NYU | TANDON SCHOOL OF ENGINEERING

SQA Society of Quantitative Analysts

JUNE 2 - JUNE 4

THE PETER CARR MEMORIAL CONFERENCE

Pseudo Sums, Contingent Claims and a Generalized Memoryless Property.

zh aw Pasquale Cirillo

Simply put, Peter was a terrific person.

I have learned a lot from him.

zh aw

@DrCirillo

With Peter, we used to have regular scientific discussions.
In person, via mail, and in video calls.

Since it is the CDF of a positive random variable (with mean one), do you know what the corresponding Lorenz curve is?
As you probably appreciate, the CPD Random Variable can be a Radon Nikodym derivative.

best
p

--
Peter Carr
Dept. Chair, NYU Finance and Risk Engineering
917 621 7733

Found in Sent - NYU Mailbox

PC **Pasquale Cirillo** 24 January 2022 at 20:33
Re: fun calculation for you
To: Peter Carr

Hi Peter,
The Lorenz function L(u) is defined as the integral of the quantile function Q(p) from 0 to u, over the mean of the random variable. Since this mean is here 1, we just have the numerator.
The quantile function of the CPD is Q(p)=(p^(1/(b-1))-1)^(-b).
Hence the Lorenz is equal to (u-u^(b/(b-1))+(u^(1/(b-1))-1)^b) (u^(1/(b-1))-1)^(-b).
It looks to me like a special case of the Lorenz for a 3-paramemter Dagum, but I have to check to be sure.

Best,

P.

See More from Peter Carr

PC **Peter Carr** 24 January 2022 at 21:28
Re: fun calculation for you
To: Pasquale Cirillo

Wow, thx but I fear a typo creeped into your final formula.
As written, the product at the end
(u^(1/(b-1))-1)^b) (u^(1/(b-1))-1)^(-b)
simplifies to
(u^(1/(b-1))-1)^(2b)
right?

See More from Pasquale Cirillo

zh aw

@DrCirillo

Since it is the CDF of a positive random variable (with mean one), do you know what the corresponding Lorenz curve is?
As you probably appreciate, the CPD Random Variable can be a Radon Nikodym derivative.

best
p

--
Peter Carr
Dept. Chair, NYU Finance and Risk Engineering
917 621 7733

Found in Sent - NYU Mailbox

Pasquale Cirillo 24 January 2022 at 20:33
Re: fun calculation for you
To: Peter Carr

Hi Peter,
The Lorenz function L(u) is defined as the integral of the quantile function Q(p) from 0 to u, over the mean of the random variable. Since this mean is here 1, we just have the numerator.
The quantile function of the CPD is Q(p)=(p^(1/(b-1))-1)^(-b).
Hence the Lorenz is equal to (u-u^(b/(b-1))+(u^(1/(b-1))-1)^b) (u^(1/(b-1))-1)^(-b).
It looks to me like a special case of the Lorenz for a 3-paramemter Dagum, but I have to check to be sure.

Best,

P.

See More from Peter Carr

With Peter, we used to have regular scientific discussions.
In person, via mail, and in video calls.

Peter Carr 24 January 2022 at 21:28
Re: fun calculation for you
To: Pasquale Cirillo

He was exceptionally competent and precise.

Wow, thx but I fear a typo creeped into your final formula.
As written, the product at the end
(u^(1/(b-1))-1)^b) (u^(1/(b-1))-1)^(-b)
simplifies to
(u^(1/(b-1))-1)^(2b)
right?

(I forgot a pair of brackets while typing)

See More from Pasquale Cirillo

@DrCirillo

But I could "return the favor" with tips about food and restaurants.
Everytime I was in NYC, we would visit at least a restaurant together.

Also, I took my wife to [redacted] last night. I had the same dish as when we ate there. I met the owner. I want to thank you for letting me know I live on the same street as this fabulous restaurant.
best
p

Bijections Between $[0, \infty)$ and the Support of a Continuous Probability Law Allow a Generalized Memoryless Property

Peter Carr and Pasquale Cirillo

FRE NYU and ZHAW

Abstract

The law of a continuous random variable X supported on an interval S enjoys the *memoryless property* if for all $x_1, x_2 \in S$, the probability measure $\mathbb{P}$ satisfies:

$$\mathbb{P}\{X \geq x_1 + x_2 | X \geq x_1\} = \mathbb{P}\{X \geq x_2\}.$$

It is well-known that the only continuous law enjoying the memoryless property is the exponential law supported on $S = [0, \infty)$. In this note, we show that the memoryless property can be generalized by replacing $+$ with a strictly-increasing associative binary operation $\oplus_G$ defined by:

$$x_1 \oplus_G x_2 \equiv G(G^{-1}(x_1) + G^{-1}(x_2)),$$

where the *generator* G is a strictly-monotonic map of all of $[0, \infty)$ onto the support interval S for the law of X. The law of a continuous random variable X supported on S enjoys the *generalized memoryless property* (GMP) if for all $x_1, x_2 \in S$, the probability measure $\mathbb{P}$ satisfies either:

$$\mathbb{P}\{X \geq x_1 \oplus_G x_2 | X \geq x_1\} = \mathbb{P}\{X \geq x_2\}$$

Preprint available here:

https://ssrn.com/abstract=4081193

@DrCirillo

What if “$1 + 1 = 1$”?

What if "$1 \oplus 1 = 1$"?

So…when do we have “$1 \oplus 1 = 1$”?

1) Define $x \oplus y := \exp(\log x + \log y) = xy$

2) $1 \oplus 1 = \exp(\log 1 + \log 1) = \exp(0+0) = \exp(0)$
$= \underline{1}$

zh
aw

@DrCirillo

Pseudo Sums

1) Take $S \in \overline{\mathbb{R}}$.
2) Let $G : \mathbb{R} \to S$ be a strictly monotonic function.
3) Define $x \oplus_G y = G(G^{-1}(x) + G^{-1}(y))$.

@DrCirillo

Pseudo Sums

1) Take $S \in \overline{\mathbb{R}}$.

2) Let $G : \mathbb{R} \to S$ be a strictly monotonic function.

3) Define $x \oplus_G y = G(G^{-1}(x) + G^{-1}(y))$.

PSEUDO SUM

Pseudo Sums

1) Take $S \in \overline{\mathbb{R}}$.
2) Let $G : \mathbb{R} \to S$ be a strictly monotonic function.
3) Define $x \oplus_G y = G(G^{-1}(x) + G^{-1}(y))$.
4) $(S : \oplus_G, e)$ is a monoid, with e identity element.

$$x \oplus_G e = e \oplus_G x = x$$
and $e = G(0)$.

Pseudo Sums

- To simplify, take $x, y \geq 0$.
- Set:
  - $G(x) = \exp(x) \longrightarrow x \oplus_G y = xy$
  - $G(x) = x^p, p \geq 0 \longrightarrow x \oplus_G y = (x^p + y^p)^{\frac{1}{p}}$

    $\downarrow$

    for $p \to \infty$ $\quad x \oplus_G y = \max(x, y)$
  - $G(x) = \log(e^x + 1) \longrightarrow x \oplus_G y = \log(e^x + e^y + e^{x+y})$
  - etc.

Pseudo Sums

- To simplify, take $x, y \geq 0$.
- Set: * $G(x) = \exp(x) \longrightarrow x \oplus_G y = xy$
  * $G(x) = x^p,\ p \geq 0 \longrightarrow x \oplus_G y = (x^p + y^p)^{\frac{1}{p}}$

    ISN'T THIS A PAYOFF?

    $\downarrow$

    for $p \to \infty$ $\quad x \oplus_G y = \max(x, y)$

  * $G(x) = \log(e^x + 1) \longrightarrow x \oplus_G y = \log(e^x + e^y + e^{x+y})$
  * etc.

@DrCirillo

A continuous random variable X shows the memoryless property if, for $x_1, x_2 \in \mathbb{R}$, one has

$$P\left(X \geq x_1 + x_2 \mid X \geq x_1\right) = P\left(X \geq x_2\right).$$

@DrCirillo

A continuous random variable X shows the memoryless property if, for $x_1, x_2 \in \mathbb{R}$, one has

$$P(X \geq x_1 + x_2 \mid X \geq x_1) = P(X \geq x_2).$$

The only continuous r.v. satisfying this is the Exponential.

@DrCirillo

But what happens if we define

$$P(X \geq x_1 \oplus x_2 \mid X \geq x_1) = P(X \geq x_2) \quad ?$$

But what happens if we define

$$P(X \geq x_1 \oplus x_2 \mid X \geq x_1) = P(X \geq x_2)\ ?$$

We call this GMP.

@DrCirillo

Why do we care?

New results

Flexibility

Because pseudo-analysis not only provides extremely useful tools from a probabilistic point of view, but it also has very interesting applications in finance.

New insights

Simplicity

Elegance

Example 1

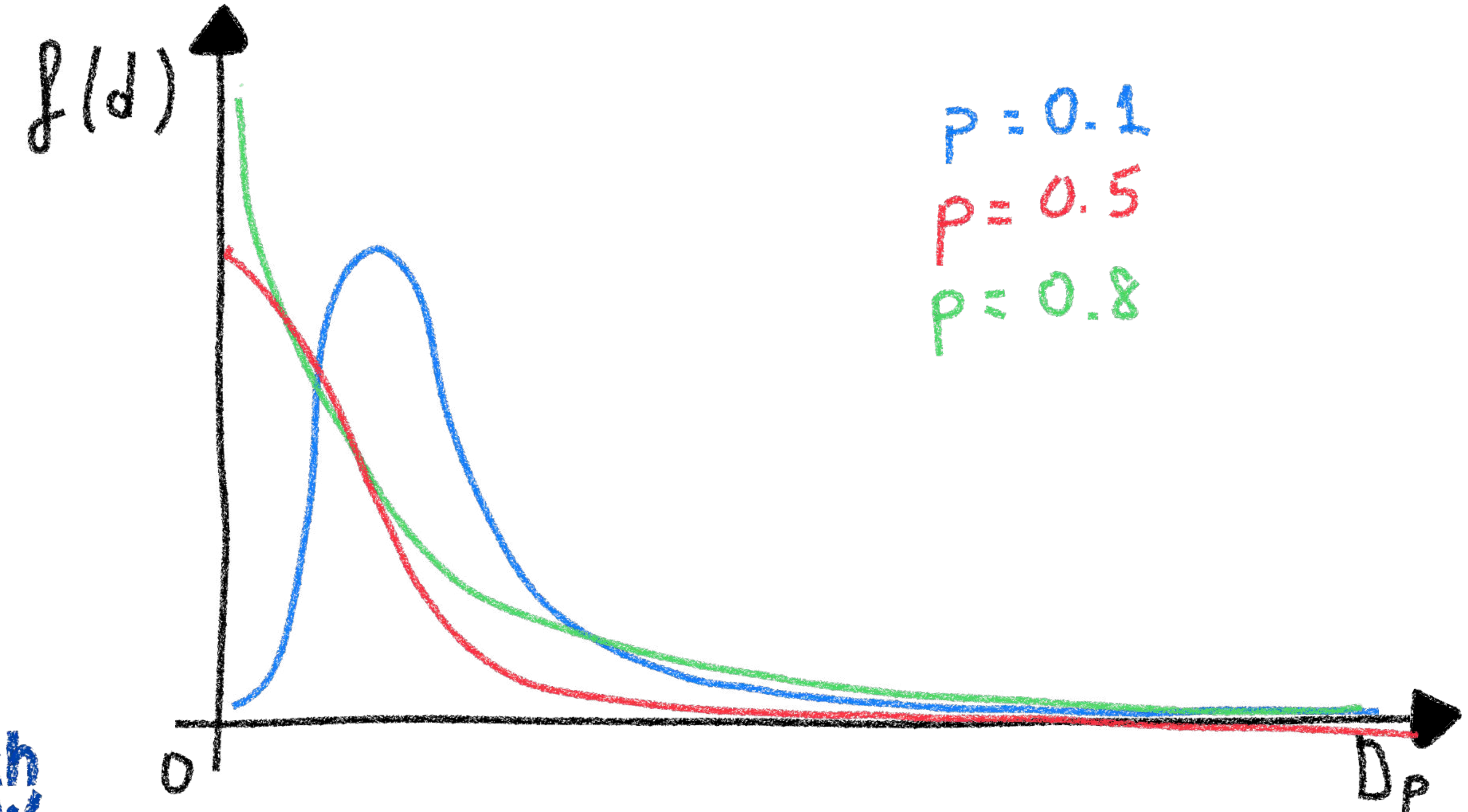
- Take $G(x) = x^p$, with $x \geq 0$ and $p \geq 0$.
- Then $x \oplus_G y = (x^p + y^p)^{\frac{1}{p}}$.
- If $P(X \geq x_1 \oplus_G x_2 \mid X \geq x_1) = P(X \geq x_2)$, then $X \sim \text{Weibull}\left(\lambda, \frac{1}{p}\right)$

$$F(x) = 1 - e^{-\lambda x^{\frac{1}{p}}}.$$

@DrCirillo

Example 1

- Moreover, if $p \in (0,1)$, $x \oplus_\alpha y = (x^p + y^p)^{\frac{1}{p}} = E[x \vee y D_p]$, where D_p is a Conjugate Power Dagum r.v. with $P(D_p \leq d) = \left(1 + d^{-\frac{1}{p}}\right)^{p-1}$, $d > 0$.



- One can show that $E[D_b] = 1$.
- Hence D_b is also a Radon-Nikodym derivative.

@DrCirillo

Example 1

- And $E[x \vee y D_p]$ is the arbitrage-free value of a zero-coupon default-free convertible bond with face-value $x \geq 0$ and conversion value $y \geq 0$.
- $D_p > 0$ is thus the gross return on the convertible bond's underlying stock.

@DrCirillo

Example 2

- If $G(x) = -b \log x$, with $x, b > 0$, then the GMP distribution is the Gumbel.
- $x \oplus_G y = -b \log\left(e^{-\frac{x}{b}} + e^{-\frac{y}{b}}\right) = E[x \wedge (y + bZ)]$, where Z is a standard logistic r.v.

Example 2

- $E[x \wedge (y + b2)]$ is the arbitrage-free value of a zero-coupon defaultable-bond with face value x, and whose underlying security has initial value y.
- b2 is the change in the value of the operations of the firm.

Thank you all,
thank you Peter.