Generalized Power-Gaussian Copula & CMS Spread Option Smiles

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Contents

Acknowledgements

- UBS Rates Options Trading – business motivation and market color
- UBS Quantitative Analytics – pricing framework for implementation and analysis

Introduction: CMS Spread Options & Copulas

- CMS spread options are options on the spread between two swap rates observed at expiry and settled as a % of notional.
- $V(t) = P(t,T)\, E^T[(X(T) - Y(T) - K)^+ |\, F_t]$
- To evaluate this, we require the joint distribution of X and Y in the T-forward measure.
- Marginal-distributions in the T-forward measure
  - The marginal distributions of X & Y are known in their respective swap-Annuity measures via standard swaption pricing models such as SABR, calibrated to European Swaptions.
  - These can be transformed to the T-forward measure using standard techniques, for example:
    - a) Represent the Radon-Nikodym derivative as a function of the swap rate
    - b) Static replication of the payoff as a continuum of swaptions (as devised by Carr, Madan, 1998)
  - $E^T[f(S) \mid F_t\,] = \frac{A(t)}{P(t,T)} E^A\left[\frac{P(T,T)}{A(T)}\, f(S)|F_t\right] = \frac{A(t)}{P(t,T)} E^A[\mathrm{g}(S)\, f(S)|F_t]$
- Copula function
  - A function $C: [0,1]^2 \rightarrow [0,1]$ is defined as a (2-dimensional) Copula if:
    - $C(u, 0) = C(0, v) = 0\ \forall\, u, v \in [0,1]$
    - $C(u, 1) = u;\ C(1, v) = v\ \forall\, u, v \in [0,1]$
    - $\big(C(u_2, v_2) - C(u_2, v_1)\big) - \big(C(u_1, v_2) - C(u_1, v_1)\big) \geq 0\ \forall\ 0 \leq u_1 \leq u_2 \leq 1\ and\ 0 \leq v_1 \leq v_2 \leq 1$
- Joint-distribution
  - Now, given marginal CDFs of X, Y denoted by F_X, F_Y, a the joint-distribution of X, Y can be specified as:
    $\boldsymbol{F_{X,Y}(x,y) = C(F_X(x), F_Y(y))}$
- $V(t) = \int\int (x\ - y)^+\ \frac{\partial F_{X,Y}}{\partial x \partial y}(x,y)dxdy$ which can be reduced to a single integral using conditioning.

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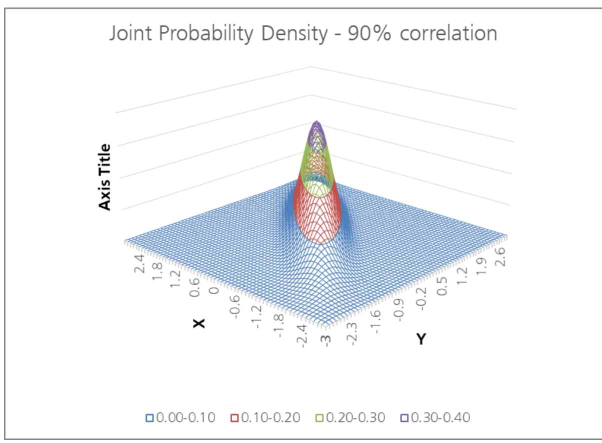
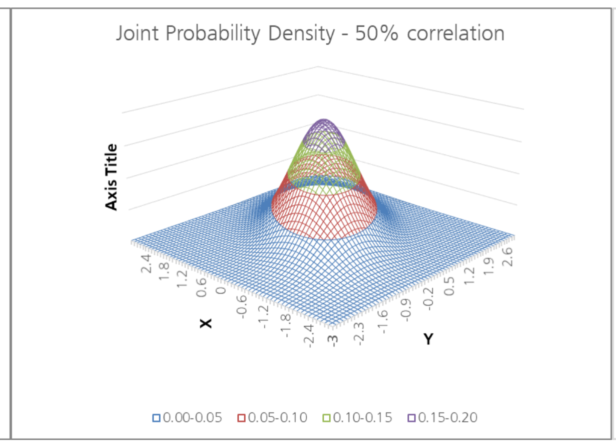
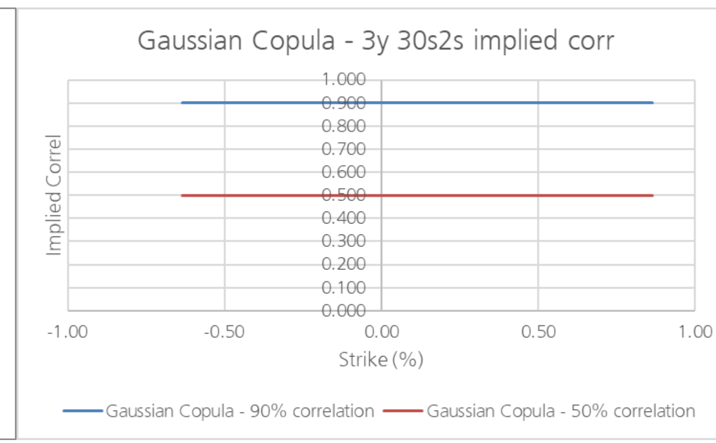
Gaussian Copula & Implied Correlations

- The Gaussian Copula, parameterized using a correlation ρ, is defined as: $C_{GC}^{\rho}(u,v) = \Phi^{\rho}(\Phi^{-1}(u), \Phi^{-1}(v))$ where Φ^{ρ} is a cumulative joint Normal distribution with correlation ρ, and Φ is a cumulative standard normal distribution
- We define implied correlation ρ_{imp} as the correlation required in a Gaussian copula to recover a given price using a Gaussian Copula model to jointly distribute the CMS indices.
- The implied correlations of prices generated using a Gaussian Copula C_{GC}^{ρ}, plotted as a function of strike will be flat with level ρ, by definition.







The joint-density charts above are illustrative, and assume Standard Normal marginal distributions.

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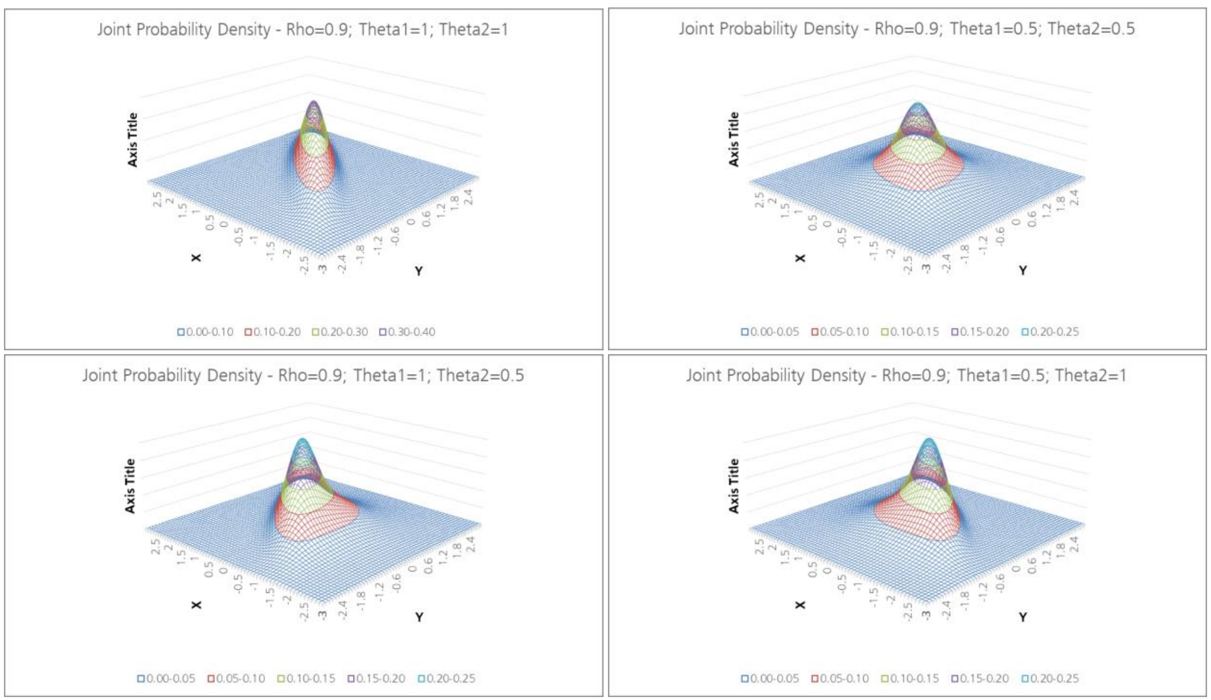
Power-Gaussian Copula & Implied Correlation Smiles

- The Power-Gaussian Copula, introduced in Andersen & Piterbarg [1], parameterized using ρ, θ_1, θ_2, is defined as follows:
  $C_{PGC}^{\rho,\theta_1,\theta_2}(u,v) = \mathrm{u}^{1-\theta_1}\mathrm{v}^{1-\theta_2}\Phi^{\rho}\left(\Phi^{-1}\left(u^{\theta_1}\right),\Phi^{-1}\left(v^{\theta_2}\right)\right)$ where Φ^{ρ} is a cumulative joint Normal distribution with correlation ρ, and Φ is a cumulative standard normal distribution.
- Implied correlation ρ_{imp} is defined as before - as the correlation required in a Gaussian copula to recover the price derived from this Power-Gaussian Copula model.
- As one would expect, the Power-Gaussian allows us to generate an implied correlation smile.
- We characterize and rationalize the domain of smile shapes that this copula can generate.



The joint-density charts above are illustrative, and assume Standard Normal marginal distributions.

- We use $\rho = 0.9$ in each of these examples.
- $\theta_1, = \theta_2 = 1, \rho = 0.9$ is equivalent to a Gaussian Copula so we have a symmetric (w.r.t X & Y) density concentrated along the diagonal X=Y.
- $\theta_1 = \theta_2 = 0.5$ maintains symmetry, introduces a strong decorrelation and fat tails.
- $\theta_1 = 1; \theta_2 = 0.5$ implies "negative" skew on X-Y.
- $\theta_1 = 0.5; \theta_2 = 1$ implies "positive" skew on X-Y.
- This is consistent with the smile plots we see in the next slide.

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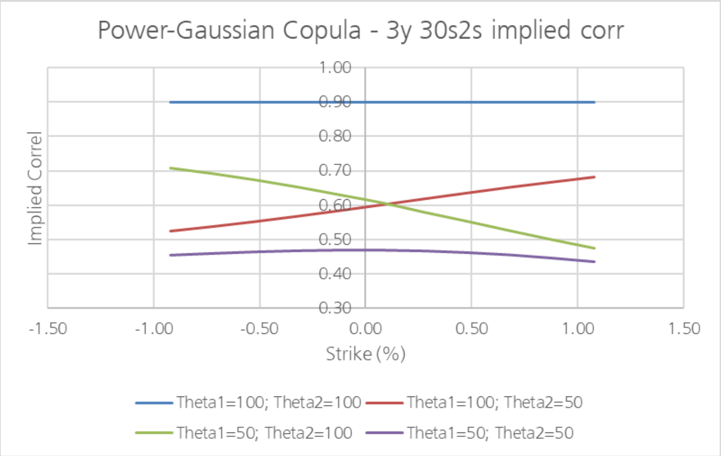
Power-Gaussian Copula & Implied Correlation Smiles (contd.)

- The four examples of PGC parameter sets generate the following implied correlation smiles:



  - $\theta_1 = 1; \theta_2 = 0.5$: The –ve skew in X-Y cheapens high-strike options on X-Y, and hence the positive slope.
  - $\theta_1 = 0.5; \theta_2 = 1$:A similar argument explains the case when parameters are flipped.
  - $\theta_1 = 0.5; \theta_2 = 0.5$: The fat tails explain a richening of the wings, and consequently the concave implied correlations.
  - This chart relies on marginal CMS distributions derived from calibrated SABR models.

- The skew injected by lowering θ_1 or θ_2 can be mathematically justified, and is an artifact of PGC.

- Reducing θ_1 or θ_2 richens options / decorrelates X & Y across strikes, the reason being two-fold. $C_{PGC}^{\rho,\theta_1,\theta_2}(u,v) = C_{GC}^{0}\left(u^{1-\theta_1}, v^{1-\theta_2}\right) C_{GC}^{\rho}\left(u^{\theta_1}, v^{\theta_2}\right)$, a product of Gaussian copulas.

  1) Reduction of θ_1& θ_2: $\lim_{\theta_1,\theta_2 \to 0} C_{PGC}^{\rho,\theta_1,\theta_2}(u,v) = C_{GC}^{0}(u,v) C_{GC}^{\rho}(1,1)$, which would make X & Y independent.

  2) Divergence of θ_1 & θ_2: $\lim_{\theta_1 \to 1; \theta_2 \to 0} C_{PGC}^{\rho,\theta_1,\theta_2}(u,v) = C_{GC}^{0}(1,v) C_{GC}^{\rho}(u,1) = uv$ which describes independence.

- Along with fat-tails (increasing decorrelation), this means "PGC" is limited to **concave** correlation smiles / increase in spread smile convexity relative to a Gaussian Copula.

- Our goal in generalizing this Copula is to increase the model's flexibility by allowing it to fit **convex** implied correlation smiles.

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Generalized Power-Gaussian Copula – motivation & intuition

- Motivation: Prior to the recent flattening of the yield-curve, the ATM correlation overpriced low-strikes, increasingly so in the wings, which required a convex and decreasing implied correlation smile.
- As noted in the previous slide, the Power-Gaussian Copula offers us two decorrelation levers:

  1) Reduction of θ_1, θ_2 uniformly to increase dependence on the independence copula.

  2) Divergence of θ_1, θ_2 to decouple the two variables.
- Notice that the Power-Gaussian Copula is a product of two Gaussian Copulas with correlation ρ & 0.
- We repurpose 1) by replacing C_{GC}^{0} with $C_{GC}^{\rho_2}$, with ρ_2 being an extra parameter.
- $\boldsymbol{C_{GPGC}^{\rho_1,\rho_2,\theta_1,\theta_2}(u,v) = C_{GC}^{\rho_2}\left(u^{1-\theta_1}v^{1-\theta_2}\right)C_{GC}^{\rho_1}\left(u^{\theta_1}, v^{\theta_2}\right)}$, verified to satisfy the Copula definition.
- Product of Gaussian Copulas is discussed more generally by Vladimir Lucic [2] in the context of equity basket options including spread options (albeit without attention given to the power-form).
- When ρ_2 is set to a high value, it can induce a convex correlation skew & smile by increasing wing correlation, as we are no longer using C_{GC}^{0}.
- Parameter intuition:
  - Divergence of θ_1, θ_2 causes decorrelation across strikes.
  - When $\rho_2 \ll \rho_1$ (e.g. PGC), $\theta_1 < \theta_2$ decorrelates high strikes, $\theta_2 < \theta_1$ decorrelates low strikes, with a concave correlation skew & smile.
  - When $\rho_2 \cong \rho_1$ we see a symmetric convex correlation smile where high and low strikes are pulled towards ρ_1, ρ_2 respy. when $\theta_1 < \theta_2$.
  - When $\rho_1 < \rho_2$; $\theta_1 < \theta_2$ with $|\theta_1 - \theta_2|$ sufficiently big so that $\rho_{imp}^{atm} < \min(\rho_1, \rho_2)$, we see a convex skew with +ve correlation slope.
  - With the ordering of either ρ or θ flipped, we would induce a –ve correlation slopes.
- We look at a few examples in the next slide.

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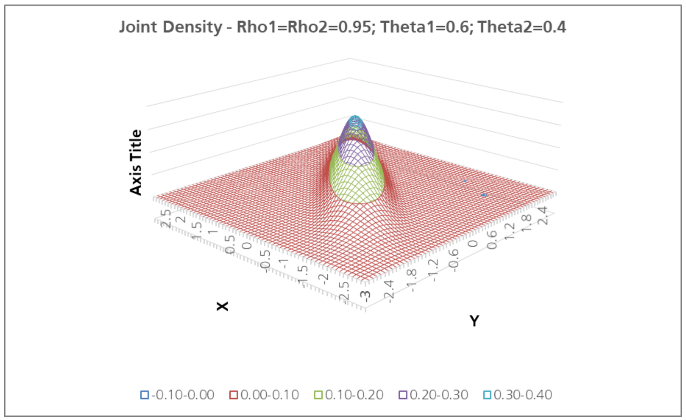
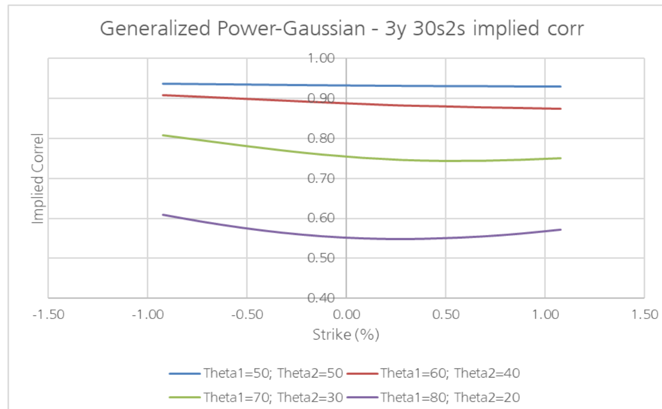
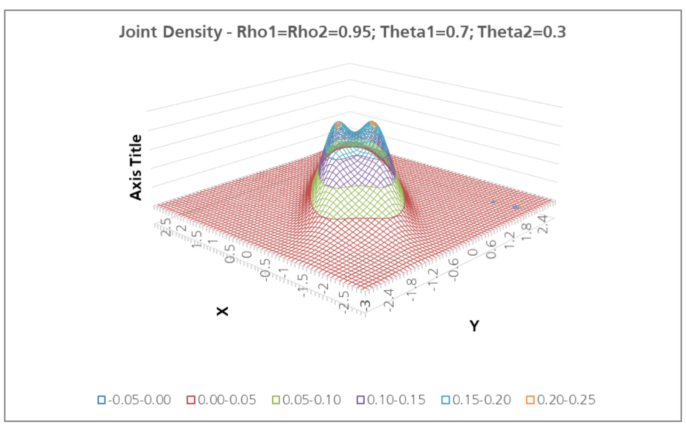
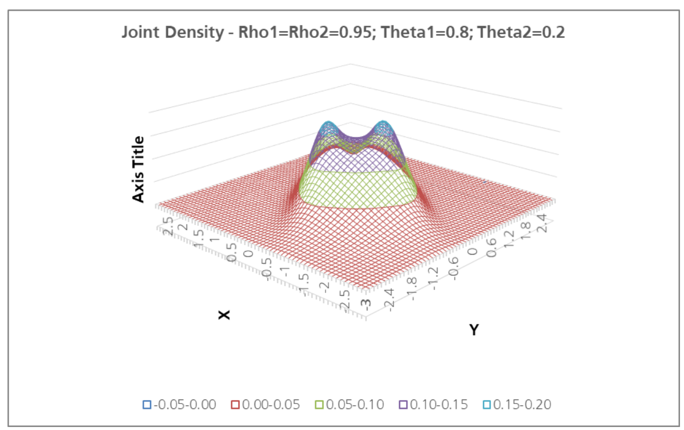
Generalized Power-Gaussian Copula (convex smiles)

- In the examples below, we use $\rho_1 = 0.95; \rho_2 = 0.95$, and try different values of $\theta_1; \theta_2$.







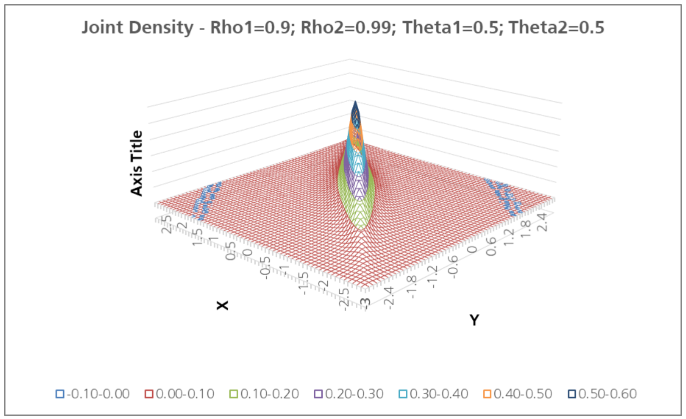


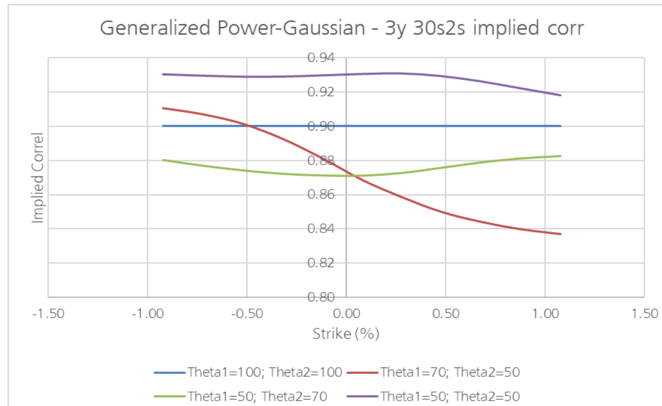
- Symmetric in θ_1 & θ_2 because $\rho_1 = \rho_2$
- When $\theta_1 = \theta_2 = 0.5$ yield ~flat 0.95 correlations mildly decorrelated by splitting u, v b/w identical copulas.
- Divergence of θ_1 and θ_2 give us decorrelation, a bi-modal density, thin-tails resulting in **convex** smiles.
- ATM strikes are the most decorrelated, wings are stretched towards the Gaussians represented by the two modes.

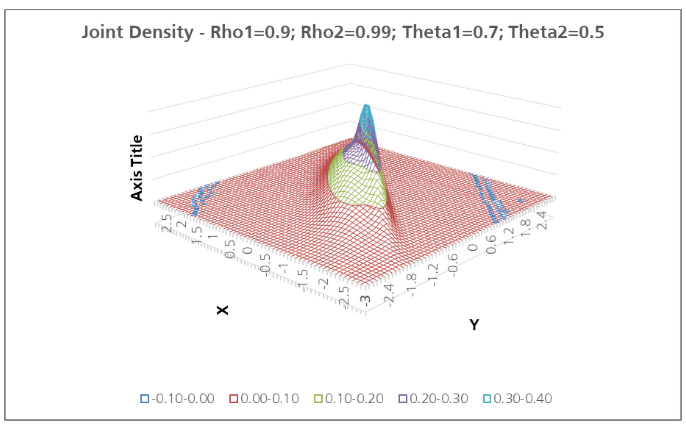
Joint-densities are generated using standard-normal marginals, spread correlations use calibrated CMS distributions.

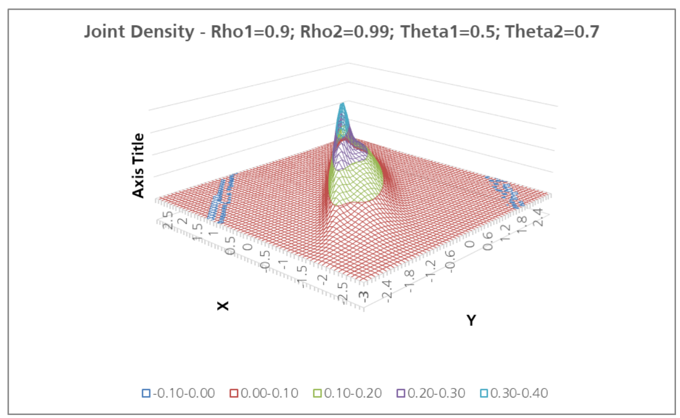
Generalized Power-Gaussian Copula (s-shaped skew)

- In the examples below, we use $\rho_1 = 0.9; \rho_2 = 0.99$.









- $\rho_1 = 0.9; \rho_2 = 0.99$ with $\theta_1 = \theta_2 = 0.5$ gives us a rough interpolation b/w 0.9 & 0.99 as $\theta_1 = \theta_2$ prevents strong decorrelation.
- A small divergence b/w θ_1, θ_2, we induces decorrelation across strikes.
- $\rho_1 < \rho_2$ induces skew. $\theta_1 > \theta_2$, richens high-strike options, and lowers high-strike correlation, and vice-versa.
- Convex smiles are consistent with the apparent thin-tailed densities.
- The inflections at the wide end of the wings are consistent with the increased density at the far end of the plot.

Joint-densities are generated using standard-normal marginals, spread correlations use calibrated CMS distributions.

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References

- [1] L. Andersen & V. Piterbarg, *Interest Rate Modeling, Vol 3*
- [2] V. Lucic, *Correlation Skew via Product Copula*

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