Drawdown Derivatives to a Hitting Time

Kevin Atteson and Peter Carr

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Drawdown Derivatives

Introduction

- *We investigate derivatives on the drawdown of a process*
- *Prior work (e.g. [DSY00], [MIAPAM04], [CZH11]) explores maximum drawdown derivatives which mature at a specified time*
- *We focus on maturity at the first time the process hits an upper barrier*
- *The value and hedge ratios of the derivatives have simpler forms and more general model-independence than fixed-maturity options*
- *An example of such a derivative is a binary option which pays if an asset draws down $d\%$ before reaching a value m*

Why is Drawdown Important

Introduction

- *The* drawdown *of a portfolio is the amount that the portfolio has lost from a prior peak*
- *Many investors place significant focus on drawdown as a measure of risk*
- *Studies have demonstrated natural cognitive biases:*
  - *Gain-loss asymmetry, e.g. most people would not enter a bet where they win $150 or lose $100 with equal probability*
  - *The endowment effect, e.g. most people place more value on something owned than the same thing not owned*
- *Some hedge funds have adopted rules by which portfolio managers are shut down if they hit a drawdown limit*
- *There are no such rules for missing out on draw ups*

Drawdown

Definition

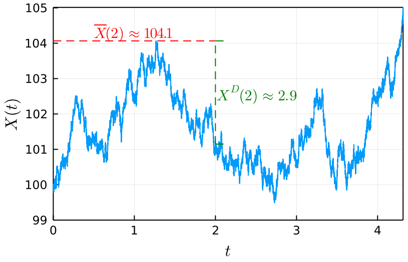
- **Running maximum**:

$$\overline{X}(t) = \sup_{s \le t} X(s)$$

- **Drawdown:**

$$X^D(t) = \overline{X}(t) - X(t)$$

Drawdown for Geometric Brownian Motion



Prior Work

Prior results

- *[DSY00] showed that for standard Brownian motion:*

$$\mathbb{P}\left(\overline{B^D}(t) < d\right) = 1 - \frac{1}{2\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \int_{-d}^{d} \left(e^{-\frac{(y+4kd)^2}{2}} - e^{-\frac{(y+2d+4kd)^2}{2}}\right) dy$$

- *[MIAPAM04] show a more complex result for Brownian motion with drift*

Prior Work

Prior results

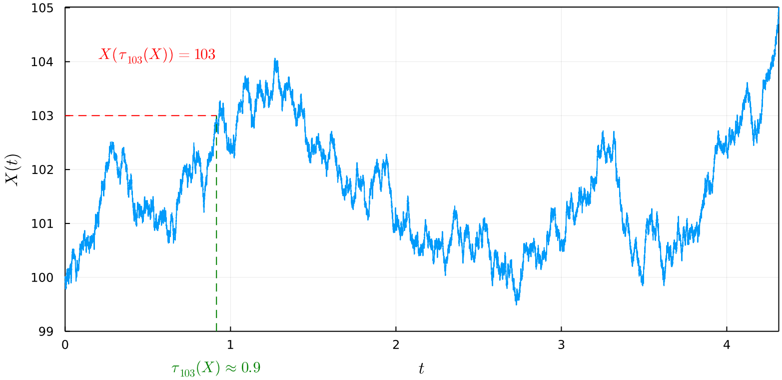
[CZH11] show:

- *Model-free "semi-static" replication of maximum drawdown insurance using one-touch knockout options*
- *"Semi-static" here means that the hedge portfolio is rebalanced only when the maximum of the process changes*
- *Makes continuity and symmetry assumptions*
- *Hedging with an infinite series of one touch or even vanilla options using stronger assumptions*
- *Hedging relative drawdown (defined later)*
- *Hedging for geometric models*

First Hitting Time

Definition

First Hitting Time: $\tau_m(X) = \inf\{t : X(t) \geq m\}$ with $\inf \emptyset = \infty$



Prior Work

Prior results

[SV07] show that for a standard Brownian motion B, drawdown at a hitting time is inverse exponentially distributed:

$$\mathbb{P}\Big(\overline{B^D}(\tau_m(B)) < d\Big) = \exp\Big(-\frac{m}{d}\Big)$$

Binary Calls on Drawdown to a Hitting Time

Definition

- Let $C^D_{m,d}(t)$ denote the price at time t of a binary call on a drawdown of d to the first hitting time of m
- Let $\tau_{m,d}$ be the stopping time corresponding to hitting m or drawing down by d, whichever comes first, that is $\tau_{m,d} = \tau_m(X) \wedge \tau_d\big(X^D\big)$
- $C^D_{m,d}$ matures at $\tau_{m,d}$ with values:

$$C^D_{m,d}(\tau_{m,d}) = \begin{cases} 1 & \text{if } X^D(\tau_{m,d}) \geq d \\ 0 & \text{if } X^D(\tau_{m,d}) < d \end{cases}$$

Binary Calls on Drawdown to a Hitting Time for Brownian Motion

Lemma

Using the result of [SV07], under standard Brownian motion, we have that $\tau_m(X) < \infty$ so that:

$$C^D_{m,d}(0) = \mathbb{P}\big(X^D(\tau_{m,d}) \geq d\big) = 1 - \exp\Big(-\frac{m}{d}\Big)$$

Remark

- *This formula holds for all continuous local martingales for which $\mathbb{Q}(\tau_{m,d} < \infty) = 1$*
- *From the fundamental theorem of asset pricing, every continuous arbitrage-free process is a continuous local martingale under the risk-neutral measure*

Binary Calls on Drawdown to a Hitting Time

Theorem

For any continuous arbitrage-free process X *such that* $X(0) = 0$ *and which hits* m *or draws down* d *in finite time with probability* 1*:*

$$\mathbb{Q}(\tau_{m,d} < \infty) = 1$$

the initial price of a binary call on a drawdown of d *to the first hitting time of* m *is given by:*

$$C^D_{m,d}(0) = \mathbb{Q}\left(X^D(\tau_{m,d}) \geq d\right) = 1 - \exp\left(-\frac{m}{d}\right)$$

where $\mathbb{Q}$ *is any equivalent local martingale measure.*

Unexpired Binary Calls on Drawdown to a Hitting Time

Corollary

For a process X with:

$$\mathbb{Q}(\tau_{m,d} < \infty) = 1$$

the value of an unexpired option at time t is given by:

$$C^D_{m,d}(t) = 1 - \frac{d - \left(\overline{X}(t) - X(t)\right)}{d} \exp\left(-\frac{m - \overline{X}(t)}{d}\right)$$

The option can be hedged by shorting $\frac{\exp\left(-\frac{m-\overline{X}(t)}{d}\right)}{d}$ shares of the underlier and the keeping the remainder in the bank account.

Example Processes

Remark

Examples of processes that either hit m or drawdown by d with probability 1:

- *Brownian motion when $0 < d$ and $X(0) < m$*
- *Geometric Brownian motion when $0 < d < X(0) < m$:*
  - *Geometric Brownian motion goes to 0 in the risk-neutral measure*
  - *Geometric Brownian motion need not hit a level $m > X(0)$*
- *Any process with (stochastic) volatility $\sigma(t)$ such that $\sigma(t) > a > 0$ whenever $\overline{X}(t) < m$ and $\overline{X^D}(t) > d$*

Time Invariance

Remark

Let $T : [0,\infty) \to [0,\infty)$ *is a strictly increasing time change with* $T(0) = 0$ *and let* $Y(t) = X(T(t))$*:*

$$\begin{aligned} Y^D(t) &= \left(\overline{Y} - Y\right)(t) = \left(\overline{X} - X\right)(T(t)) = X^D(T(t)) \\ \tau_m(Y) &= \inf\{t : Y(t) = m\} = \inf\{t : X(T(t)) = m\} \\ &= \inf\left\{T^{-1}(t) : X(t) = m\right\} = T^{-1}(\tau_m(X)) \end{aligned}$$

so that:

$$\begin{aligned} & Y^D\big(\tau_m(Y) \wedge \tau_d\big(Y^D\big)\big) \\ &= X^D\Big(T\Big(T^{-1}(\tau_m(X)) \wedge T^{-1}\big(\tau_d\big(X^D\big)\big)\Big)\Big) \\ &= X^D\big(\tau_m(X) \wedge \tau_d\big(X^D\big)\big) \end{aligned}$$

Consequences of Time Invariance

Remark

- *Any time change leaves the distribution of the random variable $X^D(\tau_{m,d})$ invariant*
- *The Dambis-Dubins-Schwartz theorem ([RY99]) shows that every continuous local martingale is a time change of a standard Brownian motion*
- *Any derivative whose terminal value is invariant to time changes will have the same value and hedges for any measure under which:*
    - *The underlier is continuous and arbitrage-free*
    - *The derivative matures at a finite time with probability 1*

Comparison with Vanilla Options

Remark

- *Advantages compared with vanilla options:*
  - *The hedges for these options have significant model-independence and so are more robust*
  - *The hedges are "semi-static" in that they only change when a new maximum is hit which happens in a time set of measure* 0
  - *The underlier is "bought back" at a new maximum rather than sold as the underlier moves down*
- *Disdvantage compared with vanilla options:*
  - *Maturity date can be arbitrarily large*
  - *This can be fixed by setting a finite maturity date and settling at the value at maturity:*
    - *The market maker can settle the hedges*
    - *The buyer of the option can reinvest in another option*

Relative Drawdown Binary Calls to a Hitting Time

Definition

- The **relative drawdown**, X^R is defined as:

$$X^R(t) = \frac{X^D(t)}{\overline{X}(t)}$$

- Let $\tau_{m,r} = \tau_m(X) \wedge \tau_r\left(X^R\right)$
- We write $C^R_{m,r}(t)$ for the value at time t of a option which matures at $\tau_{m,r}$ and pays:

$$C^R_{m,r}(\tau_{m,r}) = \begin{cases} X^D(\tau_{m,r}) & \text{if } X^R(\tau_{m,r}) \geq r \\ 0 & \text{if } X^R(\tau_{m,r}) < r \end{cases}$$

Price of Relative Drawdown Binary Calls to a Hitting Time

Theorem

For any continuous arbitrage-free process X such that:

$$\mathbb{Q}(\tau_{m,r} < \infty) = 1$$

the initial of a binary call on a relative drawdown of r to the first time to hit m is given by:

$$C^R_{m,r}(0) = \frac{r}{1-r} X(0) \left(1 - \left(\frac{X(0)}{m} \right)^{\frac{1}{r}-1} \right)$$

Unexpired Relative Drawdown Binary Calls to a Hitting Time

Corollary

For a process X as above, the value of the option at time t is given by:

$$C^R_{m,d}(t) = \frac{rX(t)}{1-r} - \frac{X(t) - \overline{X}(t)\,(1-r)}{1-r}\left(\frac{\overline{X}(t)}{m}\right)^{\frac{1}{r}-1}$$

The option is hedged by shorting $\frac{1}{1-r}\left(\left(\frac{\overline{X}(t)}{m}\right)^{\frac{1}{r}-1} - r\right)$ shares of the underlier and the keeping the remainder in the bank account.

Maximum Drawdown Call Spreads at a Hitting Point

Definition

- Let $\tau_{m,K_1} = \tau_m(X) \wedge \tau_{K_1}\left(X^D\right)$
- Let X be a continuous, arbitrage-free process which has a relative drawdown K_1 or hits m with probability 1:

$$\mathbb{Q}(\tau_{m,K_1} < \infty) = 1$$

- A **maximum drawdown call spread between strikes** K_1 **and** K_2 **at hitting point** m is the option which matures at τ_{m,K_1} with value:

$$\begin{aligned} & C^S_{m,K_1,K_2}(\tau_{m,K_1}) \\ = \; & \left(\overline{X^D}(\tau_{m,K_1}) - K_1\right)^+ - \left(\overline{X^D}(\tau_{m,K_1}) - K_2\right)^+ \end{aligned}$$

Maximum Drawdown Call Spreads to a Hitting Point

Theorem

For a continuous arbitrage-free process X *as above, the price of a*

$$C^S_{m,K_1,K_2}(\tau_{m,K_1}) = K_2 - K_1 + m\left(\Gamma\left(-1, \frac{m}{K_2}\right) - \Gamma\left(-1, \frac{m}{K_1}\right)\right)$$

where $\Gamma(s,x)$ *is the incomplete gamma function:*

$$\Gamma(s,x) = \int_x^\infty y^{s-1}\exp(-y)\,dy$$

The Proof for Maximum Drawdown Call Spreads

Proof.

$$E_{\mathbb{Q}}\left[\left(\overline{X^D}(\tau_{m,K_1}) - K_1\right)^+ - \left(\overline{X^D}(\tau_{m,K_1}) - K_2\right)^+\right]$$

$$= E_{\mathbb{Q}}\left[\int_{K_1}^{K_2} \mathbb{1}\left\{\overline{X^D}(\tau_{m,K_1}) \geq k\right\} dk\right]$$

$$= \int_{K_1}^{K_2} E_{\mathbb{Q}}\left[\overline{X^D}(\tau_{m,K_1}) \geq k\right] dk$$

$$= \int_{K_1}^{K_2} \left(1 - \exp\left(-\frac{m}{k}\right)\right) dk$$

The Integral

Proof.

Using the transformation $u = \frac{m}{k}$:

$$\begin{aligned}
&\int_{K_1}^{K_2} \left(1 - \exp\left(-\frac{m}{k}\right)\right) dk \\
&= K_2 - K_1 + \int_{\frac{m}{K_1}}^{\frac{m}{K_2}} \frac{m}{u^2} \exp(-u)\, du \\
&= K_2 - K_1 + m\left(\Gamma\left(-1, \frac{m}{K_2}\right) - \Gamma\left(-1, \frac{m}{K_1}\right)\right)
\end{aligned}$$

□

Drawdown Derivatives

Summary

- *Drawdown derivatives which expire at a hitting time have price and hedge formulas which are invariant for all arbitrage-free models for which the underlier is continuous and the option has a.s. finite maturity*
- *We have derived simple formulas for the following drawdown derivatives:*
  - *Binary calls on drawdown*
  - *Binary calls on relative drawdown*
  - *Call spreads on drawdown*

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