Convolution-FFT for option pricing in the Heston model

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Overview

- Provide an analytic expression for the joint characteristic function of the log-price and variance in the Heston model that addresses the discontinuity problem.
- Consider option valuation using the fast Fourier transform (FFT) and convolution.
- Apply shifting and damping transforms to improve boundary errors and prove an error estimate.
- Numerical experiments and comparisons to Carr and Madan (1999) illustrate the speed and accuracy of our approach.

Fourier Methods in Option Pricing

- Let S_t be the (spot) price of the underlying asset at time t.
- Write x_t for the log of the stock price.
- The characteristic function of x_T is

$$\phi_T(u) = \mathbb{E}[\exp(iux_T)].$$

- For constant interest rates r and no dividends, the price of the European call option is

$$C = S\Pi_1 - Ke^{-rT}\Pi_2$$

where

$$\Pi_2 = \mathbb{P}(S_T > K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left(\frac{e^{-iu\ln K}\phi_T(u)}{iu}\right)du$$

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left(\frac{e^{-iu\ln K}\phi_T(u-i)}{iu\phi_T(-i)}\right)du.$$

Carr and Madan (1999)

- Let k be the log strike price and $C_T(k)$ the price of the call option with strike $\exp(k)$ and maturity T.
- The risk-neutral density of the log price x_T is $q_T(x)$.
- Then the characteristic function is

$$\phi_T(u) = \int_{-\infty}^{\infty} e^{iux} q_T(x) dx.$$

- The price at time-zero of the call option is

$$C_T(k) = \int_{k}^{\infty} e^{-rT}(e^x - e^k) q_T(x) dx.$$

Carr and Madan (1999)

- Define $c_T(k) = \exp(\alpha k) C_T(k)$ for $\alpha > 0$ so that $c_T(k)$ is square integrable.
- The Fourier transform of $c_T(k)$ is

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk.$$

- Carr and Madan (1999) show that

$$\psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}.$$

- Taking the inverse Fourier transform

$$\begin{aligned} C_T(k) &= \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) dv \\ &= \frac{\exp(-\alpha k)}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_T(v) dv. \end{aligned} \tag{1}$$

Carr and Madan (1999)

- The integral (1) can be approximated as

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) \eta$$

  where $v_j = \eta(j-1)$.
- The FFT can be used to efficently calculate the sum.
- There are various numerical issues that must be handled with care and we also need to specify the model for S.
- Many papers have been written addressing numerical issues, model specific issues, and related Fourier methods.
- We shall consider the Heston (1993) model and methods inspired by Carr and Madan (1999).

Heston (1993) Model

- Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.
- The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by two independent Wiener processes satisfying the usual conditions of completeness and right continuity.
- The Heston model under$\mathbb{P}$ can be written as

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t \left(\rho dW_{1t} + \sqrt{1-\rho^2} dW_{2t} \right), \quad (2)$$

$$dv_t = \kappa \left(\theta - v_t \right) dt + \sigma \sqrt{v_t} dW_{1t}, \quad (3)$$

where $\rho \in [-1, +1]$ is the correlation coefficient between W_{1t} and W_{2t} the two independent Wiener process.

- Assume $2\kappa\theta \geq \sigma^2$, so that the zero boundary is unattainable and $v_t > 0$.

Heston Model

- Assume the market price of risk (Λ_1, Λ_2) satisfies

$$\frac{\mu - r}{\sqrt{v_t}} = \rho \Lambda_1 + \sqrt{1-\rho^2}\Lambda_2. \tag{4}$$

- Define an equivalent measure $\mathbb{Q}^\Lambda$ on $\mathcal{F}_t$ by

$$\left.\frac{d\mathbb{Q}^\Lambda}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t (\Lambda_{1s}^2 + \Lambda_{2s}^2)\,ds + \int_0^t \Lambda_{1s}\,dW_1(s) + \int_0^t \Lambda_{2s}\,dW_2(s)\right).$$

- To obtain a complete Heston model let $\Lambda_{1t} = \Lambda\sqrt{v_t}$, for $\Lambda > 0$.
- Λ_2 is uniquely determined by equation (4).

Heston Model

- By Girsanov's theorem

$$dW_1^{\Lambda}(t) = dW_1(t) + \Lambda\sqrt{v_t}dt,$$
$$dW_2^{\Lambda}(t) = dW_2(t) + \frac{\mu - r - \Lambda\rho v_t}{\sqrt{(1-\rho^2)v_t}}dt,$$

are independent Wiener processes under $\mathbb{Q}^{\Lambda}$.

- The risk-neutral Heston dynamics are

$$dS_t = rS_t dt + \sqrt{v_t}S_t\left(\rho dW_{1t}^{\Lambda} + \sqrt{1-\rho^2}dW_{2t}^{\Lambda}\right),$$
$$dv_t = \bar{\kappa}\left(\bar{\theta} - v_t\right)dt + \sigma\sqrt{v_t}dW_{1t}^{\Lambda},$$

where $\bar{\kappa} = (\kappa + \sigma\Lambda)$ and $\bar{\theta} = \kappa\theta/\bar{\kappa}$ for $\bar{\kappa} \neq 0$.

Heston Model

- Let $x_t = \log\left(\frac{S_t}{S_0}\right)$ then the joint process $X_t = (x_t, v_t)^T$ is given by

$$dX_t = \eta(v_t, t)dt + \sqrt{v_t}\xi dW_t^{\Lambda}, \tag{5}$$

  where

$$\eta(v_t, t) = \begin{pmatrix} r - \frac{1}{2}v_t \\ \bar{\kappa}(\bar{\theta} - v_t) \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \rho & \sqrt{1-\rho^2} \\ \sigma & 0 \end{pmatrix}.$$

- We consider the characteristic function of the joint variable $X = (x, v)$.

Pricing Formula

- The time-t price of the call option with strike K and expiry T is

$$\begin{aligned} C_t =& e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \,|S_t, v_t\right] \\ =& S_t\mathbb{E}^{\mathbb{S}}\left[\mathbf{1}_{S_T>K}\,|S_t, v_t\right] - Ke^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{S_T>K}\,|S_t, v_t\right], \end{aligned}$$

  where $F(t, T) = e^{r(T-t)}S_t$ is the forward price, as seen from t, and the equivalent martingale measure $\mathbb{S}$ is $\frac{d\mathbb{S}}{d\mathbb{Q}} = \frac{S_T}{F(t,T)}$.

- Write $\mathbb{P}_1 = \mathbb{S}$ and $\mathbb{P}_2 = \mathbb{Q}$, under which

$$\begin{aligned} P_1(S_T, K) =& \mathbb{P}_1(S_T \geq K), \\ P_2(S_T, K) =& \mathbb{P}_2(S_T \geq K), \end{aligned}$$

  and the pricing formula becomes

$$C_t = S_t P_1\left(S_T, K\right) - Ke^{-r\tau}P_2\left(S_T, K\right). \tag{6}$$

Characteristic Function

- The characterisitic function in the Heston (1993) model can be written in different ways and its properties and use in finance have been studied by many authors: e.g.,
  - Kahl and Jäckel (2005), Gatheral (2006), Albrecher et al. (2007),
  - Lord and Kahl (2010), Lucic (2015), Cui et al. (2017).
- The characteristic function of $X_t = (x_t, v_t)^T$ under measure $\mathbb{P}$ with parameter $U = (p, q)^T$ given the current state $X = (x, v)^T$ is defined by

$$\varphi(U, X, t) = \mathbb{E}^{\mathbb{P}_i}\left[e^{\mathrm{i}U^T X_T} | X_t = (x, v)^T\right], \tag{7}$$

with boundary $\varphi(U, X, T) = e^{\mathrm{i}U^T X}$.

Theorem 1

The joint characteristic function of $X_t = (x_t, v_t)^T$ under measure P_i is

$$\varphi_i(p,q) = \exp\left(\mathrm{i}p(x+r\tau) + \mathrm{i}q(v+a\tau) + \frac{\gamma+\lambda}{\sigma^2}(1-\zeta)v - \frac{\gamma-\lambda}{\sigma^2}a\tau + \frac{2a}{\sigma^2}\ln\zeta\right), \tag{8}$$

where $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, $a = \bar{\kappa}\bar{\theta}$, $b_1 = \bar{\kappa} + \Lambda\sigma - \rho\sigma$, $b_2 = \bar{\kappa} + \Lambda\sigma$ *for* $i = 1, 2$,

$$\gamma = \sqrt{\sigma^2\left(p^2 - 2\mathrm{i}c_i p\right) + \left(b_i - \mathrm{i}\sigma\rho p\right)^2}, \tag{9}$$

$$\lambda = b_i - \mathrm{i}\sigma\rho p - \mathrm{i}\sigma^2 q, \tag{10}$$

$$\zeta = \frac{2\gamma}{\gamma + \lambda + (\gamma - \lambda)e^{-\gamma\tau}}. \tag{11}$$

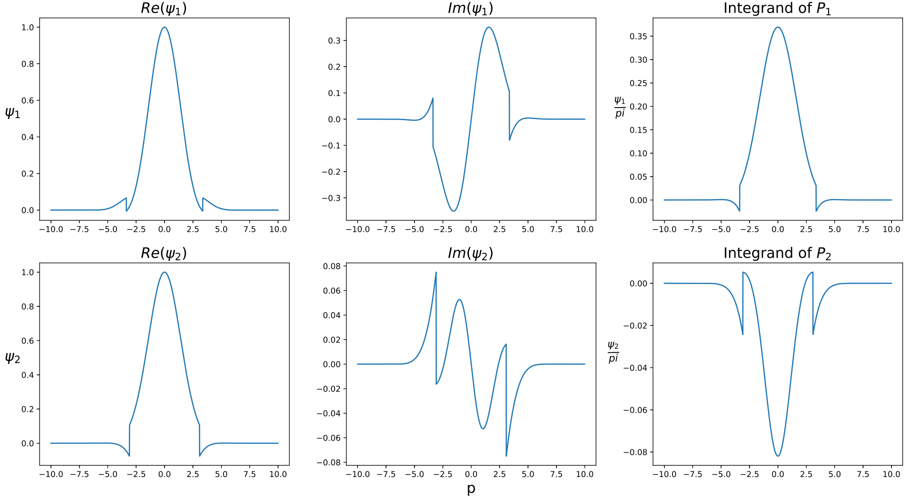
Characteristic Function

- We use the kernel functions obtained from the joint characteristic function of the increment $(X_T - X_t)$

$$\begin{aligned}\psi_i(p,q) =& \mathbb{E}\left[e^{\mathrm{i}U^T(X_T-X_t)}\middle|X_t = X\right] \\ =& e^{-\mathrm{i}U^TX}\varphi_i(p,q) \\ =& \exp\left(\mathrm{i}pr\tau + \mathrm{i}qa\tau + \frac{\gamma+\lambda}{\sigma^2}(1-\zeta)\,v - \frac{\gamma-\lambda}{\sigma^2}a\tau + \frac{2a}{\sigma^2}\ln\zeta\right).\end{aligned} \tag{12}$$

Characteristic Function

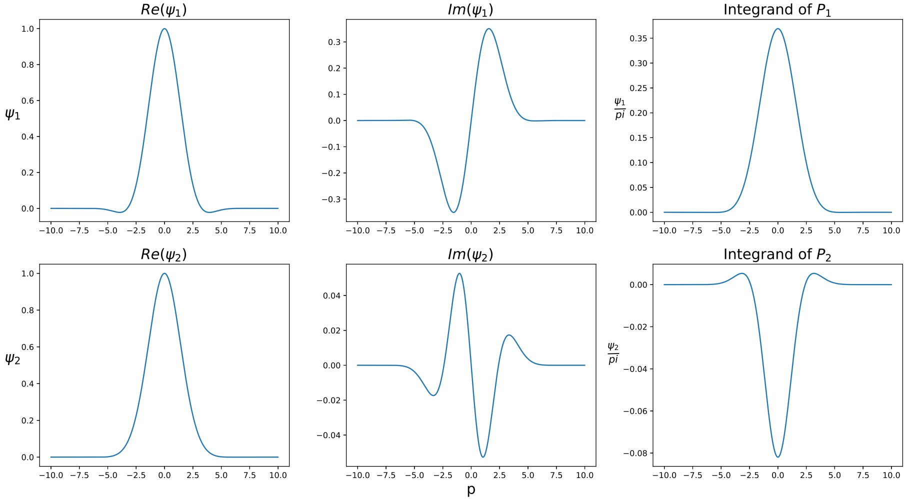
Figure 1: Heston's characteristic function



$$\Lambda = 1,\ r = 0.03,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ \tau = 5$$

Characteristic Function

Figure 2: Joint characteristic function



$$\Lambda = 1,\ r = 0.03,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ \tau = 5$$

Convolution Method

- The premise of the convolution method is that the conditional density $\phi(y|x,v)$ depends only on the difference of x and y

$$\phi(y|x) = \phi(y-x).$$

- Drăgulescu and Yakovenko (2002) showed that for small Δt, the distribution of x_t evolves in Gaussian manner in discrete time with the given variance v

$$\begin{aligned}\phi\left(x_t|x,v\right) &= \frac{1}{\sqrt{2\pi v\Delta t}}\exp\left(-\frac{\left(x_t - x - (r-\frac{1}{2}v)\Delta t\right)^2}{2v\Delta t}\right)\\ &= \phi\left(x_t - x|v\right).\end{aligned}$$

Convolution Method

- Then the Fourier transform of P_i is

$$\begin{aligned}\mathfrak{F}\left[P_i(x)\right](u) &= \mathfrak{F}\left[\mathbb{E}_i\left[\mathbf{1}_{S_T\geq K}\middle| x=\ln\frac{S}{K}\right]\right](u) \quad (13)\\ &= \mathfrak{F}\left[\int_{\mathbb{R}}\delta(y)\phi_i(y|x)dy\right](u)\\ &= \mathfrak{F}\left[\left(\delta(y)*\phi_i(y-x)\right)(x)\right](u)\\ &= \mathfrak{F}\left[\delta(y)\right](u)\mathfrak{F}\left[\phi_i(-y)\right](u), \quad (14)\end{aligned}$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.}\end{cases}$$

Convolution Method

- The Fourier transform of the density function in (14) is

$$\begin{aligned}\mathfrak{F}\left[\phi_i(-y)\right](p) &= \int_{\mathbb{R}} e^{-\mathrm{i}py}\phi(-y)dy \\ &= \int_{\mathbb{R}} e^{\mathrm{i}p(y-x)}\phi_i(y-x)dy \\ &= e^{-\mathrm{i}px}\int_{\mathbb{R}} e^{\mathrm{i}py}\phi_i(y\,|x\,)dy \\ &= e^{-\mathrm{i}px}\mathbb{E}_i\left[e^{\mathrm{i}px_T}\,|x\right] \\ &= e^{-\mathrm{i}px}\varphi_i(p) = \psi_i(p). \end{aligned} \tag{15}$$

- We simplify (14) as

$$\mathfrak{F}\left[P_i(x)\right](p) = \mathfrak{F}\left[\delta(x)\right](p)\psi_i(p),$$

and recover P_i by

$$P_i(x) = \mathfrak{F}^{-1}\left[\mathfrak{F}\left[\delta(x)\right](p)\psi_i(p)\right]. \tag{16}$$

Convolution Method

- Apply the change of variables to $x = \ln \frac{S}{K}$ with varying S the pricing formula (6) becomes

$$C(S,K,v,t) = SP_1\left(S,K\right) - Ke^{-r\tau}P_2\left(S,K\right)$$
$$= S\ \mathfrak{F}^{-1}\left[\mathfrak{F}\left[\delta(x)\right](p)\psi_1(p)\right](x) - Ke^{-r\tau}\mathfrak{F}^{-1}\left[\mathfrak{F}\left[\delta(x)\right](p)\psi_2(p)\right](x)$$

- Discretize the real space as

$$x_n = \left(n - \frac{N}{2}\right)\Delta x,\ \text{ for } n = 0, 1, \cdots, N-1,\ \text{ and } \Delta x = \frac{L}{N},$$

and the frequency space as

$$p_n = \left(n - \frac{N}{2}\right)\Delta p,\ \text{ for } n = 0, 1, \cdots, N-1,\ \text{ and } \Delta p = \frac{2\pi}{L}.$$

Convolution Method

- The cFFT estimation of $\tilde{P}_i$ using the formula given in equation (16) are given by

$$\tilde{P}_i = (-1)^n \mathfrak{D}^{-1} \left[\left\{ w_k \mathfrak{D} \left[\{ w_n (-1)^n \delta(x_n) \}_{n=0}^{N-1} \right] (p_k) \psi_i (p_k) \right\}_{k=0}^{N-1} \right]_n ,$$

for some weight scheme w_n.

- Then the pricing formula is approximated by

$$C(S, K, v, t) \approx S \tilde{P}_1 - K e^{-r\tau} \tilde{P}_2.$$

- We refer to this as cFFT Scheme I.

cFFT Scheme II

- Scheme II approaches the pricing formula similar to Carr and Madan (1999), but we apply the Fourier transform on the log-price region.
- Let $\alpha < 0$ be a dampling parameter, we obtain the following Fourier transform

$$\begin{aligned}\mathfrak{F}\left[e^{\alpha x}C(x)\right] =& e^{-r\tau}\int_{\mathbb{R}} \text{Re}\left(e^{-\mathrm{i}ux}e^{\alpha x}\mathbb{E}^{\mathbb{Q}}\left[(Ke^{x_T}-K)^+ \middle| x=\ln(\tfrac{S}{K})\right]\right)dx \\ =& e^{-r\tau}\text{Re}\left(\int_{\mathbb{R}} e^{-\mathrm{i}ux}e^{\alpha x}\int_{\mathbb{R}} g(y)\tilde{\phi}_2(x-y)dydx\right) \\ =& e^{-r\tau}\mathfrak{F}\left[e^{\alpha x}g(x)\right]\psi_2(u+\alpha\mathrm{i}),\end{aligned} \tag{17}$$

where

$$g(x) = (Ke^x - K)^+ \text{ and } \tilde{\phi}(x) = \phi(-x).$$

- The call option can be recovered from reverting and undamping (17)

$$C(x) = e^{(-r\tau-\alpha x)}\mathfrak{F}^{-1}\left[\mathfrak{F}\left[e^{\alpha x}g(x)\right]\psi_2(u+\alpha\mathrm{i})\right](x). \tag{18}$$

- Scheme II can be implemented similar to Scheme I.

Error Analysis

- Two error sources:
  - Truncation error associated with the sampling region $(-\frac{L}{2}, \frac{L}{2})$.
  - Discretization error associated with the sampling frequency $(\Delta x, \Delta u)$.
- Let $|e_i| = \left|P_i - \tilde{P}_i\right|$, for $i = 1, 2$.
- For the cFFT-I Scheme I

$$\begin{aligned}|e(x)| &= \left|C(x) - \tilde{C}(x)\right| \\ &= \left|Ke^x\left(P_1(x) - \tilde{P}_1(x)\right) - Ke^{-r\tau}\left(P_2(x) - \tilde{P}_2(x)\right)\right| \\ &\leq Ke^x|e_1| + Ke^{-r\tau}|e_2|. \end{aligned} \tag{19}$$

- cFFT-I and cFFT-II have the following error estimates

$$|e| \leq \mathcal{O}\left(e^{-\frac{\pi D}{L}N}\right) + \mathcal{O}\left(N^{-m}\right),$$

for $m \geq 2$.

Error Analysis

- Discretization error is at least order two, which is the same as Lord et al. (2008).
- Truncation error is negative exponential to the frequency.
- From (19) error would increase when x approaches the boundary.
- We introduce the boundary control schemes to improve the boundary error.

Boundary control: damping and shifting

- For a target function f write

$$C(x) = \mathbb{E}^{\mathbb{P}_i}\left[f(x_T)\,|x_0 = x\right]$$

  we add a damping parameter $\alpha < 0$ making $e^{\alpha x}C(x)$ integrable.

- Hyndman and Oyono Ngou (2017) introduced a shifting method on the target function to address the boundary error.

- The basic idea of shifting the target function is to map it from non-periodic to a periodic function which would be considered as the signal.

Boundary control: damping and shifting

- Consider $h(x)$ with explicit expectation $\mathbb{E}\left[h(x_t)\,|x\right]$.
- Shifting: $f^{\alpha}(x) \to \tilde{f}^{\alpha}(x) = e^{\alpha x}\left(f(x) - h(x)\right)$, $x \in [x_0, x_n]$.
- The candidate for shifting function $h(x)$ such that the damping of the shifted target function $\tilde{f}^{\alpha}(x) = e^{\alpha x}\left(f(x) - h(x)\right)$ is smoothly connected at the boundaries

$$
\begin{aligned}
\tilde{f}^{\alpha}(x_0) &= \tilde{f}^{\alpha}(x_n), \\
\frac{d\tilde{f}^{\alpha}}{dx}(x_0) &= \frac{d\tilde{f}^{\alpha}}{dx}(x_n).
\end{aligned}
$$

Boundary control: damping and shifting

cFFT-I	cFFT-II
$\alpha = 0$	$\alpha < -1$
$h(x) = Ax + B$	$h(x) = Ae^x + B$
$A = \frac{f(x_N) - f(x_0)}{x_N - x_0}$	$A = \frac{e^{\alpha x_N} f'_N - e^{\alpha x_0} f'_0}{e^{(\alpha+1)x_N} - e^{(\alpha+1)x_0}}$
$B = \frac{x_N f(x_0) - x_0 f(x_N)}{x_N - x_0}$	$B = \frac{x_N f(x_0) - x_0 f(x_N)}{x_N - x_0}$
	$f'_0 = \frac{-3f(x_0) + 4f(x_1) - f(x_2)}{2\Delta x}$
	$f'_N = \frac{3f(x_N) - 4f(x_{N-1}) + f(x_{N-2})}{2\Delta x}$
$\tilde{f}^\alpha(x) = f(x) - h(x)$	$\tilde{f}^\alpha(x) = e^{\alpha x}\left(f(x) - h(x)\right)$

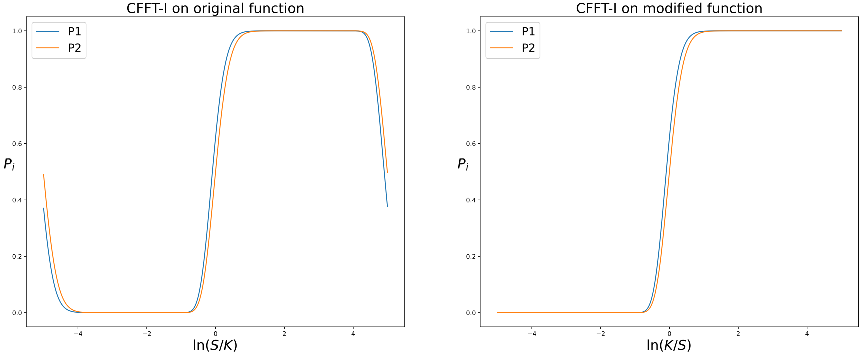
- We can recover cFFT-I (cFFT-II) by reversing the shifting (and dampling) scheme.

Numerical Results

- To illustrate accuracy, we compare our method to the numerical results using the integral method.
- To illustrate performance, we compare our method to the Carr and Madan (1999) FFT method.
- First, we present the results of CFFT-I method that is applied to estimate the probabilities in the Heston model.
- Then we apply CFFT-II to price the European call with Heston model and show the effect of different boundary control schemes.
- At the end, we present a table that summarizes the performance of CFFT-II method in certain cases.

Numerical Results

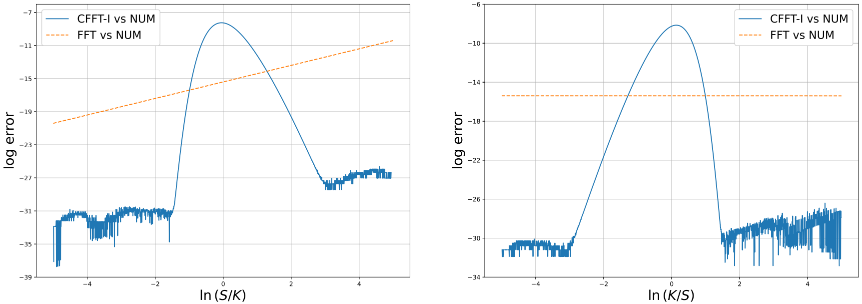
Figure 3: P_i by CFFT-I



$r = 0.03,\ v = 0.1,\ \Lambda = 1,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ T = 1,\ L = 10,\ N = 2000$

Numerical Results

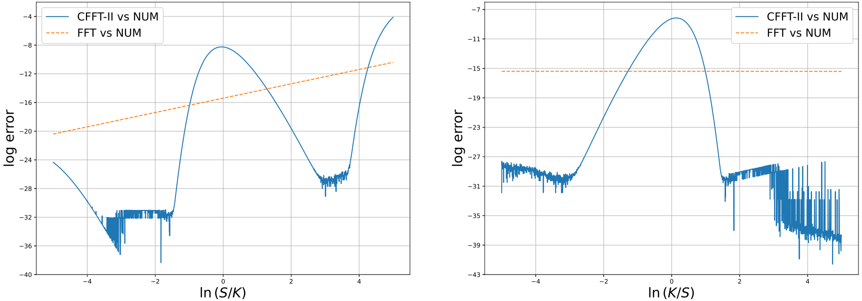
Figure 4: Error of CFFT-I



$r = 0.03,\ v = 0.1,\ \Lambda = 1,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ T = 1,\ L = 10,$
$N = 2000$

Numerical Results

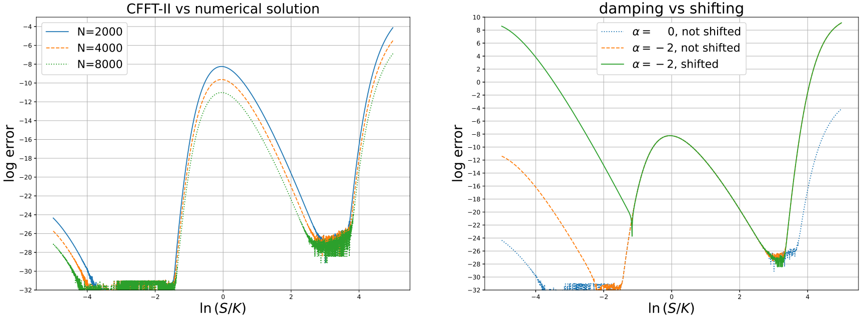
Figure 5: Error of CFFT-II



$r = 0.03,\ v = 0.1,\ \Lambda = 1,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ T = 1,\ L = 10,$
$N = 2000,\ \alpha = -2$

Numerical Results

Figure 6: CFFT-II with different schemes



$$r = 0.03,\ v = 0.1,\ \Lambda = 1,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ T = 1,\ L = 10$$

Numerical Results

Table 1: Heston model: CPU time, Call option and error

	CPU time (ms)		S=100,K=80		S=100,K=100		S=100,K=120	
	CFFT-II	FFT	call	error	call	error	call	error
N=2000	0.124	0.155	25.77846	5.93e-05	13.45867	2.60e-04	5.97903	1.40e-04
N=4000	0.175	0.294	25.77841	8.04E-06	13.45887	6.50e-05	5.97885	4.29e-05
N=8000	0.251	0.544	25.77841	4.60e-06	13.45892	1.63e-05	5.97889	4.73e-06

$r = 0.03,\ v = 0.1,\ \Lambda = 1,\ \rho = -0.8,\ \kappa = 3,\ \theta = 0.1,\ \sigma = 0.25,\ T = 1,\ L = 10,$
$\alpha = -2$

Conclusion

Mathematical and implementation details can be found in:

- Gao, Xiang. *Stochastic control, numerical methods, and machine learning in finance and insurance.* PhD Thesis, Concordia University, May 2021.
- `https://spectrum.library.concordia.ca/988412/` or the forthcoming arXiv preprint.
- Thank you!

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