Generalized Compounding and Growth Optimal Portfolios

Reconciling Kelly and Samuelson

P. Carr - U. Cherubini

New York University - University of Bologna

The Peter Carr Memorial Conference
New York, June 2-4, 2022

Peter Paul Carr

Outline

Cointegrated interests

The 90s

Delivery option in futures contracts

The 20s

- Generalized algebras (pseudo-addition, pseudo-multiplication) (Aczél, 1966, Sklar-Schweizer-Sklar, 1961, Ling, 1965, Sugeno, 1988, Pap...).
- Representation of associative operations in terms of *generators*:

$$a \perp b = G\left(G^{-1}(a) + G^{-1}(b)\right)$$

Italian Society of Probability and Mathematical Statistics

"Options, Algebras and Probability: in Memory of Peter Carr", June 14th, 3rd Meeting, Bologna.

Example from physics: Tsallis algebra and q-calculus

Redefinition of the exponential and logarithm function

- Exponential function:

$$\psi(x) \equiv \exp_\gamma(x) = (1 + \gamma x)^{1/\gamma}$$

- Logarithm function

$$\psi^{-1}(x) \equiv \log_\gamma(x) = \frac{x^\gamma - 1}{\gamma}$$

Tsallis product and sum

$$x \otimes_\gamma y = (x^\gamma + y^\gamma - 1)^{1/\gamma}$$

$$x \oplus_\gamma y = x + y + \gamma xy$$

Kolmogorov averages

Theorem

If a function $g(x_1, x_2, \ldots, x_N)$ satisfies the definition of regular average, it can be written as

$$g(x_1, x_2, \ldots, x_N) = \psi\left(\frac{\psi^{-1}(x_1) + \psi^{-1}(x_2) + \ldots, \psi^{-1}(x_N)}{N}\right)$$

where $\psi(x)$ is a continuous, strictly monotone function.

Regular Averages

Kolmogorov axioms

Definition

A sequence of functions $g(x_1, x_2, \ldots, x_N)$ for $N \geq 2$ defines a *regular* kind of average if the following conditions are met

1. $g(x_1, x_2, \ldots, x_N)$ is continuous and monotone in each variable
2. $g(x_1, x_2, \ldots, x_N)$ is exchangeable
3. The function evaluated at $x_1 = x_2 = \ldots = x_N = \bar{x}$ yields $\bar{x}$
4. If $g(x_1 = \bar{x}, x_2 = \bar{x}, \ldots, x_k = \bar{x}) = g(x_1, x_2, \ldots, x_k)$, for $k < N$, then $g(x_1 = \bar{x}, x_2 = \bar{x} \ldots, x_k = \bar{x}, \ldots, x_N) = g(x_1, x_2 \ldots, x_k, \ldots x_N)$

Compounding: back to the basics

- Geometric compounding (taken for granted)

$$W_N = W_0 R_1 R_2 \ldots R_{N-1} R_N$$

- Using the standard algebra of exponential and log functions, and setting $W_0 = 1$:

$$W_N = \exp(\log R_1 + \log R_2 + \ldots + \log R_{N-1} + \log R_N)$$

The basic idea

- What could happen in a different algebra?

$$W_N = \psi\left(\psi^{-1}(R_1) + \psi^{-1}(R_2) + \ldots + +\psi^{-1}(R_{N-1}) + +\psi^{-1}(R_N)\right)$$

- What could be the economic rationale for a different algebra?

The Kelly criterion

- A gambler must decide which fraction f of wealth to bet on a repeated lottery in which he can gain or lose f on each bet.

Log-wealth Maximization

The gambler maximize:

$$G = \lim_{N \to \infty} \frac{1}{N} \log \frac{W_N}{W_0} = p \log(1 + f) + q \log(1 - f)$$

with $q = 1 - p$

Solution and Shannon entropy

The solution is $f^* = p - q = 2p - 1$ for $p > q$ and 0 otherwise, at which:

$$G^* = p \log(p) + q \log(q) + \log 2 = -H + \log 2$$

where H is Shannon entropy.

Samuelson vs Kelly

Kelly (1959)

" The gambler introduced here follows an essentially different criterion from the classical gambler. At every bet he maximize the expected value of the logarithm of capital. The reason has nothing to do with the value function which he attached to his money but merely with the fact that it is the logarithm which is additive in repeated bets and to which the law of large numbers applies."

Samuelson, 1971, on Kelly's log-wealth maximization

"Its essential defect is that it attempts to replace the pair of the 'asymptotically sufficient parameters' $E(\log X)$ and Variance$(\log X)$ by the first of these alone, thereby gratuitously ruling out arbitrary γ in the family $u(x) = x^\gamma/\gamma$ in favor of $u(x) = \log(x)$"

The Kelly criterion (in a different algebra)

Log-wealth Maximization

The gambler maximize:

$$G = \lim_{N \to \infty} \frac{1}{N} \psi^{-1} \left(\frac{W_N}{W_0} \right) = p\psi^{-1}(1+f) + q\psi^{-1}(1-f)$$

with $q = 1 - p$

Tsallis algebra

The Kelly criterion maximises the power function of wealth:

$$\max_{f \in [0,1]} \log_\gamma W_N = \max_{f \in [0,1]} \frac{p(1+f)^\gamma + q(1-f)^\gamma - 1}{\gamma}$$

with solution, in the feasible region ($p \in [0.5, 1)$)

$$f = \frac{1 - (p/q)^{1/(\gamma-1)}}{1 + (p/q)^{1/(\gamma-1)}}$$

Between Kelly and Samuelson

The Long Term Investment Problem Debate in the 60s

- Economics: Expected Utility (Markowitz, Merton, Samuelson)
- Information theory: law of large numbers (Kelly, Breiman, Latané)

Common feature: geometric compounding

There is no disagreement that

- Growth-Optimal-Portfolio (GOP) is expected log-wealth maximization of wealth

What we have done

- We apply the standard Kelly approach in Tsallis algebra and we find Samuelson. Unlike Samuelson, we do not use expected utility.
- GOP is not expected log wealth maximization, it is expected power wealth maximization.

The Debate Today: Ergodicity Economics

Ergodicity Economics (Peters (2019), Peters-Adamou (2021))

- Agents make decisions applying the law of large numbers to suitable transforms of the stochastic process
- Different dynamics lead to different choices of the decision maker
- Experimental evidence (Copenaghen experiment)

An Economic Model for the Compounding Algebra

In the standard geometric compounding we assume

$$R_k = \exp(s_k \Delta t)$$

where s_k is the continuous time average return in the constant time period of length Δt.

MGF and time change

Assume the gross return R_k is affected by a random variable Z_k, with $E(Z_k) = \Delta t$, representing *operational time* or *business time*. Denote $\psi(.)$ the Laplace transform, or MGF, of these random variables. Then, when we compute the return dynamics we have to take into account the stochastic clock, and integrating out Z_k we have

$$R_k = E(\exp(s_k * Z_k)) = \psi(s_k)$$

Notice that function $\psi(y)$ extends the case $\psi(y) = \exp(y)$ which is obtained if the stochastic time degenerates to the constant $Z_k = \Delta t$.

Stochastic Clock: The Discrete Time Model Setting

- Time interval $[0, T]$
- N uniform sub-intervals each of length $\triangle t \equiv T/N$
- positive discrete-time stochastic process $\{R_i, i = 0, 1, \ldots, N\}$

Assumptions

- $Z_i, i = 1, 2, \ldots, N$ positive identically distributed rv
- cumulated time process $\tau_i, i = 1, 2, \ldots, N$

$$\tau_i = Z_1 + Z_2 + \ldots + Z_i$$

- $G(z_1, \ldots, z_N)$ joint cdf of $\{Z_1, \ldots, Z_N\}$ and joint Laplace transform $\psi_N(s_1, \ldots, s_N)$.

Stochastic Time with i.i.d. Increments

Sketch of proof (Marshall-Olkin like)

$$
\begin{aligned}
E\frac{W_N}{W_0} &= Ee^{\bar{s}\tau(N)} \\
&= Ee^{\bar{s}\sum_{i=1}^{N} Z_i} \\
&= \prod_{i=1}^{N} Ee^{\bar{s}Z_i} \text{business time increments are independent} \\
&= \prod_{i=1}^{N} Ee^{\bar{s}Z_1} \text{business time increments identically distributed} \\
&= \left(Ee^{\bar{s}Z_1}\right)^N \\
&= \psi\left(\frac{\sum_{i=1}^{N} s_i}{N}\right)^N = \psi\left(\frac{\sum_{i=1}^{N} \psi^{-1}(R_i)}{N}\right)^N
\end{aligned}
$$

Generalized Compounding with i.i.d. Clocks

- Assume the stochastic clock is represented by a sequence of i.i.d. random variables Z_k with MGF $\psi(.)$

$$\left(E_Z \frac{W_N}{W_0}\right)^{\frac{1}{N}} = \psi\left(\frac{\sum_{i=1}^{N} \psi^{-1}(R_i)}{N}\right)$$

- Note that the geometric mean of generalized compounding is represented by a Kolmogorov average with generator represented by the MGF of the stochastic clock.
- With generalized compounding, the Kelly principle maximizes

$$\lim_{N\to\infty} \psi^{-1}\left(E_Z \frac{W_N}{W_0}\right)^{\frac{1}{N}} = \lim_{N\to\infty} \frac{\sum_{i=1}^{N} \psi^{-1}(R_i)}{N}$$

The VG Model

- Upward movements: gamma distributed, mean Δt, variance ν
- Downward movements: gamma distributed, mean Δt, variance ν
- θ is the variance of the clock normalized by its square means

$$\frac{\nu}{\Delta t^2} = \theta$$

Then, assuming w.l.g. $\Delta t = 1$:

$$\psi(s) = E(\exp(s * Z)) = (1 - \theta s)^{-1/\theta}$$

VG Compounding ($\gamma \equiv -\theta$)

$$\psi(s) = E(\exp(s * Z)) = (1 + \gamma s)^{1/\gamma}$$

$$\psi^{-1}(R) = \frac{R^\gamma - 1}{\gamma}$$

GOP with Gamma i.i.d. stochastic clock

Proposition

*(**Power GOP**)The growth optimal portfolio in the VG model is the constant portfolio that maximizes*

$$\log_\gamma \left(E \frac{W_N}{W_0} \right)^{1/N} = \frac{\frac{1}{N} \sum_{t=1}^{N} R_t^\gamma - 1}{\gamma} \tag{1}$$

- GOP is isomorphic to a CRRA utility optimization problem
- Since $\theta > 0$, and so $\gamma < 0$
  - optimal investment in the risky asset is lower than under log-wealth maximization (fractional Kelly)
  - θ is a measure of the variance of Z_k
  - log-wealth maximization cannot be reached

Inverse Gaussian (IG) i.i.d. stochastic clock

IG MGF

$$\psi(s_k) = E(\exp(s_k Z)) = \exp\left(\theta\left(1 - \sqrt{1 - \frac{2}{\theta}s_k}\right)\right)$$

$$\psi^{-1}(R_k) = \log(R_k) - \frac{1}{2\theta}(\log(R_t))^2$$

- The Z_k clock has mean 1 and variance $1/\theta$

GOP with IG i.i.d. stochastic clock

Proposition

*(**Mean-Variance GOP**) The growth optimal portfolio in the IG model is the constant weights portfolio that maximizes*

$$\log_{IG}\left(E\frac{W_N}{W_0}\right)^{1/N} = \frac{1}{N}\sum_{t=1}^{N}\log(R_t) - \frac{1}{N}\frac{1}{2\gamma}\sum_{t=1}^{N}(\log(R_t))^2 \qquad (2)$$

- GOP is isomorphic to a mean-variance utility optimization problem
- Since $\theta = \gamma > 0$,
  - optimal investment in the risky asset is lower than under log-wealth maximization (fractional Kelly)
  - $1/\gamma$ is the variance of Z_k and is isomorphic to risk aversion
  - log-wealth maximization cannot be reached

Summary

Our approach:

- Assume markets with i.i.d stochastic clock dynamics
- Then compounding needs not be geometric
- Average returns are Kolmogorov means
- The Generalised GOP is not log-wealth maximizing
  - Lower investment in risky assets (fractional Kelly)
  - Log-wealth maximization cannot be reached

Well known speculative price dynamics models lead to Generalised GOP isomorphic to very well known expected utility models. In these models the variance of the clock plays the role of market price of risk.

- Variance Gamma (VG) models generate power wealth maximization
- Inverse Gaussian (IG) models generate mean variance wealth maximization.

Other ideas

- Asset pricing with time change (permanent/transitory decomposition)
- Option pricing with generators.
- Hyperarchimedean copula

$$C(u, v) = \phi\left(\phi^{-1}(u) \oplus \phi^{-1}(v)\right)$$

- A copula from running max and max drawdown.