Semi-analytical pricing of barrier options in the time-dependent Heston model

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- The classical Heston model was introduced in [Heston 1993]. It immediately drew a lot of attention since the characteristic function (CF) of the log-spot price x_t for this model can be found in closed form. Thus, pricing of European options becomes almost straightforward by using the well-known FFT methods, see, e.g., survey in [Schmelzle 2010]. The Heston model belongs to the class of SV models and introduces an instantaneous variance v_t as a mean-reverting square-root process correlated to the underlying stock price process S_t.
- All parameters in the original Heston model are assumed to be **time-homogeneous**. If the so-called Feller condition $2\kappa\theta > \sigma^2$ is satisfied, the process v_t is strictly positive, $v_t \in [0, \infty)$; otherwise its behavior at the origin should be additionally identified, see e.g., [Feller 1954; Carr and Linetsky 2006; Lucic 2008].

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- Using this model in practice is still a challenge because its CF is known in closed form only when the model parameters are constant (this can be relaxed for piecewise constant coefficients, [Mikhailov and Nogel 2003; Guterding and Boenkost 2018]). For the time **inhomogeneous** parameters there is no a semi-analytical formula for the European option price, and one has to use numerical methods (either a Monte Carlo simulation, [Andersen 2008] or a finite-difference (FD) approach, [Kluge 2002; Itkin and Carr 2011] and references therein). To improve this, in [Benhamou, Gobet, and Miri 2010] a small volatility of volatility expansion and Malliavin calculus techniques are used to derive an analytic approximation for the price of vanilla options for any time-dependent Heston model, see also a survey in [Rouah 2015]. The time-dependent correlation function was also considered in [Teng 2021].

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However, for exotic options, such as e.g., barrier options, not so many analytical results have been obtained even for the case of constant coefficients, where they are available in two basic cases.

- **Zero drift and zero correlation**. Then, the option price is given by a 1D formula for the corresponding barrier option which should be further integrated with the density of the integrated variance, see [Lipton and McGhee 2002; Lipton 2001]. Often, this density is known in closed form, but for some models only the Laplace transform of the density is known that brings additional complexity (as applied to the CIR variance process, see [Cont and Tankov 2004; Belomestny and Schoenmakers 2016]). Aside of technical problems, this approach potentially can be further applied to the time dependent SV model with no correlation and drift.

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- The model has **a small parameter** ϵ, so the solution can be constructed asymptotically, see [Lipton and McGhee 2002; Ilhan, Jonsson, and Sircar 2004; Kato, Takahashi, and Yamada 2013; Lipton, Gal, and Lasis 2014; Barger and Lorig 2017] among others. For instance, in [Lipton and McGhee 2002] this is done by assuming that $vT \ll 1$. However, for the time-dependent model construction of such semi-analytical solution could become problematic.

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To emphasize, **the reflection principle would not have been valid for time-dependent barriers**red. Also, time-dependent coefficients of the model make it hard to derive the joint pdf (it is possible if, e.g. only some coefficients are functions of time while κ, ρ are constants, as in [Carr and Sun 2007]), but numerical complexity of this approach is close to that proposed here.

Therefore, practitioners who need to price barrier options using the whole time-dependent Heston model with no simplifications yet have to use **numerical methods**. In this paper we develop **an alternative approach** to this problem by using **the generalized integral transform method (GIT)** originally developed in physics and then introduced into mathematical finance by the authors in [Carr and Itkin 2021; Itkin and Muravey 2020; Carr, Itkin, and Muravey 2020; Itkin and Muravey 2022] and also in cooperation with Alex Lipton in [Itkin, Lipton, and Muravey 2020; Itkin, Lipton, and Muravey 2022; Itkin, Lipton, and Muravey 2021]. To shorten the references, in what follows we cite just a recent book, [Itkin, Lipton, and Muravey 2021], having in mind that the corresponding materials could also be found in the above referenced papers.

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Despite our methods can be applied to any sort of barrier options, here, as an example, we consider only a Down-and-Out barrier Put option written on the underlying process $S_t \in [L(t), \infty]$, which follows the Heston dynamics

$$\begin{aligned} dS_t &= S_t(r-q)dt + S_t\sqrt{v}dW_t^{(1)}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^{(2)}, \\ d\langle W_t^{(1)}, W_t^{(2)}\rangle = \rho dt, &\quad [S,t] \in [0,\infty)\times[0,\infty), \quad S_0 = S, \quad v_0 = v, \end{aligned} \tag{1}$$

with all the model coefficients $\kappa, \theta, \sigma, \rho$ being functions of the time t, and where $L(t) > 0$ is the lower barrier. We also discuss other types of the barrier options.

We assume that once S_t hits the barrier, the contract is terminated and the option expires worthless, i.e.

$$P(t, L(t), v) = 0, \tag{2}$$

where $P(t, S, v)$ is the option price. In other words, in this case we assume no rebate is paid either at the option maturity T, or at hit. This assumption can be easily relaxed, see [Itkin and Muravey 2022]. At the other boundary we assume the standard condition

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If the process S_t survives till $t = T$, the Put option provides its holder with the payoff

$$P(T, S, v) = (K - S)^+, \tag{4}$$

where $K > 0$ is the strike. The Eq.(4) is the terminal condition for our problem. We also assume that $L(T) < K$.

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We develop the GIT method for pricing barrier options in the time-dependent Heston model (with the time-dependent barrier) and derive a **semi-analytical solution** of this problem which is expressed via a two-dimensional integral. This integral depends on **yet unknown function** $\bar{\Phi}(t, v)$ which is the gradient of the solution at the moving boundary $S = L(t)$ and solves a linear mixed Volterra-Fredholm (LMVF) equation of the second kind also derived in the paper.

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In other words, we show that the GIT method can be developed not only for one-factor models, but for the SV models (two drivers with inhomogeneous correlation) as well. As such, this 2D method should naturally inherit all advantages of the corresponding 1D methods, e.g., speed and accuracy.

Let us introduce a new variable $x = \log(S/K)$. By the standard argument, [Cont and Voltchkova 2005], under the risk neutral measure the Put option price $P(t,x,v)$ with x, v being the initial values of processes x_t, v_t at the time $t=0$ solves the partial differential equation (PDE)

$$\frac{\partial P}{\partial t} + \frac{1}{2} v \frac{\partial^2 P}{\partial x^2} + \left[r(t) - q(t) - \frac{1}{2} v \right] \frac{\partial P}{\partial x} + \frac{1}{2}\sigma^2(t) v \frac{\partial^2 P}{\partial v^2} + \kappa(t)(\theta(t) - v)\frac{\partial P}{\partial v} + \rho(t)\sigma(t) v \frac{\partial^2 P}{\partial x \partial v} = r(t) P, \tag{5}$$

subject to the terminal condition

$$P(T,x,v) = K(1-e^x)^+, \tag{6}$$

and the boundary conditions

$$P(t, y(t), v) = 0, \quad y(t) = \log(L(t)/K) < 0, \tag{7}$$

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Following the idea of the method of generalized integral transforms (GIT) for $S \in [L(t), \infty)$, [Itkin and Muravey 2020], we introduce the following integral transform

$$\bar{u}(t,p,v) = \int_{y(t)}^{\infty} P(t,x,v) e^{-\sqrt{p}x} dx, \tag{8}$$

where $p = a + i\omega$ is a complex number. It might look that we also need to request $\mathrm{Re}(p) = \beta > 0$ for the transform to exist. However, usually the solution $u(t,x,v)$ converges to zero as $u(t,x,v) \propto e^{-ax^2}$, $a > 0$, see e.g., [Itkin and Muravey 2020], hence the integral in the RHS of Eq.(8) is well-behaved.

Then, multiplying both parts of Eq.(5) by $e^{-x\sqrt{p}}$ and integrating on x from $y(t)$ to infinity with allowance for the boundary conditions, we obtain

$$0 = \frac{\partial}{\partial t}\bar{u} + [a(t,p) + c(p)v]\bar{u}(t,v) + \frac{1}{2}\sigma^2(t)v\frac{\partial^2 \bar{u}}{\partial v^2} + \bar{\kappa}(t,p)(\bar{\theta}(t,p) - v)\frac{\partial \bar{u}}{\partial v} - ve^{-y(t)\sqrt{p}}\Phi(t,v), \tag{9}$$

$$a(t,p) = r(t)(\sqrt{p} - 1) - q(t)\sqrt{p}, \quad c(p) = \frac{1}{2}(p - \sqrt{p}), \quad \bar{\kappa}(t,p) = \kappa(t) - \rho(t)\sigma(t)\sqrt{p},$$

$$\bar{\theta}(t,p) = \theta(t)\frac{\kappa(t)}{\bar{\kappa}(t,p)}, \quad \Phi(t,v) = \frac{1}{2}P_x(t,y(t),v) + \rho(t)\sigma(t)P_v(t,y(t),v),$$

$$\bar{u}(T,p) = K\left[\frac{e^{-y(T)\sqrt{p}} - 1}{\sqrt{p}} - \frac{e^{-y(T)\left(\sqrt{p}-1\right)} - 1}{\sqrt{p} - 1}\right].$$

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Assuming that the function $u(t,x,v)$ is smooth enough at the boundary $x = y(t)$, it follows that

$$\lim_{x \to y(t)} P_v(t,x,v) = \partial_v \lim_{x \to y(t)} P(t,x,v) = \partial_v P(t,y(t),v) = 0, \tag{10}$$

and, hence, the second term in the definition of $\Phi(t,v)$ vanishes. Hence $\Phi(t,v)$ **is half of the option Delta at the barrier**.

Then, multiplying both parts of Eq.(5) by $e^{-x\sqrt{p}}$ and integrating on x from $y(t)$ to infinity with allowance for the boundary conditions, we obtain

$$0 = \frac{\partial}{\partial t}\bar{u} + [a(t,p) + c(p)v]\bar{u}(t,v) + \frac{1}{2}\sigma^2(t)v\frac{\partial^2 \bar{u}}{\partial v^2} + \bar{\kappa}(t,p)(\bar{\theta}(t,p) - v)\frac{\partial \bar{u}}{\partial v} - ve^{-y(t)\sqrt{p}}\Phi(t,v), \tag{9}$$

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The Eq.(9) is an **inhomogeneous PDE** and also **exponentially affine** in the variable v. Its solution can be constructed if the Green's function of the homogeneous PDE is known. It can be observed that a similar homogeneous PDE is considered in [Carr, Itkin, and Muravey 2020] with respect to pricing barrier options under the CIR model. Therefore, we can proceed in the same way.

Theorem

The Eq.(9) can be transformed to the form

$$\frac{\partial \bar{U}}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \bar{U}}{\partial z^2} + \frac{b}{z}\frac{\partial \bar{U}}{\partial z} + \bar{\Psi}(\tau, z), \tag{11}$$

where $b = m - 1/2$ *is some constant,* $\bar{U} = \bar{U}(\tau, z)$ *is the new dependent variable, and* $(\tau, z) \in [0, \infty) \times [0, \infty)$ *are the new independent variables, if*

$$\frac{\kappa(t)\theta(t)}{\sigma^2(t)} = \frac{m}{2}, \tag{12}$$

where $m \in [0, \infty)$ *is some constant. The homogeneous version of Eq.(11) is the PDE associated with the one-dimensional Bessel process,* $dX_t = dW_t + \frac{b}{X_t}dt$, *[Revuz and Yor 1999].*

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Here

$$\bar{u}(t, v) = \bar{U}(t, z) e^{\alpha(t,p)v + \beta(t,p)}, \ z = g(t, p)\sqrt{v}, \ \tau(t, p) = \frac{1}{4}\int_t^T g^2(s, p)\sigma^2(s)\, ds, \tag{13}$$

$$g(t, p) = \exp\left[\frac{1}{2}\int_0^t \left(\bar{\kappa}(s, p) - \alpha(s, p)\sigma^2(s)\right) ds\right], \ \beta(t, p) = \int_t^T [a(s, p) + \kappa(s)\theta(s)\alpha(s, p)] ds,$$

and $\alpha(t, p)$ solves the Riccati equation

$$\alpha'(t, p) = -c(p) + \bar{\kappa}(t, p)\alpha(t, p) - \frac{1}{2}\alpha(t, p)^2\sigma(t)^2. \tag{14}$$

As follows from Proposition 1, for the Heston model the transformation from Eq.(9) to Eq.(13) **cannot be done unconditionally**. However, even with the restriction in Eq.(12) the model still makes sense. Indeed, the model parameters already contain the independent mean-reversion rate $\kappa(t)$ and vol-of-vol $\sigma(t)$. Since m is an arbitrary constant, it could be calibrated to the market data together with $\kappa(t)$ and $\sigma(t)$. Therefore, even in this form the Heston model should be capable to be **calibrated to the market option prices**.

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The terminal condition in Eq.(9) doesn't depend on v which means

$$\frac{\partial}{\partial v}\left[\bar{U}(0,z)e^{\alpha(T,p)v+\beta(T,p)}\right] = e^{\alpha(T,p)v+\beta(T,p)}\left[\frac{\partial \bar{U}(0,z)}{\partial z}\frac{g^2(T,p)}{2z} + \bar{U}(0,z)\alpha(T,p)\right] = 0, \quad (15)$$

or

$$\bar{U}(0,z) = \bar{u}(T,p)e^{-Bz^2}, \qquad B = \alpha(T,p)/g^2(T,p). \quad (16)$$

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Since Eq.(11) is an inhomogeneous PDE, it can be solved if the Green's function of the corresponding homogeneous PDE is known. Since this homogeneous counterpart with $\Psi(\tau,z,p) = 0$ is the Bessel equation defined at the semi-infinite domain $z \in [0,\infty)$, this Green's function is known in closed form assumed that the Bessel process stops when it reaches the origin. In more detail, it is relatively easy to show that the boundary $v_t = 0$ is an attainable regular boundary by Feller's classification, [Lipton 2001]. Therefore, similar to [Gorovoi and Linetsky 2004] we always make regular boundaries instantaneously reflecting, and include regular reflecting boundaries into the state space. We also assume that infinite boundaries are unattainable..

By definition, $m \geq 1$ implies $b \geq 1/2$. It is known, [Lawler 2018; Linetsky and Mendoza 2010], that in case $b \geq 1/2$ the density $G(\tau, z, \zeta)$ is a good density with no defect of mass, i.e., it integrates into 1. The explicit representation reads, [Cox 1975; Emanuel and Macbeth 1982]

$$G(\tau, z, \zeta) = \frac{\sqrt{z\zeta}}{\tau} \left(\frac{\zeta}{z}\right)^b e^{-\frac{z^2+\zeta^2}{2\tau}} I_{b-1/2}\left(\frac{z\zeta}{\tau}\right). \tag{17}$$

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Alternatively, $0 < m < 1$ implies $-1/2 < b < 1/2$. Then by another change of variables, $\bar{U}(\tau, z) = z^{2(1-m)} \bar{U}_1(\tau, z)$, [Polyanin 2002], Eq.(13) transforms to the same equation with respect to $\bar{U}_1(\tau, z)$ but now with $b = (3 - 2m)/2$. Accordingly, since $0 < m < 1$ we have $b > 1/2$, Therefore, again the Green's function is represented by Eq.(17).

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Since the Green's function of the homogeneous form of Eq.(11) is known, the solution of Eq.(13) can be represented as, Polyanin 2002

$$\bar{U}(\tau, z) = \int_0^\infty \bar{U}(0, \zeta) G(\tau, z, \zeta) d\zeta + \int_0^\tau \int_0^\infty G(\tau - s, z, \zeta) \Psi(s, \zeta) ds\, d\zeta. \tag{18}$$

Using the definition of $\bar{U}(0, z)$ in Eq.(16), the first integral can be computed in closed form

$$I_1(\tau, z, p) \equiv \int_0^\infty \bar{U}(0, \zeta) G(\tau, z, \zeta) d\zeta = K \left[\frac{e^{-y(T)\sqrt{p}} - 1}{\sqrt{p}} - \frac{e^{-y(T)\left(\sqrt{p}-1\right)} - 1}{\sqrt{p} - 1} \right] \frac{e^{-\frac{Bz^2}{2B\tau+1}}}{(2B\tau + 1)^{b+\frac{1}{2}}},$$

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And finally

$$\bar{u}(t, v, p) = e^{\alpha(t,p)v+\beta(t,p)} I_1(t, v, p) + \frac{1}{2} e^{\alpha(t,p)v+\beta(t,p)} \int_t^T \int_0^\infty \sqrt{v'} g(s, p) \Bigg\{ G\left(\int_t^s \frac{1}{4} g^2(\gamma, p)\sigma^2(\gamma) d\gamma, g(t, p)\sqrt{v}, g(s, p)\sqrt{v'}\right) \\ \times e^{-y(s)\sqrt{p} - [\beta(s,p) + v'\alpha(s,p)]} \Phi\left(s, v'\right) \Bigg\} ds\, dv'. \tag{19}$$

In case the model coefficients are time-homogeneous, i.e. $\bar{\kappa}(t, \sqrt{p}) = \bar{\kappa}(\sqrt{p}),\ \sigma(t) = \sigma$, Eq.(14) subject to the terminal condition $\alpha(T, p) = \alpha_T$ can be solved analytically. The solution reads

$$\alpha(t,\xi) = \frac{1}{\sigma^2}\left\{\bar{\kappa}(\sqrt{p}) + \sqrt{2c((\sqrt{p}))\sigma^2 - \bar{\kappa}^2(\sqrt{p})}\tan\left[\tan^{-1}\left(\frac{\alpha_T\sigma^2 - \bar{\kappa}(\sqrt{p})}{\sqrt{2c(\sqrt{p})\sigma^2 - \bar{\kappa}^2(\sqrt{p})}}\right) + \frac{1}{2}(T-t)\sqrt{2c(\sqrt{p})\sigma^2 - \bar{\kappa}^2(\sqrt{p})}\right]\right\}$$

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If the model coefficients are functions of the time t, we can use the method of [Guterding and Boenkost 2018]. The idea is to split the entire time interval $t \in [0, T]$ into N subintervals of the length $\Delta t = T/N$, and approximate time-dependent model parameters by piecewise constant coefficients. Accordingly, the solution $\alpha_i(t, p)$ for every interval i is given by Eq.(20). Hence, Eq.(14) can be solved backward in time starting with $\alpha(T, p) = 0$ in a **fully analytic way**.

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Also, in a complex plane the function $C(t,p)$, as it is defined in Eq.(20), is a multivalued function which can easily be seen if we represent it in the form

$$C(t,p) = \sqrt{(\sqrt{p} - p_+)(\sqrt{p} - p_-)}, \qquad p_\pm = \frac{\sigma - 2\kappa\rho \pm \sqrt{4\kappa^2 - 4\kappa\rho\sigma + \sigma^2}}{2(1-\rho^2)\sigma}. \tag{21}$$

Thus, both branch (critical) points $p_\pm$ are **pure real**.

Since functions $\sin[\xi(x - y(t))]$ form an orthonormal basis in $[y(\tau), \infty)$ we can look for the solution $u(t, x, v)$ in the following form

$$P(t, x, v) = \int_0^\infty \chi(\xi, t, v) \sin[\xi(x - y(t))] d\xi, \tag{22}$$

where $\chi(\xi, t, v)$ are some weights to be determined. Note, that this definition automatically respects the vanishing boundary conditions for $P(t, x, v)$. For $x = y(t)$ this is obvious. For $x \to \infty$ this can be seen looking at the final solution of a similar problem which is obtained in [Itkin and Muravey 2020]. We assume that the integral in Eq.(22) converges absolutely and uniformly $\forall x \in [y(t), \infty)$ for any $t > 0$ and $v > 0$.

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Applying Eq.(8) to both parts of Eq.(22) and integrating yields

$$\bar{u}(t, v, p) = \int_{y(t)}^\infty e^{-\sqrt{p}x} \int_0^\infty \chi(\xi, t, v) \sin(\xi(x - y(t))) d\xi dx = e^{-\sqrt{p}y(t)} \int_0^\infty \chi(\xi, t, v) \frac{\xi d\xi}{\xi^2 + p}, \tag{23}$$

or

$$\int_0^\infty \chi(\xi, \tau, v) \frac{\xi d\xi}{\xi^2 + p} = \bar{u}(t, v, p) e^{\sqrt{p}y(t)}. \tag{24}$$

Now, similar to a standard construction of inverse operators, e.g., the inverse Laplace transform, we need an **analytic continuation** of the transform parameter p into the complex plane. Let us integrate both sides of Eq.(24) on p along the so-called keyhole contour, [Itkin and Muravey 2020].

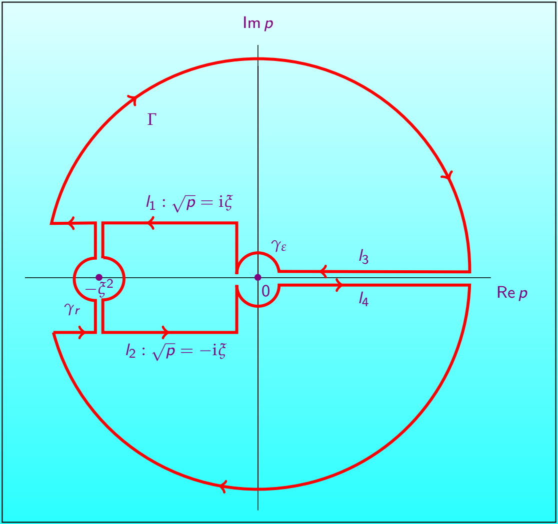

Figure 1: Contour of integration of Eq.(24) in a complex plane of p.

Theorem

Let us consider a time-dependent Heston stochastic volatility model defined in Eq.(1) with the additional condition in Eq.(12) that $\frac{\kappa(t)\theta(t)}{\sigma^2(t)} = \frac{m}{2}$, *where* $m \in [0, \infty)$ *is some constant. Also, let us consider a Down-and-Out barrier Put option with the lower barrier* $L(t)$ *be time-dependent as defined in Eq.(2), and let* $y(t) = \log(L(t)/K)$. *Given the values of the log-spot* $x = \log(S/K)$ *and the instantaneous variance* v *at* $t = 0$, *we get*

$$
\begin{aligned}
P(t,x,v) &= -\frac{2}{\pi}\int_0^\infty \sin\left[\xi(x-y(t))\right]\Big\{\cos(\xi y(t))\,\mathrm{Im}[P_s(t,v,-i\xi)] + \sin(\xi y(t))\,\mathrm{Re}[P_s(t,v,-i\xi)]\Big\}d\xi, \\
&= -\frac{1}{\pi}\int_0^\infty |P_s(t,v,-i\xi)|\left\{\cos[\phi - \xi(x-2y(t))] - \cos[\phi + x\xi]\right\}d\xi, \qquad (25) \\
P_s(t,v,-i\xi) &= P_1(t,v,-i\xi) + P_2(t,v,-i\xi), \quad |P_s(t,v,-i\xi)|^2 = [\mathrm{Re}\,P_s(t,v,-i\xi)]^2 + [\mathrm{Im}\,P_s(t,v,-i\xi)]^2, \\
P_1(t,v,\sqrt{p}) &= K\left[\frac{e^{-y(T)\sqrt{p}}-1}{\sqrt{p}} - \frac{e^{-y(T)(\sqrt{p}-1)}-1}{\sqrt{p}-1}\right]e^{\beta(t,p)+\gamma(t,p)v}, \quad \phi = \arg(P_s(t,v,-i\xi)), \\
P_2(t,v,\sqrt{p}) &= \frac{1}{2}\int_t^T ds\int_0^\infty dv'\Bigg[\Phi(s,v')\sqrt{v'}g(s,p) \\
&\quad \times G\left(\frac{1}{4}\int_t^s g^2(\zeta,p)\sigma^2(\zeta)d\zeta,\ g(t,p)\sqrt{v},\ g(s,p)\sqrt{v'}\right)e^{-y(s)\sqrt{p}+\alpha(t,p)v+\beta(t,p)-[\beta(s,p)+v'\alpha(s,p)]}\Bigg], \\
B_1 &= \frac{\alpha(T,p)}{g^2(T,p)}, \quad \gamma(t,p) = \alpha(t,p) - \frac{\alpha(T,p)}{1+2B_1\tau}\frac{g^2(t,p)}{g^2(T,p)}.
\end{aligned}
$$

Once the function $\Phi(t,v)$ *is known (which is a half of the gradient (in* x*) of the solution* $P(t,v,x)$ *at the boundary* $x = y(t)$*), the solution of this pricing problem is obtained via Eq.(25) by computing the integrals in the RHS.*

It can be directly checked that $\mathrm{Re}(P_1(t, v, \mathrm{i}\zeta)) = \mathrm{Re}(P_1(t, v, -\mathrm{i}\zeta))$ and, hence, the difference of P_1 is pure imaginary. Therefore, $[P_1(t, v, -\mathrm{i}\zeta)) - P_1(t, v, -\mathrm{i}\zeta)]/\mathrm{i}$ is real. Same should be true for the difference of P_2, however, this can be verified only numerically.

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Similar to the one-dimensional case described in detail in [Itkin, Lipton, and Muravey 2021], **the function $\Phi(t,v)$ solves a linear mixed Volterra-Fredholm (LMVF) integral equation of the second kind**. It can be obtained by differentiating both sides of Eq.(25) with respect to x and setting $x = y(t)$. Assuming that $\Phi(t,v) \in \mathbb{R}$, this yields

$$f(t,v) = \Phi(t,v) + \frac{1}{2\pi}\int_t^T ds \int_0^\infty dv' \Phi(s,v')\,\mathcal{K}(s,v',t,v), \tag{26}$$

$$f(t,v) = -\frac{1}{\pi}\int_0^\infty \xi \Big\{ \cos(\xi y(t))\,\mathrm{Im}[P_1(t,v,-\mathrm{i}\xi)] + \sin(\xi y(t))\,\mathrm{Re}[P_1(t,v,-\mathrm{i}\xi)] \Big\} d\xi,$$

and $\mathcal{K}(s,v',t,v)$ is the kernel of this LMVF integral equation which reads

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$$\times \exp\left[-y(s)\sqrt{p} + \alpha(t,p)v + \beta(t,p) - \left(\beta(s,p) + v'\alpha(s,p)\right)\right].$$

It can be directly checked that $\mathrm{Re}(P_1(t,v,\mathrm{i}\xi)) = \mathrm{Re}(P_1(t,v,-\mathrm{i}\xi))$ and, hence, the difference of P_1 is pure imaginary. Therefore, $[P_1(t,v,-\mathrm{i}\xi)) - P_1(t,v,-\mathrm{i}\xi)]/\mathrm{i}$ is real. Same should be true for the difference of P_2, however, this can be verified only numerically.

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Compared with the 1D case where the RHS of the Volterra equation is a 1D integral, here for the 2D case it is a 3D integral since the integral on ξ cannot be taken analytically. At the first glance this should significantly slow down computation of the gradient $\Phi(t,v)$.

It can be directly checked that $\mathrm{Re}(P_1(t,v,\mathrm{i}\xi)) = \mathrm{Re}(P_1(t,v,-\mathrm{i}\xi))$ and, hence, the difference of P_1 is pure imaginary. Therefore, $[P_1(t,v,-\mathrm{i}\xi)) - P_1(t,v,-\mathrm{i}\xi)]/\mathrm{i}$ is real. Same should be true for the difference of P_2, however, this can be verified only numerically.

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However, in this paper for solving Eq.(26) we also propose variation of the RBF method which allows reduction of the 3D integral to a 2D one in variables (t,ξ). Therefore, our approach seems to be a natural extension of the GIT method to the 2D case while preserving all nice features of the method.

- We proposed a generalization of the GIT method to price Down-and-Out barrier Put options P_{do} under the Heston stochastic volatility model where **all coefficients and the barrier are deterministic functions of the time** (subject to the condition Eq.(12)). The method requires solving a two-dimensional mixed Volterra-Fredholm equation for the gradient $\Phi(t, v)$ of the solution at the moving boundary $x = y(t)$. Once it is found, the option price P_{do} follows since it was expressed in a **semi-analytical** form via a two-dimensional integral of $\Phi(t, v)$. Note, that this integral is computed as a part of the system matrix $\|A\|$ for the LMVF equation, and hence doesn't require extra time.
- We focus on a Down-and-Out Put, but other types of barrier options are also covered by using the in-out parity since for the Heston model a closed-form solution for European options (via a FFT transform) is known.For the Up-and-Out barrier Put option P_{uo} a simple change of variables $x \to -x$ reduces the pricing problem to that one which we consider in this paper. The Call options can be priced in a similar way by using a covered Call instead of a Put.

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- Since the dependence of $\Phi(t, v)$ on the strike K appears only in the LHS of the LMVF equation Eq.(26), the gradient $\Phi(t, v)$ for different strikes can be found by solving a single system of linear equations with multiple RHS. Also, taking T large enough (for stock and index options traded at the market $T \le 1$ year, so we can choose, e.g. $T < T_* = 2$ years) one can solve the LMVF equation, and find $\Phi(t, v)$ for all $t \le T_*$ in one sweep. Then the barrier option prices can be obtained for all maturities $T \le T_*$ by computing the RHS in Eq.(25). Therefore, our method is similar to solving the forward PDE (where the density of the underlying can be found in one sweep and then the option prices for various K and T come by integrating this density with the payoff), rather than the backward one.

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While item 3 is certainly important for practical applications of our approach, we believe that **the main contribution of the paper is given in items 1,2**. Item 4 is provided more for an illustrative purpose.

Our numerical approach used for item 3 is kind of **simplistic** and can be definitely improved by using modern RBF techniques. Therefore, **construction of an efficient method** for solving LMVF equations (with high accuracy and speed) is **subject of a separate investigation** which we intend to provide in the future, but not in this paper.

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Since we don't calibrate the model to market quotes, in these tests (without any loss of generality) we choose an artificial (test) dependencies, namely:

$$\theta(t) = \theta_0 e^{-\theta_k t}, \qquad \sigma(t) = \sigma_0 e^{-\sigma_k t}, \qquad \rho(t) = \rho_0, \qquad \kappa(t) = \frac{m\sigma^2(t)}{2\theta(t)}, \tag{28}$$

S_0	m	θ_0	σ_0	ρ_0	θ_k	σ_k	L	v_0	r	q
60	2	0.1	0.3	-0.7	0.3	0.2	40	0.5	0.02	0.01

Table 1: Parameters of the test.

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We compare our results with those obtained by solving Eq.(5) using the FD method described in detail in [Itkin 2015]. As the collocation points we choose a uniform grid in $t \in [0, T]$ and $v \in [v_0 - v_m, v_0 + v_m]$ with $v_m = 0.1$. We take $N_t = 10, N_v = 4$. The integrals in time are computed by using the Simpson quadratures. Also, following the discoveries in [Itkin and Muravey 2021], to solve a system of linear equations obtained via the RBF method we use a *minres* iterative solver which is good when the matrix is not positive definite, but symmetric (our matrix is almost symmetric with $|a(i,j) - a(j,i)| \approx 0.001$.

We use a set of maturities $T \in [1/24, 1/12, 0.25, 0.5, 1, 2]$ years and strikes $K \in [45, 50, 60, 70, 80, 90]$. The Down-and-Out barrier Put option prices and elapsed times obtained in these experiments are presented in the Table below. For the FD method the time step is 0.01 year to preserve the method accuracy in time. Also, the relative percentage error ε between the FD and GIT solutions is presented in the second Table.

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T	0.042	0.083	0.25	0.5	1	2	0.042	0.083	0.25	0.5	1	2
K	**GIT**						**FD**					
45	0.0343	0.0466	0.0352	0.0227	0.1298	0.5616	0.0288	0.0621	0.0450	0.0252	0.0763	0.7668
50	0.2760	0.4390	0.3402	0.0932	0.1642	1.2165	0.3187	0.5249	0.3602	0.1967	0.1596	0.7836
60	2.9707	3.3150	2.4670	1.3160	0.3904	1.0277	3.3253	3.6601	2.2717	1.2213	0.6829	1.0200
70	10.0135	9.5921	6.9704	4.0235	1.6431	1.1620	10.3518	9.7413	5.8385	3.3366	1.7720	1.1771
80	19.3932	18.1225	13.3700	8.9262	4.3228	1.6871	19.6622	17.8838	10.8061	6.3585	3.4695	1.9798
90	29.2469	27.4915	20.9084	14.8801	7.5348	2.7396	29.4417	26.8270	16.6750	10.1083	5.6790	3.2311
Elapsed time	2.45	2.02	1.91	1.87	2.19	2.53	0.13	0.23	0.65	1.3	2.7	5.4

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K	0.042	0.083	0.25	0.5	1.0	2.0
45	-19.10	24.96	21.78	9.92	-70.12	26.76
50	13.40	16.37	5.55	52.62	-2.88	-55.25
60	10.66	9.43	-8.60	-7.75	42.83	-0.75
70	3.27	1.53	-19.39	-20.59	7.27	1.28
80	1.37	-1.33	-23.73	-40.38	-24.59	14.78
90	0.66	-2.48	-25.39	-47.21	-32.68	15.21

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It can be seen the relative error ε of the GIT method as compared with the FD reference solution **varies across strikes and maturities**. For large K and short T, ε is of order of few percents, while for intermediate maturities it is in the range [20%,40%]. However, for some strikes, say ATM, it is about 8%. For large T we need to use more integration points while the error varies from few percents and up to 2-30% depending on the strike. Big relative error at $K = 45$ and high T is due to the small price value, hence even small absolute errors could produce high relative errors.

A formal (**theoretical**) comparison of our approach (**a numerical part**) with the FD method reveals the following.

- The FD method requires a 3D grid for temporal t and two spatial x, v variables. In our method, since we derived a semi-analytical expression for the barrier option price, we need a 2D grid in (t, v) to solve the LMVF equation numerically. Therefore, we dropped off one dimension that gives rise to acceleration of computations.

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Thank you!