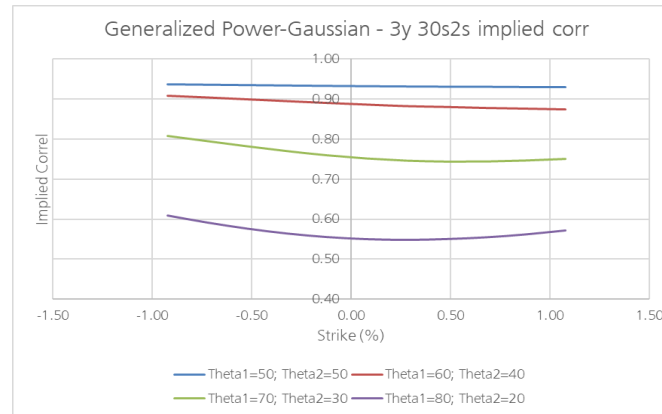
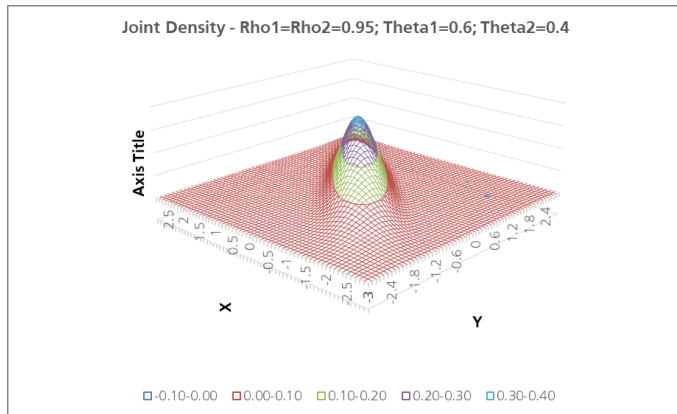
- Along with fat-tails (increasing decorrelation), this means "PGC" is limited to **concave** correlation smiles / increase in spread smile convexity relative to a Gaussian Copula.
- Our goal in generalizing this Copula is to increase the model's flexibility by allowing it to fit **convex** implied correlation smiles.

Generalized Power-Gaussian Copula – motivation & intuition

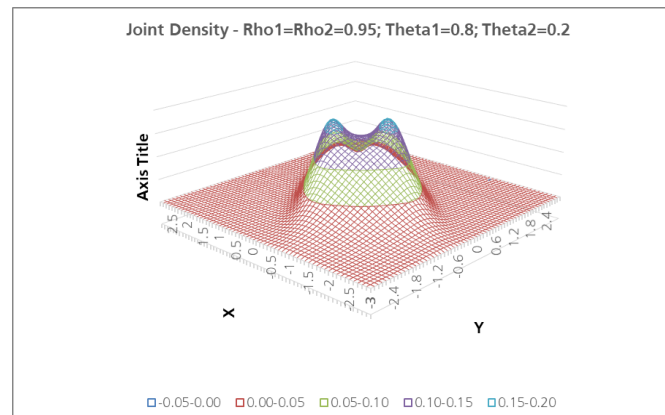
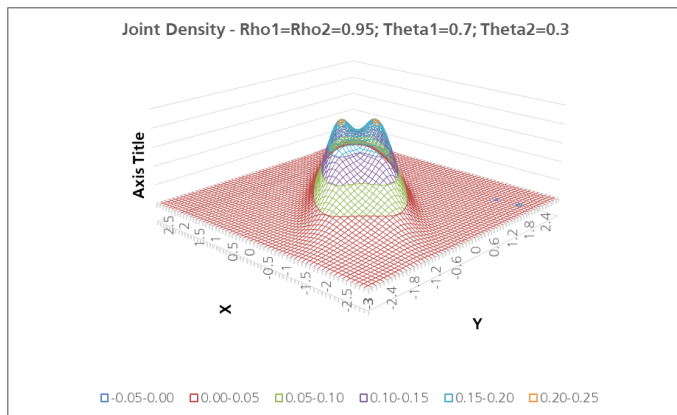
- Motivation: Prior to the recent flattening of the yield-curve, the ATM correlation overpriced low-strikes, increasingly so in the wings, which required a convex and decreasing implied correlation smile.
- As noted in the previous slide, the Power-Gaussian Copula offers us two decorrelation levers:
 - 1) Reduction of θ_1, θ_2 uniformly to increase dependence on the independence copula.
 - 2) Divergence of θ_1, θ_2 to decouple the two variables.
- Notice that the Power-Gaussian Copula is a product of two Gaussian Copulas with correlation ρ & 0.
- We repurpose 1) by replacing C_{GC}^0 with $C_{GC}^{\rho_2}$, with ρ_2 being an extra parameter.
- $C_{GPGC}^{\rho_1, \rho_2, \theta_1, \theta_2}(\mathbf{u}, \mathbf{v}) = C_{GC}^{\rho_2}(\mathbf{u}^{1-\theta_1} \mathbf{v}^{1-\theta_2}) C_{GC}^{\rho_1}(\mathbf{u}^{\theta_1}, \mathbf{v}^{\theta_2})$, verified to satisfy the Copula definition.
- Product of Gaussian Copulas is discussed more generally by Vladimir Lucic [2] in the context of equity basket options including spread options (albeit without attention given to the power-form).
- When ρ_2 is set to a high value, it can induce a convex correlation skew & smile by increasing wing correlation, as we are no longer using C_{GC}^0 .
- Parameter intuition:
 - Divergence of θ_1, θ_2 causes decorrelation across strikes.
 - When $\rho_2 \ll \rho_1$ (e.g. PGC), $\theta_1 < \theta_2$ decorrelates high strikes, $\theta_2 < \theta_1$ decorrelates low strikes, with a concave correlation skew & smile.
 - When $\rho_2 \cong \rho_1$ we see a symmetric convex correlation smile where high and low strikes are pulled towards ρ_1, ρ_2 respy. when $\theta_1 < \theta_2$.
 - When $\rho_1 < \rho_2$; $\theta_1 < \theta_2$ with $|\theta_1 - \theta_2|$ sufficiently big so that $\rho_{imp}^{atm} < \min(\rho_1, \rho_2)$, we see a convex skew with +ve correlation slope.
 - With the ordering of either ρ or θ flipped, we would induce a -ve correlation slopes.
- We look at a few examples in the next slide.

Generalized Power-Gaussian Copula (convex smiles)

- In the examples below, we use $\rho_1 = 0.95; \rho_2 = 0.95$, and try different values of $\theta_1; \theta_2$.



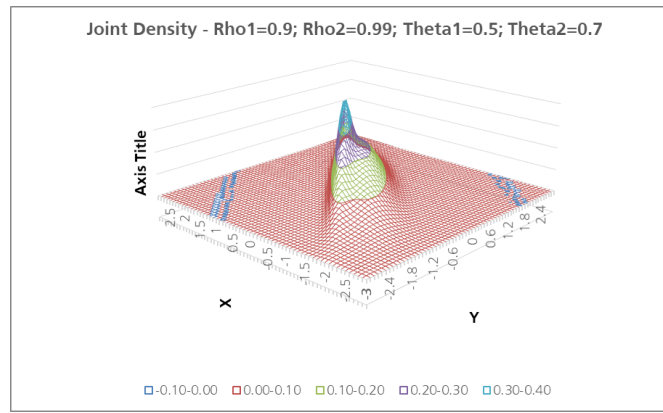
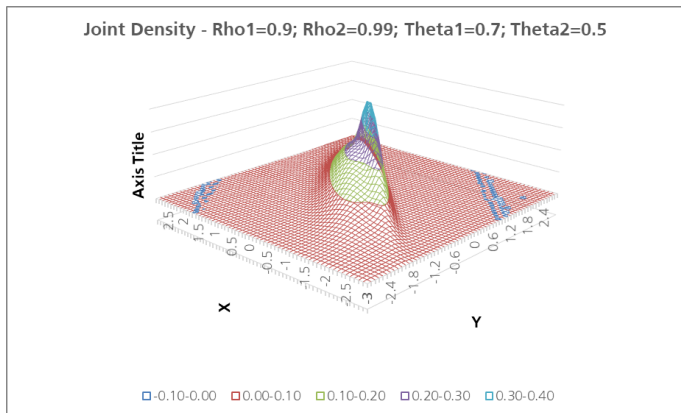
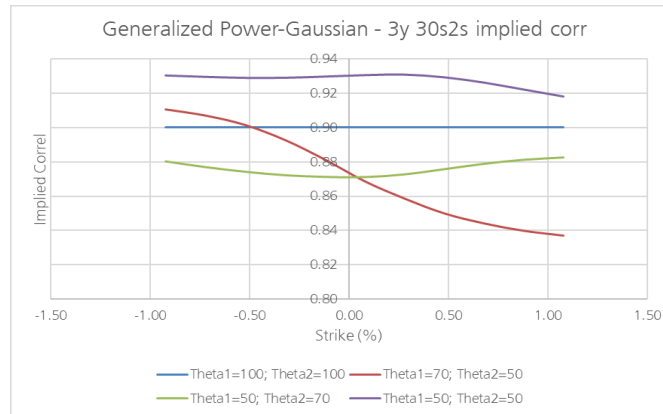
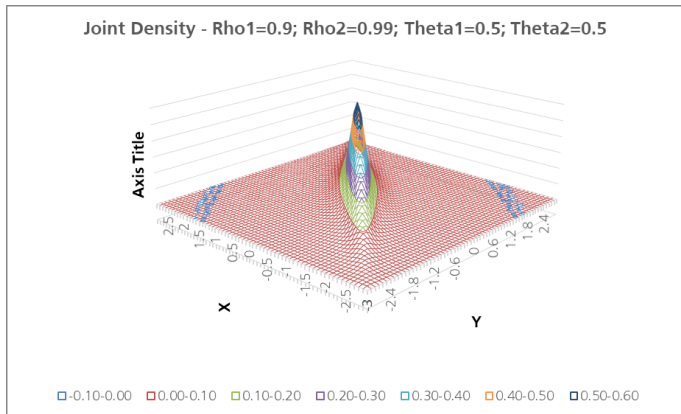
- Symmetric in θ_1 & θ_2 because $\rho_1 = \rho_2$
- When $\theta_1 = \theta_2 = 0.5$ yield ~flat 0.95 correlations mildly decorrelated by splitting u, v b/w identical copulas.
- Divergence of θ_1 and θ_2 give us decorrelation, a bi-modal density, thin-tails resulting in **convex** smiles.
- ATM strikes are the most decorrelated, wings are stretched towards the Gaussians represented by the two modes.



Joint-densities are generated using standard-normal marginals, spread correlations use calibrated CMS distributions.

Generalized Power-Gaussian Copula (s-shaped skew)

- In the examples below, we use $\rho_1 = 0.9$; $\rho_2 = 0.99$.



- $\rho_1 = 0.9$; $\rho_2 = 0.99$ with $\theta_1 = \theta_2 = 0.5$ gives us a rough interpolation b/w 0.9 & 0.99 as $\theta_1 = \theta_2$ prevents strong decorrelation.
- A small divergence b/w θ_1, θ_2 , we induces decorrelation across strikes.
- $\rho_1 < \rho_2$ induces skew. $\theta_1 > \theta_2$, richens high-strike options, and lowers high-strike correlation, and vice-versa.
- Convex smiles are consistent with the apparent thin-tailed densities.
- The inflections at the wide end of the wings are consistent with the increased density at the far end of the plot.

Joint-densities are generated using standard-normal marginals, spread correlations use calibrated CMS distributions.

References

- [1] L. Andersen & V. Piterbarg, *Interest Rate Modeling, Vol 3*
- [2] V. Lucic, *Correlation Skew via Product Copula*

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