JUNE 2 - JUNE 4

THE PETER CARR MEMORIAL CONFERENCE

Generalizations of the Carr-Madan spanning formula

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Joint work with Peter Carr & Andrew Papanicolaou

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Executive Summary 1/3

- A great deal of mathematics is about breaking down complex structures into simpler ones, and then applying the *totum summa partium* principle.
  - For example, Taylor series break down a complex function into a sum of simpler, power functions.
- The Carr-Madan spanning formula (1998) breaks down any single-asset European option as a weighted sum of cash, forward contracts, and vanilla options:

European payoff = cash + forward contracts

$$+ \sum_K q_K \, \mathbf{call}(K) + \sum_K q_K \, \mathbf{put}(K)$$

  - Most successful application: log-contract → VIX

Executive Summary 2/3

- After expounding the spanning formula in connection with related results, we explain how it can be generalized to two classes of multi-asset European options:
  - Dispersion options
  - European options with homogeneous payoff, e.g. best-of
- The main mathematical tool for breaking down the multi-asset option as a sum of basket vanillas, known as Radon transform, is also used in medical imaging to reconstruct e.g. the slice of a patient's brain.
  - The Radon transform is closely related to the multidimensional Fourier transform.

Executive Summary 3/3

- When a pure theorist may be satisfied with an abstract, general solution leaving out technical details, we tested our theory with relevant applications and derived first-time explicit solutions

Source: Redbubble.com | Credit: LeMuesch

Outline

Introduction

1. **The Carr-Madan spanning formula**

2. **Generalization to multi-asset options**

3. **Main application results**

Conclusion, References & Appendices

Introduction

Sample term sheet for a dispersion option

On the Redemption Date, the Issuer will pay to the holder the following amount in U.S. dollars:

$$N \times \text{Bonus}$$

$$\text{Bonus} = \max\left(0\%, \text{Dispersion} - \text{Strike}\right)$$

$$\text{Dispersion} = \sum_{i=1}^{5} w_i \times \text{abs}\left(\frac{\text{Share}^i_{\text{final}}}{\text{Share}^i_{\text{initial}}} - \text{Basket}_{\text{final}}\right)$$

/arrant on five shares in USD quanto

e the investor receives a Bonus linked to the performance of five
trike Level. The product has no capital protection at any time
f any capital invested. Investment is therefore highly speculative
ors who can afford to lose their entire investment amount.

e Bankers, N.A. (credit rating Aa3, unsecured)

),000

= USD 1,000

to

Issue Price per Warrant USD 60
Listing None
Trade Date (T) [today]
Strike Date T
Issue Date T + 5 days
Redemption Date T + 3 years
Underlying Shares

i	Name	Ticker	$\text{Share}^i_{\text{initial}}$	Weight w_i
1	Apple	AAPL	[114]	20%
2	Microsoft	MSFT	[210]	20%
3	Airbus	AIR	[64]	20%
4	Yamaha	7951	[5000]	20%
5	Beyond Meat	BYND	[170]	20%

Settlement Amount
On the Redemption Date, the Issuer will pay to the holder the following amount in U.S. dollars:

$$N \times \text{Bonus}$$

Where
$$\text{Bonus} = \max\left(0\%, \text{Dispersion} - \text{Strike}\right)$$

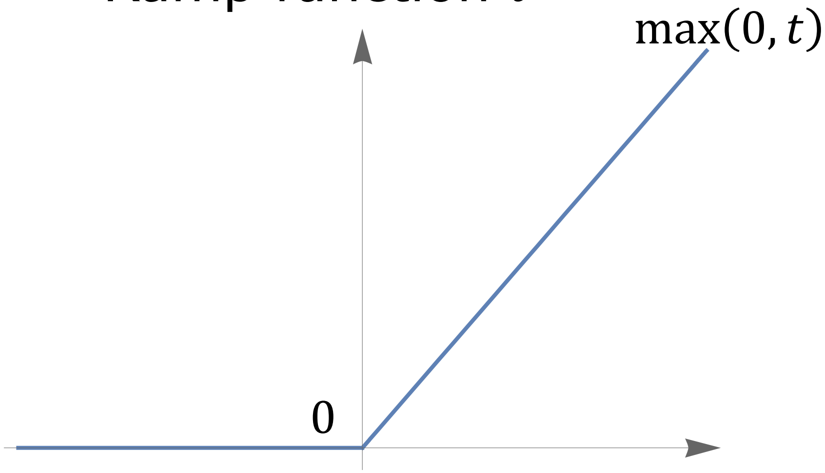
$$\text{Dispersion} = \sum_{i=1}^{5} w_i \times \text{abs}\left(\frac{\text{Share}^i_{\text{final}}}{\text{Share}^i_{\text{initial}}} - \text{Basket}_{\text{final}}\right)$$

With
Strike = 20%
$\text{Basket}_{\text{initial}} = 1$
$\text{Basket}_{\text{final}} = \sum_{i=1}^{5} w_i \times \frac{\text{Share}^i_{\text{final}}}{\text{Share}^i_{\text{initial}}}$
$\text{Share}^i_{\text{initial}}$ with i from 1 to 5 is the official closing price of Underlying Share i on the Strike Date
$\text{Share}^i_{\text{initial}}$ with i from 1 to 5 is the official closing price of Underlying Share i on the Redemption Date

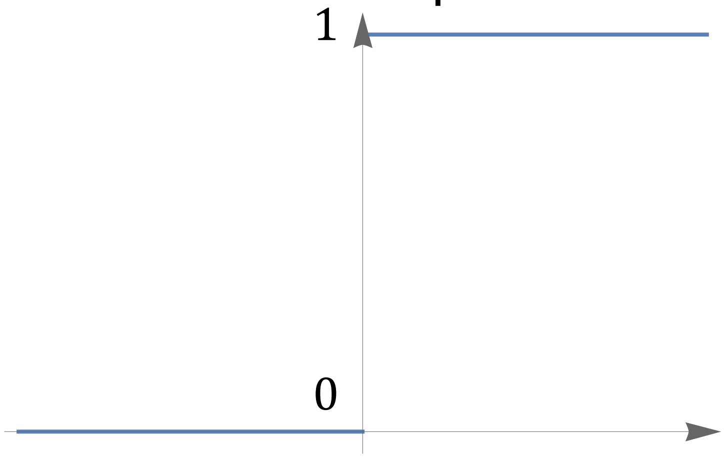
Business Day Convention Following Business Day
Governing law U.S. law

Notations

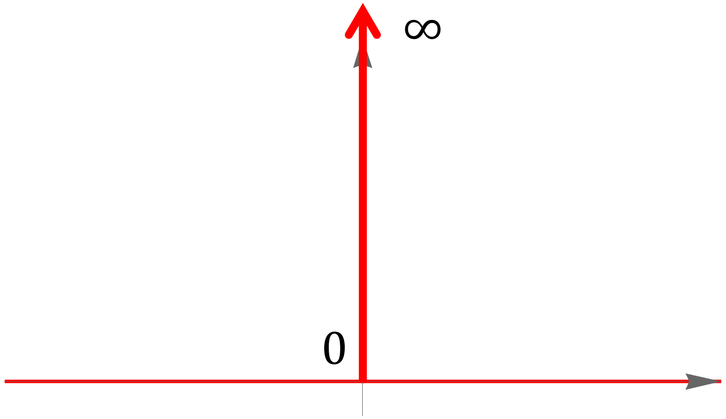
- Ramp function t^+



- Heaviside's step function $H(t)$



- Dirac's delta function $\delta(t)$



Part 1

The Carr-Madan spanning formula

Carr-Madan spanning formula 1/4

- Any European option with payoff function f is statically replicated with a continuum of vanilla options

$$f(S_T) = \underbrace{f(S_0)}_{\text{cash}} + f'(S_0) \underbrace{(S_T - S_0)}_{\text{fwd contract}} + \int_0^{S_0} f''(K) \underbrace{(K - S_T)^+}_{\text{put payoff}} \, dK$$

$$+ \int_{S_0}^{\infty} f''(K) \underbrace{(S_T - K)^+}_{\text{call payoff}} \, dK$$

  - S_T = terminal price of underlying asset
  - f(x) = option payoff at time T when terminal price = x
  - S_0 = any price (typically current underlying price)
  - K = call or put strike price

Carr-Madan (1998) proof 2/4

- Set $S_0 = 0$ for ease (span only with calls): we want to prove

$$f(S_T) = f(0) + f'(0)S_T + \int_0^\infty f''(K)(S_T - K)^+ dK$$

- By sifting property of Dirac's delta function $\delta(\cdot)$

$$f(S_T) = \int_0^\infty f(K)\delta(S_T - K)dK$$

- Integrate by parts

$$f(S_T) = \underbrace{[f(K)H(S_T - K)]_0^\infty}_{=f(0)} + \int_0^\infty f'(K)H(S_T - K)dK$$

- Integrate by parts again

$$f(S_T) = f(0) + \underbrace{[f'(K)(S_T - K)^+]_0^\infty}_{=f'(0)\,S_T} + \int_0^\infty f''(K)(S_T - K)^+ dK$$

Another proof 3/4

- Start with

$$\text{RHS} = f(0) + f'(0)S_T + \int_0^\infty f''(K)(S_T - K)^+ dK$$

- For $K > S_T$ the call payoff is zero, and for $K < S_T$ the payoff is simply $S_T - K$

$$\text{RHS} = f(0) + f'(0)S_T + \int_0^{S_T} f''(K)(S_T - K) dK$$

- Integrate by parts:

$$\text{RHS} = f(0) + f'(0)S_T + [f'(K)(S_T - K)]_0^{S_T} + \int_0^{S_T} f'(K) dK$$

- After calculations and simplifications RHS = $f(S_T)$

Yet another proof

- F.O. Taylor series with remainder in integral form

$$f(x) = f(a) + f'(a)(x-a) + \underbrace{\int_a^x f''(t)(x-t)dt}_{\text{remainder}}$$

- Take $x = S_T, a = 0, t = K$

$$f(S_T) = f(0) + f'(0)S_T + \int_0^{S_T} f''(K)(S_T - K)\,dK$$

- Rewrite the integral as

$$\int_0^{\infty} f''(K)(S_T - K)^{+}\,dK$$

- Done!

Application to the log-contract 1/2

- The log-contract introduced by Neuberger (1990) pays the log-return to maturity of the underlying asset:

$$f(S_T) = \ln \frac{S_T}{S_0}$$

- This contract is interesting because it has constant dollar gamma → variance swaps → VIX
- Problem: the log-contract does not trade…
- Solution: Carr-Madan replication

$$\ln \frac{S_T}{S_0} = \frac{S_T - S_0}{S_0} - \int_0^{S_0} \frac{dK}{K^2}(K - S_T)^+ - \int_{S_0}^{\infty} \frac{dK}{K^2}(S_T - K)^+$$

Application to the log-contract 2/2

- Carr-Madan:

$$\ln\frac{S_T}{S_0}=\frac{S_T-S_0}{S_0}-\int_0^{S_0}\frac{dK}{K^2}(K-S_T)^+-\int_{S_0}^{\infty}\frac{dK}{K^2}(S_T-K)^+$$

- Ito-Doeblin:

$$\ln\frac{S_T}{S_0}=\int_0^T\frac{dS_t}{S_t}-\frac{1}{2}\int_0^T\sigma_t^2dt\,,\quad \sigma_t=\text{instant volatility}$$

- Hence realized variance is replicated with
  - Dynamic position of 1/S in underlying asset until maturity
  - Static positions in cash, forward contracts and OTM calls and puts

$$\frac{1}{2}\int_0^T\sigma_t^2dt=\int_0^T\frac{dS_t}{S_t}-\frac{S_T-S_0}{S_0}+\int_0^{S_0}\frac{dK}{K^2}(K-S_T)^++\int_{S_0}^{\infty}\frac{dK}{K^2}(S_T-K)^+$$

- And the price of realized variance (= variance swap strike) is determined in a model-free way → VIX

2021 generalization to non-vanilla "replicant" options

- Bossu-Carr-Papanicolaou (2021) generalized the spanning formula to single-asset, non-vanilla "replicant" options:

$$f(S_T) = c + qS_T + \int_a^b \phi(K) G(S_T, K)\, dK$$

  - c = unknown cash quantity
  - q = unknown asset quantity
  - $\phi(K)$ = unknown replicant option quantity
  - $G(S_T, K)$ = known payoffs of a family of replicant options indexed by $K \in (a, b)$, e.g.
    - Straddles: $G(S_T, K) = |S_T - K|$
    - Butterflies: $G(S_T, K) = (c - |S_T - K|)^+, c > 0$
- Integral equation $F(x) = \int G(x, y)\phi(y) dy$ with integral kernel $G(x, y)$

Part 2

2022 generalizations to European multi-asset options

- Bossu-Carr-Papanicolaou, *Quantitative Finance* (*feature article*)
- Bossu, *Applied Mathematical Finance* (forthcoming)

Quantitative Finance, 2022
https://doi.org/10.1080/14697688.2022.2040743

Static replication of European standard dispersion options

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Dispersion options may be replicated using vanilla basket calls whose basket weights span an n-dimensional continuum

1. Introduction

Over the past few decades, an array of derivative instruments and trading strategies have appeared where the payoff is based on some measure of statistical dispersion of one or more underlying assets. In the single-asset category, realized volatility and variance swaps appeared in the 1990s, then VIX futures and options in the 2000s as well as other volatility-related exotic options. In the multi-asset category, examples include vanilla price dispersion trades, realized variance dispersion trades, correlation swaps, or call and put options written on cross-sectional price dispersion†† as illustrated in figure 1. Significant market activity for dispersion instruments can be observed in annual reports of many large quantitative hedge funds.‡‡ Accurate pricing and hedging of these instruments is notoriously more complex compared to other multi-asset options such as basket options (e.g. Brigo *et al.* 2004) or worst-of and best-of options.

In our preceding publication (2021) we considered the inverse problem of replicating a single-asset European option with cash, the asset and a 'continuous portfolio' of arbitrary 'replicant' options indexed by a single real variable such as a strike price. In this paper we extend our framework to the multi-asset class of 'standard dispersion' options written

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‖We dedicate this article to the memory of Peter Carr, a prodigious researcher, NYU colleague, mentor, advisor and friend who just passed on March 1st, 2022—our deepest sympathies go to his family. A man of ideas, Peter was a role model and source of inspiration for us all, always interested in new ways of bridging the gap between theory and practice. His knowledge of the quantitative finance literature and the people in the field was unparalleled. We will always remember his vision and ethos as we continue to work in the field that he taught us so much about.

††In the financial industry, price dispersion is more commonly defined as mean absolute deviation corresponding to the 'taxicab' ℓ_1-norm, whereas our approach is based on the Euclidean ℓ_2-norm for ease of mathematics. We do not discuss to what extent our ℓ_2 approach may approximate ℓ_1 instruments because an exact replication of the latter will be derived in a follow-up paper using different mathematical methods.

‡‡For example: Infinity Q Alpha Fund SEC Form N-CSR 31 Aug. 2020, pp. 5, 8–, Assenagon Alpha Annual Report 31 Jan. 2020, p. 7.

2022 generalizations to European multi-asset options 1/4

- Carr-Madan (1998): single-asset European payoffs are replicable with vanilla calls and puts whose strikes span a continuum of $\mathbb{R}$

$$f(S_T) = [\text{cash+forward contracts}] + \int_0^{S_0} f''(K) \underbrace{(K - S_T)^+}_{\text{put payoff}} dK + \int_{S_0}^{\infty} f''(K) \underbrace{(S_T - K)^+}_{\text{call payoff}} dK$$

- Bossu-Carr-Papanicolaou (2022) and Bossu (2022, forth.): multi-asset European payoffs are replicable with vanilla basket calls whose weights span a continuum of $\mathbb{R}^n$

$$f(x_1, \cdots, x_n) = [\text{cash+forward contracts}] + \int \cdots \int \underbrace{(\Sigma_i w_i x_i - k)^+}_{\text{vanilla basket call payoff}} \phi(w_1, \cdots, w_n) dw_1 \cdots dw_n$$

  - $x_1, \cdots, x_n$ = performances (e.g. price ratios or returns to maturity) of n underlying assets
  - $(\Sigma_i w_i x_i - k)^+$ = known payoff of a **vanilla basket call** with basket weights $w_1, \cdots, w_n$ and fixed moneyness k
  - $\phi(w_1, \cdots, w_n)$ = unknown quantity of vanilla basket calls that replicate the target option f

2022 generalizations to European multi-asset options 2/4

$$f(x_1, \cdots, x_n) = \int \cdots \int (\Sigma_i w_i x_i - k)^+ \phi(w_1, \cdots, w_n) dw_1 \cdots dw_n$$

- Benefits:
  - Theoretical: Multi-asset option f can be priced with the same model used for vanilla basket calls
  - Practical: May help identify approximate static hedges, harmonize pricing methodology…
- Limitations:
  - Need to discretize (similar to Carr-Madan / VIX)
  - Weights w_i may be negative → spread of baskets
  - Vanilla basket calls are OTC → limited market information
    - BUT vanilla basket calls are the most liquid options within multi-asset exotics
  - Some quantities $\phi(w_1, \cdots, w_n)$ may be infinite
    - May be resolved with integration by parts, especially in 3D and 2D

2022 generalizations to European multi-asset options 3/4

- We solved this multidimensional replication problem for two classes of multi-asset options:
  - "Standard dispersion options" with payoff

$$f(x_1, \cdots, x_n) = f_0\left(\sqrt{\sum_i x_i^2}\right)$$

    - (Bossu-Carr-Papanicolaou, 2022)
  - Any multi-asset option with absolutely homogeneous payoff (Bossu, 2022, forthcoming)

$$f(\lambda x_1, \cdots, \lambda x_n; \lambda k) = |\lambda| f(x_1, \cdots, x_n; k)$$

    - Application: best-of / worst-of options
- Work is ongoing for a complete extension to any European multi-asset option

2022 generalizations to European multi-asset options 4/4

- General solution for absolutely homogeneous payoff $f(x_1,\cdots,x_n;k)$

$$\phi(w_1,\cdots,w_n) = \left(\mathcal{R}^{-1}\left[\frac{\partial^2 f}{\partial k^2}(x_1,\cdots,x_n;k)\right]\right)(w_1,\cdots,w_n)$$

  - $\mathcal{R}^{-1}[g(x_1,\cdots,x_n;k)]$ is the inverse Radon transform of a function g

- Proof: We want ϕ such that

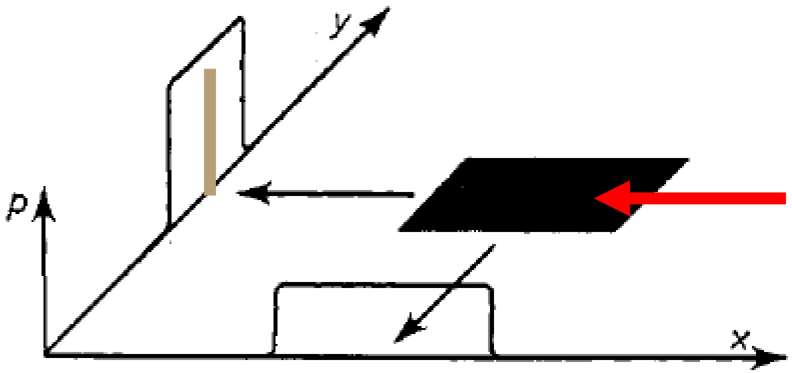
$$f(x_1,\cdots,x_n;k) = \int\cdots\int(\Sigma_i w_i x_i - k)^+ \phi(w_1,\cdots,w_n)dw_1\cdots dw_n$$

  - Differentiate both sides twice against k

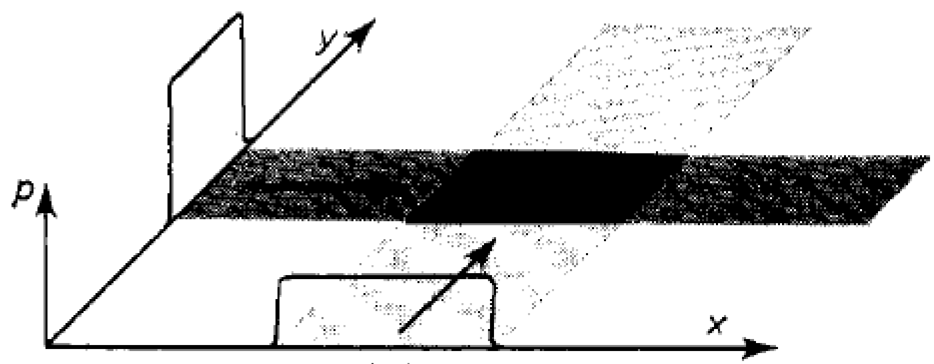
$$\frac{\partial^2 f}{\partial k^2}(x_1,\cdots,x_n;k) = \int_{\mathbb{R}^n}\delta(\Sigma_i w_i x_i - k)\phi(w_1,\cdots,w_n)dw_1\cdots dw_n$$

  - RHS is the Radon transform of ϕ = total sum of ϕ over the hyperplane $\mathcal{H}_{\mathbf{x},k} \coloneqq \{(w_1,\cdots,w_n)\in\mathbb{R}^n : \Sigma_i w_i x_i = k\}$ (dim. $n-1$)
  - ϕ = inverse Radon transform

2D inverse Radon transform 1/2



(a)



(b)

- (a) horizontal and vertical rays go through a rectangular solid
  - Detector reads the total mass
- (b) solid reconstruction by backprojecting mass information from each axis

2D inverse Radon transform 2/2

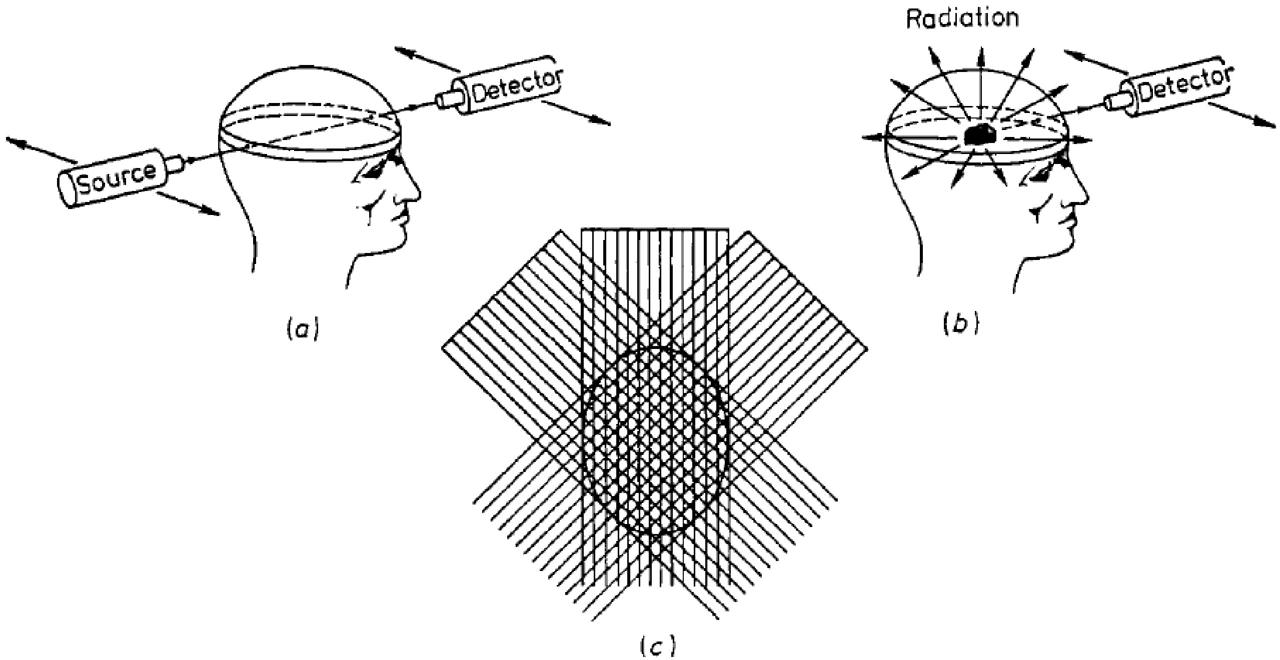
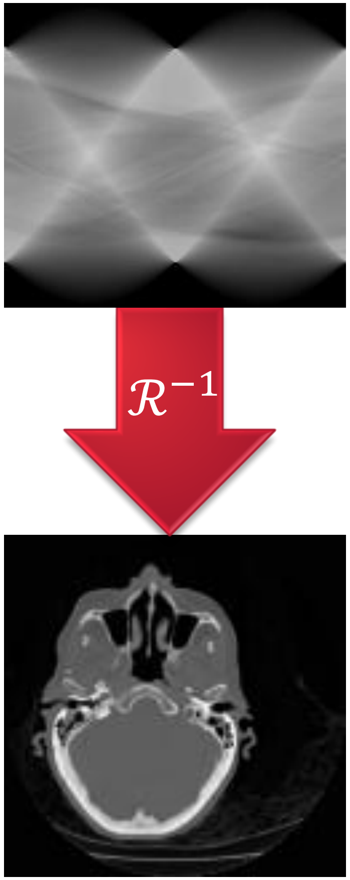


Fig. 3. Reconstructive tomography as applied to (*a*) transmission imaging, and (*b*) emission imaging. (*c*) A typical scanning pattern consists of linear translations at successive angular increments.



Source: Brooks & Di Chiro (1976) "Principles of Computer Assisted Tomography"

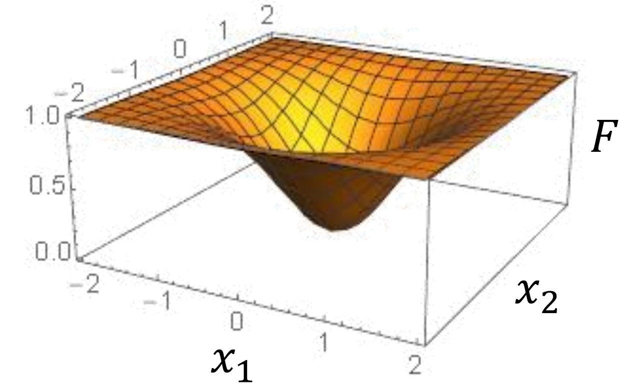
Part 3

Main application results

Application 1

"Mexican hat" dispersion straddle in 2D

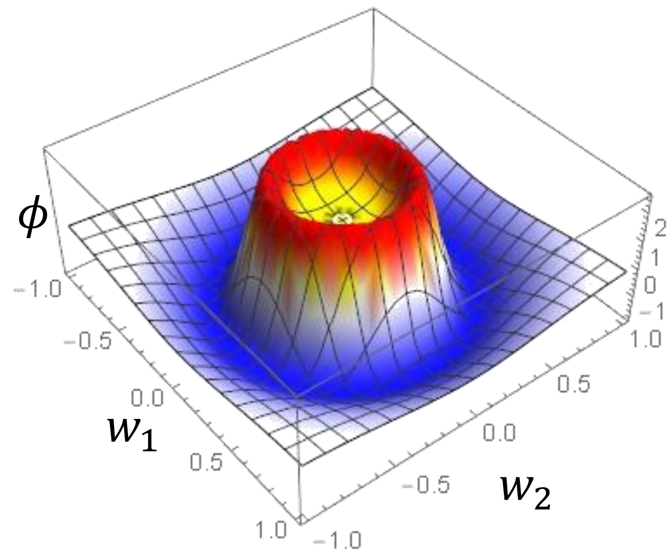
$$F(x_1, x_2) = 1 - e^{-x_1^2 - x_2^2}$$



sol.

$$\phi_{\text{mex}}(r; k) = \frac{4k}{\pi r^4}\left(1 - \frac{k^2}{r^2}\right) - \frac{8k^2}{\pi r^5}\left(\frac{3}{2} - \frac{k^2}{r^2}\right)\mathfrak{D}\left(\frac{k}{r}\right)$$

- $r := \sqrt{w_1^2 + w_2^2}$
- $\mathfrak{D}(t) := e^{-t^2} \int_{-\infty}^{t} e^{s^2} ds$ (Dawson's special function)



- Static decomposition into basket calls:

$$1 - e^{-x_1^2 - x_2^2}$$
$$= \iint (w_1 x_1 + w_2 x_2 - k)^+ \phi_{\text{mex}}\left(\sqrt{w_1^2 + w_2^2}; k\right) dw_1 dw_2$$

Application 2
Standard dispersion call in 3D

$$\left(\sqrt{x_1^2+x_2^2+x_3^2}-k\right)^+ = (\text{const.})\times$$
$$\iiint (w_1x_1+w_2x_2+w_3x_3-k)^+\delta'\left(1-w_1^2-w_2^2-w_3^2\right)dw_1dw_2dw_3$$

- The 3-asset standard dispersion call is replicated by a portfolio of (infinitely leveraged) vanilla basket calls whose weights span the sphere of radius 1
- Integration by parts → alternative decomposition

$$\left(\sqrt{x_1^2+x_2^2+x_3^2}-k\right)^+$$
$$=\iiint_{w_1^2+w_2^2+w_3^2=1} dw_1dw_2dw_3\left[\frac{1}{\pi}(w_1x_1+w_2x_2+w_3x_3-k)^+ - \frac{k}{2\pi}H(w_1x_1+w_2x_2+w_3x_3-k)\right]$$

Application 3

Standard dispersion call in 2D

$$\left(\sqrt{x_1^2 + x_2^2} - k\right)^+ = (\text{const.}) \times \iint_{w_1^2 + w_2^2 < 1} \frac{(w_1 x_1 + w_2 x_2 - k)^+}{(1 - w_1^2 - w_2^2)^{3/2}} dw_1 dw_2$$

- 2-asset SD call is replicated by a portfolio of vanilla basket calls whose weights span the disc of radius 1
- This integral does not converge in a classical sense (the denominator is 0 at the boundary) and must be regularized.

$$= (\text{const.}) \times \iint_{w_1^2 + w_2^2 \le 1} \frac{\arcsin\sqrt{w_1^2 + w_2^2}}{(w_1^2 + w_2^2)^{3/2}} \delta(w_1 x_1 + w_2 x_2 - k) dw_1 dw_2$$

$$+ (\text{const.}) \times \iint_{w_1^2 + w_2^2 = 1} [(w_1 x_1 + w_2 x_2 - k)^+ + k\, H(w_1 x_1 + w_2 x_2 - k)] dw_1 dw_2$$

Application 4

Standard dispersion put

- SD put-call parity: SD Put = SD Call – SD Forward

$$(k-s)^+ = k - s + (s-k)^+, \quad s := \sqrt{x_1^2 + \cdots + x_n^2}$$

- NB: k – s is combo of cash and zero-strike SD Call
  - **Q**: is zero-strike SD Call replicable with vanilla basket calls?
  - **A1**: general solution for positive-strike SD Call vanishes as k→0
    - Not a valid solution for zero-strike call s
  - **A2**: zero-strike call s turns out to replicated by an equally weighted portfolio of zero-strike basket calls

$$s = \sqrt{x_1^2 + \cdots + x_n^2} = (\text{const.}) \times \iint\limits_{w_1^2+\cdots+w_n^2=1} (w_1 x_1 + \cdots + w_n x_n)^+ dw_1 \cdots dw_n$$

Application 5
Replication of best-of call in 2D

$$(\max(x_1, x_2) - k)^+$$

$$= (\text{const.}) \times \iint \frac{(w_1 x_1 + w_2 x_2 - k)^+}{(1-(w_1-w_2)^2)(1-(w_1+w_2)^2)} dw_1 dw_2$$

- 2-asset best-of call is replicated by a portfolio of vanilla basket calls whose weights span the entire plane $\mathbb{R}^2$
- This integral does not converge in a classical sense and must be regularized with care.
  - Denominator = 0 along the four diagonals $w_1 + w_2 = \pm 1$, $w_1 - w_2 = \pm 1$.

Conclusion

- European multi-asset options can be replicated with a continuum of vanilla basket calls.
- A major implication is that they should be priced with the same model used for vanilla basket calls, under penalty of arbitrage.
- Our results are part of ongoing research effort to extend the Carr-Madan and Breeden-Litzenberger formulas to multi-asset options, leveraging on advanced mathematical tools and theory such as Radon transforms that have vast potential for further applications in quantitative finance and indeed other scientific fields.

Ongoing further research

- Dimension 1: discrete Carr-Madan
  - Single underlying asset
  - Perfect hedge with an infinite series of calls and puts
  - Imperfect hedge with a finite number of calls and puts
- Dimension n: static hedging with a finite number of basket calls/puts
  - Multi-asset: best-of, dispersion, …
  - Numerical methods: least squares, machine learning…
  - Risk analysis and delta-hedging of residual error
- Extensions of Ross recovery theorem
  - Perron-Frobenius…

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Appendix A
The mathematics of dispersion option replication

- Replication problem in vector notations:

$$F(\|\mathbf{x}\|_2) = \int_{\mathbb{R}^n} (\mathbf{x}\cdot\mathbf{w} - k)^+ \phi(\|\mathbf{w}\|_2) d\mathbf{w}$$

- Substitute $\mathbf{x} = s\mathbf{v}$ where $s = \|\mathbf{x}\|_2$ and $\mathbf{v} = \mathbf{x}/\|\mathbf{x}\|_2$ is a unit vector

$$F(s) = \int_{\mathbb{R}^n} (s\mathbf{v}\cdot\mathbf{w} - k)^+ \phi(\|\mathbf{w}\|_2) d\mathbf{w}$$

- Differentiate twice against s

$$F''(s) = \int_{\mathbb{R}^n} (\mathbf{v}\cdot\mathbf{w})^2 \delta(s\mathbf{v}\cdot\mathbf{w} - k) \phi(\|\mathbf{w}\|_2) d\mathbf{w}$$

- Sifting property of Dirac's delta function: $f(t)\delta(t-k) \equiv f(k)\delta(t-k)$

$$F''(s) = \int_{\mathbb{R}^n} \frac{k^2}{s^2} \delta(s\mathbf{v}\cdot\mathbf{w} - k) \phi(\|\mathbf{w}\|_2) d\mathbf{w}$$

- Rearrange to get a Radon transform inverse problem

$$f(\|\mathbf{x}\|_2) = \int_{\mathbb{R}^n} \delta(\mathbf{x}\cdot\mathbf{w} - k) \phi(\|\mathbf{w}\|_2) d\mathbf{w}$$

Appendix A
The mathematics of dispersion option replication

- Radon transform inverse problem for radial functions can be shown to simplify to a fractional integral equation

$$f_0(s) = \int_0^s \underbrace{(s^2 - r^2)^{\frac{n-3}{2}}}_{G(s,r)} 2r\phi_0(r)dr$$

- When $n \geq 3$ is odd the exponent $\frac{n-3}{2}$ is an integer and the equation may be solved by standard calculus techniques (e.g. Leibniz rule)

$$\left(\frac{d}{ds^2}\right)^{\frac{n-3}{2}} [f_0(s)] = (\text{const}) \times \int_0^s 2r\phi_0(r)dr$$

$$\frac{d}{ds}\left(\frac{d}{ds^2}\right)^{\frac{n-3}{2}} [f_0(s)] = (\text{const}) \times 2s\phi_0(s)$$

$$\left(\frac{d}{ds^2}\right)^{\frac{n-1}{2}} [f_0(s)] = (\text{const}) \times \phi_0(s)$$

- When n is even there is a residual half-integral.

Appendix B
Inverse Radon transform of radial functions

- If $\varphi(\mathbf{y}) = \phi(|\mathbf{y}|)$ is radial, then $\mathcal{R}\varphi$ is also radial and collapses to a one-dimensional integral

$$(\mathcal{R}\phi)(\mathbf{x}, k) := \int_{\mathbb{R}^n} \delta(\mathbf{x}\cdot\mathbf{y} - k)\phi(|\mathbf{y}|)d\mathbf{y}$$

$$= \int_{\mathbf{0}}^{\infty} r^{n-1}\phi(r)dr \int_{|\mathbf{u}|=1} \delta(r\mathbf{x}\cdot\mathbf{u} - k)d\mathbf{u}$$

- The inner integral is a line integral over the unit circle (2D) / surface integral over the unit (hyper-)sphere (*n*D)
- Slice integration (“Catalan formula”)
  - For any unit vector **v**

$$\int_{|\mathbf{u}|=1} g(\mathbf{u}\cdot\mathbf{v})d\mathbf{u} = A_n \times \int_{-1}^{1} g(t)(1-t^2)^{\frac{n-3}{2}}dt$$

  - $A_n = 2\pi^{\frac{n-1}{2}}/\Gamma[\frac{n-1}{2}]$ = surface area of unit sphere of $\mathbb{R}^{n-1}$

Appendix B
Inverse Radon transform of radial functions

- Slice integration: Substitute $\mathbf{x} = |\mathbf{x}|\mathbf{v}$ where $|\mathbf{v}| = 1$

$$\int_{|\mathbf{u}|=1} \delta(r\mathbf{x}\cdot\mathbf{u} - k)d\mathbf{u} = A_n \times \int_{-1}^{1} \delta(r|\mathbf{x}|t - k)(1 - t^2)^{\frac{n-3}{2}} dt$$

$$= \frac{A_n}{r|\mathbf{x}|^{n-2}} \left[\left(|\mathbf{x}|^2 - \frac{k^2}{r^2}\right)^{+}\right]^{\frac{n-3}{2}}$$

- Radon transform of radial ϕ:

$$(\mathcal{R}\phi)(\mathbf{x}, k) = \frac{A_n}{|\mathbf{x}|^{n-2}} \int_{k/|\mathbf{x}|}^{\infty} r^{n-2}\phi(r)\left(|\mathbf{x}|^2 - \frac{k^2}{r^2}\right)^{\frac{n-3}{2}} dr$$

which is also radial as a function of $|\mathbf{x}|$ only.

Appendix B
Inverse Radon transform of radial functions

- After change of variable $r \mapsto k/r$ and other simplifications, the Radon transform inverse problem

$$f(|\mathbf{x}|, k) = (\mathcal{R}\phi)(\mathbf{x}, k)$$

collapses to a 1D integral equation of the form

$$f_0(s) = \int_0^s r\phi_0(r)\,(s^2 - r^2)^{\alpha-1} dr$$

where $\alpha = \frac{n-1}{2} > 0,\; s = |\mathbf{x}|, f_0(s) = (\text{const.}) \times k^2 s^{n-2} f(s,k),$

$\phi_0(r) = (\text{const.}) \times \left(\frac{k}{r}\right)^{n+1} \phi\left(\frac{k}{r}\right)$

- RHS = **left-sided modified Erdelyi-Kober fractional integral** of the function ϕ_0

- Solution: ϕ_0 = fractional derivative of f_0 against s^2

Appendix B
Inverse Radon transform of radial functions

- Fractional integral equation solution:

$$\phi_0(r) = \begin{cases} \left(\dfrac{d}{dr^2}\right)^{\frac{n-1}{2}} f_0(r) & n \text{ odd} \\ \dfrac{2}{\sqrt{\pi}}\left(\dfrac{d}{dr^2}\right)^{n/2} \displaystyle\int_0^r \dfrac{s f_0(s)}{\sqrt{r^2 - s^2}}\, ds & n \text{ even} \end{cases}$$

- For a dispersion option $f_0(s) = s^n F''(s)/\pi^{\frac{n-1}{2}}$

$$F(|\mathbf{x}|) = \int_{\mathbb{R}^n} (\mathbf{x}\cdot\mathbf{y} - k)^+ \frac{\phi_0(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}\, d\mathbf{y} + c + q|\mathbf{x}|$$