Optionality as a Binary Operation

- Peter Carr
New York University
1 MetroTech, 10th floor New York, NY 11201 pc73@nyu.edu

- Doug Costa
Susquehanna International Group 401 City Avenue
Bala Cynwyd, PA 19004 doug.costa@sig.com

Peter loved to make connections

- Between ideas and concepts (options, finance, physics, mathematics...)
- Between people (Like Paul Erdős, Peter’s “brain was open” to all. Now we each have a Carr Number.)

This project explores connections between option-pricing and basic algebraic structures

- It is partly inspired by his work on a new kind of contract he called "stoptions," so read his paper on that, too.

- With zero rates and drift, Peter observed that the risk-neutral expectation value of a "married put" (put + stock) with strike k and initial stock price s, $E[k \vee sX] = k \oplus s = h(k, s)$, where X is a random variable on $[0, \infty)$ representing the stock's absolute return, defines a binary operation on $[0, \infty)$ which is continuous and monotone increasing in each coordinate, has 0 as identity element, and is homogeneous in k, s.

Married Put Function makes $[0, \infty)$ a Magma

- In algebra, any set with a binary operation defined on it is a "magma," basically "formless and void," i.e., having little structure.
- If the operation is associative [$(a \oplus b) \oplus c = a \oplus (b \oplus c)$], then it is a "semigroup."
- If it has an "Identity element" [$0 \oplus a = a \oplus 0 = a$ for all a], then it is a"monoid."
- If every element has an inverse, then it is a "group."

Conjugate Power Dagum Distribution makes [0, ∞) a Monoid

- For random variable X defined on [0, ∞), the conjugate power Dagum (CPD) distribution with parameter b is defined by the cumulative distribution function $F(x) = (1 + x^{(-1/b)})^{b-1}$.
- Peter discovered that if X is a CPD random variable, then the married put operation is $k \oplus s = E[k \vee sX] = (k^{1/b} + s^{1/b})^{b}$.
- This is patently associative, commutative, with 0 as identity element, so makes [0, ∞) a monoid. In fact, since ordinary multiplication distributes over $\oplus$, it is a "semi-field."
- Peter: "Option pricing is just a change of arithmetic."

Logistic Distribution makes $[-\infty, \infty)$ a Monoid

- For random variable X defined on $[-\infty, \infty)$, the zero-mean logistic distribution with parameter b is defined by the cumulative distribution function $F(x) = (1 + e^{(-x/b)})^{-1}$.
- Peter discovered that if X is a zero-mean logistic random variable, and we consider options on cumulative log-returns (or futures spreads, etc.) then the married put operation is $k \oplus s = E[k \vee (s + X)]$
  $= b\mathrm{Ln}(e^{k/b} + e^{s/b})$.
- This is also clearly associative, commutative, with $-\infty$ as identity element, so makes $[-\infty, \infty)$ a monoid. In fact, now ordinary addition distributes over $\oplus$, so it is again a "semi-field."

Critical result from analysis

- J. Aczel (ca. 1949) showed that if $h(x, y) = x \oplus y$ is a binary operation defined on an interval J of real numbers such that it is continuous and monotone increasing in each coordinate, then $\oplus$ makes J a monoid if and only if there is a monotone continuous function $g: J^* \rightarrow J$ such that for all x, y in J, $x \oplus y = g(g^{-1}(x) + g^{-1}(y))$, where J* is either $[0, \infty)$ or $(-\infty, \infty)$. We then call g a "generator" for $\oplus$.
- This shows that the only associative operations on intervals of real numbers are isomorphic transforms of ordinary addition via the isomorphism g.
- Bonus: defining a "pseudo-product" by $x \otimes y = g(g^{-1}(x)g^{-1}(y))$ makes J into a semi-field isomorphic to either $[0, \infty)$ or $(-\infty, \infty)$ via g.

Consequences of Aczel

- If $\bigoplus$ is a continuous, order-preserving, associative operation on J, then it must be commutative.
- And the identity element must be g(0).

Example 1

- $J = J^* = [0, \infty)$; for all u in J^*, $g(u) = u^b$.
- Then for x, y in J, $x \oplus y = (x^{1/b} + y^{1/b})^b$ and $x \otimes y = xy$.
- This is the Dagum distribution-based married put semi-field on J.

Example 2

- $J = [-\infty, \infty)$, $J^* = [0, \infty)$; for u in J^*, $g(u) = Ln_B(u)$.
- Then for x, y in J, $x \oplus y = bLn(e^{x/b} + e^{y/b})$ and $x \otimes y = x + y$, where $b = 1/Ln(B)$.
- This is the Logistic distribution-based married put semi-field on J.

Excitement: what are the possibilities?

- What are the possible algebraic structures on intervals of real numbers defined by married puts, or other contracts with optionality?
- What are the distributions possible, given properties of pseudo-arithmetics which *could* represent married puts?
- What distributions on what intervals correspond to which algebraic structures?

Peter’s 2020 question

- Can we find all intervals J with a distribution on J such that married puts define an associative operation?

- This raises a philosophical question: “The Associativity Problem.” Why does associativity, in particular matter?
- Possible answers: computational efficiency, opening the door to use of representation theory,…?

Answers Part 1

- On $J = [-\infty, \infty)$, $x \bigoplus y = E[\, x \vee (y + bZ)\,]$ defines an associative operation if and only if Z is a standard logistic random variable.

Flavor of Proof

- Suffices to do the case b =1.
- Write $h(x, y) = x \oplus y = E[\, x \vee (y + Z)]$ and express this as an integral. Then associativity is the functional equation $h(h(x, y), z) = h(x, h(y, z))$.
- Use distributivity of + over $\oplus$ to ”pull out” y:
  $y + h(h(x-y, 0), z-y) = y + h(x-y, h(0, z-y))$.
- So, associativity is: $h(h(u, 0), v) = h(u, h(0, v))$, for all u, v.
- Use commutativity (symmetry of distribution) and put-call parity to see that this is equivalent to $\mathrm{Call}(\, \mathrm{Call}(-u) - v) = \mathrm{Call}(u - \mathrm{Call}(-v)) - \mathrm{Call}(u)$

Flavor of Proof continued

- Take partial derivatives wrt u (strike) to get functional equation for the "survival function" (SF =1- CDF). (This involves a lot of calculation using symmetry again.)
- (**) SF(u+v) = SF(u)SF(v)/[1-SF(u)-SF(v)+2SF(u)SF(v)]
- Define f(z) = (1/SF(z)) -1. Inverting both sides of (**), see that f(u+v) = f(u)f(v). Conclude that f(z) = e^z.
- SF(z) = 1/(1 + e^z), i.e., Z is standard logistic.

Answers Part 2

- Realize that by Aczel, every associative pseudo-sum $\oplus$ defined on an interval J isomorphic to $[0, \infty)$, is defined by a "generator" from the logistic pseudo-sum: $g: [-\infty, \infty) \dashrightarrow J$, $x \oplus y = g(g^{-1}(x) \oplus^{b} g^{-1}(y))$, where $\oplus^{b}$ is the logistic pseudo-sum with parameter $b>0$.
- Since g may be increasing or decreasing, we must allow married puts OR buy-writes to define the operation $\oplus$.
- **We need to find all increasing g such that (with $u = g^{-1}(x)$, $v = g^{-1}(y)$), $g(u \oplus^{b} y) = E[\, g(u) \vee (g(v) \otimes W)] = E[\, g(u) \vee g(v + g^{-1}(W))]$ for some random variable W.**
- Similarly for decreasing g, but V replaced by $\wedge$.

Answers Part 2 continued

- Answers: the only functions g for which there is a random variable W defining the pseudo-sum as a married-put or buy-write are those of the form $g(x) = Ae^{mx} + B$, or $g(x) = Ax + B$. **In each case the distribution of W is uniquely determined.**
- If $m > 0$, then W is a scaled, translated CPD random variable.
- If $m < 0$, then W is a scaled, translated Singh-Maddala (1/CPD) random variable.
- If $g(x) = Ax + B$, then W is a scaled, translated Logistic random variable.

Concluding Observations

- We have determined all distributions under which married puts or buy-writes create associative binary operations. Associativity is surprisingly restrictive.
- We have determined all pseudo-arithmetics isomorphic to $J^* = [0, \infty)$ which can be realized as married put or buy-write valuations.
- We have NOT done anything in the case $J^* = (-\infty, \infty)$. [The difficulty seems to be thinking of derivative contracts which would define well-behaved operations. Married puts, for example, won't be order-preserving.]

More Concluding Observations

- Peter had hoped to find a wide variety of distributions making the married put operation associative, thus creating monoids. So it was somewhat disconcerting to find associativity so limiting.
- Looking deeper, nature becomes more interesting: the binary operations defined by married puts are mostly non-associative and non-commutative and define algebras in which the distributive law holds, but *addition is non-associative and non-commutative.* What can be said about these structures? And what relations among underlying distributions correspond to isomorphism classes of these structures?

The Strange Case of Black-Scholes Part 1

With zero rates and drift, the BS married put with strike k and initial

stock price s is: $k \oplus^{BS} s = sN(d(s,k)) + kN(d(k,s))$,

where $d(s,k) = LN(s/k)/\sigma + \sigma/2$ and σ^2 is the variance to expiration.

This is clearly a commutative operation.

But since the underlying distribution is lognormal, our results imply that it is not associative.

The Strange Case of Black-Scholes Part 2

- We can now ask whether a “generator”(isomorphism) applied to this operation is realizable as a married put value, and if so, under what distribution?
- For $g:[0, \infty) \rightarrow [- \infty, \infty)$ defined by $g(x) = Ln(x)$, the answer is YES and our machinery shows the distribution on $[- \infty, \infty)$ has CDF
- $H(w) = (1 + e^{-w}[N(-w/\sigma + \sigma/2)/N(w/\sigma + \sigma/2])^{-1}$. INTERESTING!