SCUOLA NORMALE SUPERIORE

Elementary compound derivative pricing

Peter Carr[1], Federico Maglione[2]

[1]NYU Tandor School of Engineering, [2]Scuola Normale Superiore

2-4 June 2022 – The Peter Carr Memorial Conference

SCUOLA
NORMALE
SUPERIORE

Research question

Is it possible to price compound derivatives using just a calculator?

If so, can we also price an American call on an asset that pays a discrete known dividend?

SCUOLA NORMALE SUPERIORE

Related literature

- Carr and Torricelli (2021) propose a class of distributions consistent with risk-neutral pricing for which the pricing of vanilla options is elementary
- We wish to investigate whether such elementary pricing can be extended to the pricing of compound options
- Compound options are used for:
  - the pricing of *options on levered equity* (Geske 1979*b*, Toft and Prucyk 1997)
  - the modelling of the *firm's equity* in structural models of default (Geske 1977, Hull et al. 2004, Geske et al. 2016)
  - the evaluation of *American claims* (Roll 1977, Geske 1979*a*, Whaley 1981, Prekopa and Szantai 2010)
  - the pricing of some *exotic derivatives* (Carr 1988, Buraschi and Dumas 2001, Barone 2013)

SCUOLA NORMALE SUPERIORE

Dagum distribution

We say that a positive random variable X follows a Dagum distribution, i.e. $X \sim \mathrm{D}(m, p, q)$, with $m > 0$, $p > 0$, $q > 0$ if its PDF and CDF are

$$f_X(x; m, p, q) = \frac{p}{q} \frac{\left[1 + \left(\frac{x}{m}\right)^{-p}\right]^{-\frac{1}{q}-1}}{x} \left(\frac{x}{m}\right)^{-p}$$

$$F_X(x; m, p, q) = \left[1 + \left(\frac{x}{m}\right)^{-p}\right]^{-\frac{1}{q}}$$

- m controls the location of X and $\mathbb{E}X = m$ if $p > 1$; if $0 < p < 1$ the mean does not exist as the integral diverges
- p controls the precision (that is p^{-1} controls the dispersion)
- q controls the skewness of $\ln X$: for $0 < q < 1$ odds moments are positive, for $q > 1$ odds moments are negative, for $q = 1$, $\ln X$ is symmetric

Skew-logistic and Singh-Maddala distributions

SCUOLA NORMALE SUPERIORE

A random variable Y is skew-logistically distributed, i.e. $Y \sim \mathrm{SL}(\mu, \sigma, \alpha)$, with $\mu \in \mathbb{R}$, $\sigma > 0$, $\alpha > 0$, and PDF and CDF

$$f_Y(y; \mu, \sigma, \alpha) = \frac{\alpha}{\sigma} \exp\left(-\frac{y-\mu}{\sigma}\right)\left(1+\exp\left(-\frac{y-\mu}{\sigma}\right)\right)^{-\alpha-1}$$

$$F_Y(y; \mu, \sigma, \alpha) = \left(1+\exp\left(-\frac{y-\mu}{\sigma}\right)\right)^{-\alpha}$$

and

$$X \sim \mathrm{D}(m, p, q) \iff \ln X \sim \mathrm{SL}(\ln m, 1/p, 1/q)$$

The Singh-Maddala random variable $Z \sim \mathrm{SM}(\lambda, \gamma, \kappa)$ with $\lambda > 0$, $\gamma > 0$, $\kappa > 0$, has PDF and CDF given by

$$f_Z(z; \lambda, \gamma, \kappa) = \frac{\gamma}{\kappa} \frac{\left[1+\left(\frac{z}{\lambda}\right)^{\gamma}\right]^{-\frac{1}{\kappa}-1}}{z} \left(\frac{z}{\lambda}\right)^{\gamma}$$

$$F_Z(z; \lambda, \gamma, \kappa) = 1 - \left[1+\left(\frac{z}{\lambda}\right)^{\gamma}\right]^{-\frac{1}{k}}$$

and

$$X \sim \mathrm{D}(m, p, q) \iff 1/X \sim \mathrm{SM}(1/m, p, q)$$

Conjugate-power Dagum, skew-logistic and Singh-Maddala distributions

We will refer to *conjugate-power* Dagum (CPD), *conjugate-power* skew-logistic (CPSL) and *conjugate-power* Singh-Maddala (CPSM) random variables. These variables are defined via $\mathrm{D}(m, p, q)$, for which we further assume that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

that is the dispersion and skew parameters of the Dagum distribution are *Hölder conjugate*. Letting

$$b := \frac{1}{p} \in (0, 1) \iff 1 - b = \frac{1}{q} \in (0, 1)$$

we have $X \sim \mathrm{CDP}(m, b) \equiv \mathrm{D}(m, 1/b, 1/(1 - b))$.

- as $p = 1/b > 1$, the mean of the conjugate-power Dagum is finite and equal to m
- given $q = 1/(1 - b) > 1$, the conjugate-power skew-logistic random variable is negatively skewed
- Carr and Torricelli (2021) show that the Dagum distribution exhibit mild excess kurtosis

Conjugate-power Dagum asset prices: Assumptions

- filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and representing a financial market
- no-arbitrage conditions in the economy and that there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$
- similarly to Carr and Torricelli (2021), we assume that a zero risk-free interest rate is paid
- $S_T \overset{\mathbb{Q}}{\sim} \text{CPD}\,(S_0, b(T))$, with $S_0 > 0$ and $b(T) \in (0,1)$
- $b(T)$ is referred as the *bewilderment function* as it controls the dispersion of the distribution. Also, it must to be increasing with respect to the maturity with $\lim_{T\downarrow 0} b(T) = 0$
- Carr and Torricelli (2021) propose to link the bewilderment function to the volatility via

$$b(T) = \sqrt{1 - \exp\left(-\sigma^2 T\right)}$$

Conjugate-power Dagum asset prices: Risk-neutrality

We need to show that the distribution of S_T under $\mathbb{Q}$ is actually risk-neutral. If $S_T \overset{\mathbb{Q}}{\sim} \text{CPD}\left(S_0, b(T)\right)$, let $b(T) = b$ for convenience and

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(S_T\right) &= \int_0^{\infty} u f_S^{\mathbb{Q}}(u)\,\mathrm{d}u \\
&= \int_0^{\infty} u \frac{1-b}{b} \frac{\left[1+\left(\frac{u}{S_0}\right)^{-\frac{1}{b}}\right]^{b-2}}{u} \left(\frac{u}{S_0}\right)^{-\frac{1}{b}} \mathrm{d}u \\
&= \left(-S_0^{\frac{1}{b}} \left(u^{\frac{1}{b}} + S_0^{\frac{1}{b}}\right)^{b-1} \Bigg|_0^{\infty} \right) = 0 + S_0^{\frac{1}{b}} S_0^{\frac{b-1}{b}} = S_0
\end{aligned}$$

as $\lim_{u\uparrow\infty} \left(u^{\frac{1}{b}} + S_0^{\frac{1}{b}}\right)^{b-1} = \lim_{u\uparrow\infty} u^{\frac{b-1}{b}} = \lim_{u\uparrow\infty} 1/u^{\frac{1}{b}} = 0$, given $b \in (0, 1)$.
Hence, the conjugate-power Dagum CDF is risk-neutral.

Married put

Payoff of a married put written struck at $K > 0$, with maturity $T > 0$

$$MP_T(S, K, T) = \max\{S_T, K\} = S_T \mathbb{1}_{S_T \geq K} + K \mathbb{1}_{S_T < K},$$

Using risk-neutral valuation, the price of this claim is

$$\begin{aligned} MP_0(S, K, T) &= \mathbb{E}^{\mathbb{Q}}\left(MP_T(S, K, T)\right) = \mathbb{E}^{\mathbb{Q}}\left(S_T \mathbb{1}_{S_T \geq K} + K \mathbb{1}_{S_T < K}\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left(S_T \mathbb{1}_{S_T \geq K}\right) + K \mathbb{Q}(S_T < K) \end{aligned}$$

Letting the Radon-Nikodym derivative $\eta_T = \frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{S_T}{S_0}$, with $\mathbb{Q} \sim \widehat{\mathbb{Q}}$, the main pricing equation follows

$$\begin{aligned} MP_0(S, K, T) &= S_0 \mathbb{E}^{\mathbb{Q}}\left(\eta_T \mathbb{1}_{S_T \geq K}\right) + K \mathbb{Q}(S_T < K) \\ &= S_0 \widehat{\mathbb{Q}}\left(S_T \geq K\right) + K \mathbb{Q}(S_T < K) \end{aligned} \quad (1)$$

Trivially,

$$\begin{aligned} c_0(S, K, T) &= MP_0(S, K, T) - K \\ p_0(S, K, T) &= MP_0(S, K, T) - S_0 \end{aligned}$$

Married put: Log-normal asset prices

It is well known that if $R^A_{[0,T]} = R^A_T := \ln S_T - \ln S_0$, is distributed as

$$R^A_T \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\mu^{\mathbb{Q}}(T), \sigma(T)^2\right)$$

then

$$R^A_T \overset{\widehat{\mathbb{Q}}}{\sim} \mathcal{N}\left(\mu^{\widehat{\mathbb{Q}}}(T), \sigma(T)^2\right) \equiv \mathcal{N}\left(\mu^{\mathbb{Q}}(T) + \sigma(T)^2, \sigma(T)^2\right)$$

that is the change of measure translate into *shifting* the mean of the distribution of log-returns. Hence

$$MP_0(S, K, T) = S_0 \Phi\left(\frac{\ln \frac{S_0}{K} + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}\right) + K\Phi\left(\frac{\ln \frac{K}{S_0} + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}\right) \qquad (2)$$

whith $\Phi(\cdot)$ the CDF of a standard Normal. Such formula is obviously **not elementary**.

Married put: Conjugate-power Dagum asset prices

To price the married put, $\mathbb{Q}(S_T < K)$ is nothing but the CDF of S_T under $\mathbb{Q}$. To obtain the distribution of S_T under $\widehat{\mathbb{Q}}$ notice that

$$\widehat{\mathbb{Q}}\left(S_T < x\right) = \Lambda\left(\mathbb{Q}\left(S_T < x\right)\right) = \frac{\int_0^x u f_S^{\mathbb{Q}}(u; S_0, b)\, \mathrm{d}u}{\int_0^\infty u f_S^{\mathbb{Q}}(u; S_0, b)\, \mathrm{d}u} = \frac{\int_0^x u f_S^{\mathbb{Q}}(u; S_0, b)\, \mathrm{d}u}{S_0}$$

given that S_0 is the risk-neutral mean of S_T and $\Lambda : [0,1] \to [0,1]$ is the Lorenz curve applied to the risk-neutral CDF $F_S^{\mathbb{Q}}$.

In financial terms, the Lorenz curve relates the share measure distribution function $F_S^{\widehat{\mathbb{Q}}}$ to the risk-neutral measure distribution function $F_S^{\mathbb{Q}}$. Essentially, calculating the Lorenz curve of the risk-neutral CDF allows to calculate probabilities under the measure that has S as a numéraire.

Married put: Conjugate-power Dagum asset prices

SCUOLA NORMALE SUPERIORE

Solving the integral

$$\widehat{\mathbb{Q}}\left(S_T < x\right) = \frac{1}{S_0}\int_0^x u\,\frac{1-b}{b}\,\frac{\left[1+\left(\frac{u}{S_0}\right)^{-\frac{1}{b}}\right]^{b-2}}{u\left(\frac{u}{S_0}\right)^{\frac{1}{b}}}\,\mathrm{d}u$$

$$= \frac{1}{S_0}\left(-S_0^{\frac{1}{b}}\left(u^{\frac{1}{b}}+S_0^{\frac{1}{b}}\right)^{b-1}\Big|_0^x\right)$$

$$= 1-\left[1+\left(\frac{x}{S_0}\right)^{\frac{1}{b}}\right]^{b-1}$$

That is S_T under the spot measure is distributed as a conjugate-power Singh Maddala random variable, i.e. $S_T \overset{\widehat{\mathbb{Q}}}{\sim} \text{CPSM}\left(S_0, b(T)\right)$,

Married put: Conjugate-power Dagum asset prices

The price of a married put under the conjugate-power Dagum model is **elementary**

$$
\begin{aligned}
MP_0(S, K, T) &= S_0\widehat{\mathbb{Q}}\left(S_T \geq K\right) + K\mathbb{Q}(S_T < K) \\
&= S_0\left[1 + \left(\frac{K}{S_0}\right)^{\frac{1}{b(T)}}\right]^{b(T)-1} + K\left[1 + \left(\frac{K}{S_0}\right)^{-\frac{1}{b(T)}}\right]^{b(T)-1} \\
&= \left(S_0^{\frac{1}{b(T)-1}}\frac{S_0^{\frac{1}{b(T)}} + K^{\frac{1}{b(T)}}}{S_0^{\frac{1}{b(T)}}}\right)^{b(T)-1} + \left(K^{\frac{1}{b(T)-1}}\frac{S_0^{-\frac{1}{b(T)}} + K^{-\frac{1}{b(T)}}}{S_0^{-\frac{1}{b(T)}}}\right)^{b(T)-1} \\
&= \dots \\
&= \left(S_0^{\frac{1}{b(T)}} + K^{\frac{1}{b(T)}}\right)^{b(T)}
\end{aligned}
\tag{3}
$$

as shown in Carr and Torricelli (2021).

SCUOLA
NORMALE
SUPERIORE

Compound married put

Let $T_2 > T_1$ and $K_2 \leq K_1$. The payoff of the compound married put is defined as

$$MP_{T_1}\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right) = MP_{T_1}(MP_{T_1}(S, K_2, T_2), K_1, T_1) = \max\{MP_{T_1}(S, K_2, T_2), K_1\}.$$

Using risk-neutral valuation,

$$MP_0\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right) = \mathbb{E}^{\mathbb{Q}}\left(MP^2_{T_1}\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right)\right) = \mathbb{E}^{\mathbb{Q}}\left(MP_{T_1}(S, K_2, T_2)\mathbb{1}_{MP_{T_1}(S,K_2,T_2)\geq K_1}\right) + K_1\mathbb{Q}(MP_{T_1}(S, K_2, T_2) < K_1).$$

As the married put is an increasing function of its argument, there exists a critical value $K_\star > 0$ such that

$$MP_{T_1}(S, K_2, T_2) \lesseqgtr K_1 \iff S_{T_1} \lesseqgtr K_\star$$

.

Compound married put

Therefore,

$$MP_0\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right) = \mathbb{E}^{\mathbb{Q}}\left(MP_{T_1}(S, K_2, T_2)\mathbb{1}_{S_{T_1} \geq K^\star}\right) + K_1\mathbb{Q}(S_{T_1} < K^\star). \quad (4)$$

The value $K_\star$ is model-dependent and in the case of log-normal asset prices must be computed numerically. We show that in the conjugate-power Dagum setting is elementary instead.

In the following we must assume $T := T_1 = T_2/2$, which we refer as *symmetric compound married put*.

The reason for such restriction is that the bewilderment function is not a pre-specified function, but only require to vanish as the time-to-maturity vanishes and to be bounded in $(0, 1)$. As it depends on the size of time intervals $[0, T_1]$ and $[T_1, T_2]$ only, the only way to work with *identical* bewilderment functions is to assume symmetry in the time steps.

Symmetric compound married put: Log-normal asset prices

Let $T := T_1 = T_2/2$. Under the assumption of log-normality, log returns are idd normal random variables such that

$$R^A_{[0,T]} \overset{\mathbb{Q}}{\sim} R^A_{[T,2T]} \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\mu^{\mathbb{Q}}(T), \sigma(T)^2\right)$$

By the reproductive property of the normal distribution, it also follows that the total return is

$$R^A_{[0,2T]} = R^A_{[0,T]} + R^A_{[T,2T]} \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\mu^{\mathbb{Q}}(2T), \sigma(2T)^2\right)$$

and

$$\begin{aligned} MP_0\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T \\ 2T \end{pmatrix}\right) &= S_0 \Phi_2\left(\frac{\ln \frac{S_0}{K_\star} + \sigma^2 \frac{T}{2}}{\sigma\sqrt{T}}, \frac{\ln \frac{S_0}{K_2} + \sigma^2 T}{\sigma\sqrt{2T}}; \sqrt{\frac{1}{2}}\right) \\ &+ K_2 \Phi_2\left(-\frac{\ln \frac{K_\star}{S_0} + \sigma^2 \frac{T}{2}}{\sigma\sqrt{T}}, \frac{\ln \frac{K_2}{S_0} + \sigma^2 T}{\sigma\sqrt{2T}}; -\sqrt{\frac{1}{2}}\right) \\ &+ K_1 \Phi\left(\frac{\ln \frac{K_\star}{S_0} + \sigma^2 \frac{T}{2}}{\sigma\sqrt{T}}\right) \end{aligned} \tag{5}$$

where $\Phi_2(\cdot, \cdot, \rho)$ is the bivariate CDF of a 2-dimensional standardized Gaussian random vector with correlation ρ.

Symmetric compound married put: Conjugate-power Dagum asset prices

Let $R^G_{[0,T]} = \frac{S_T}{S_0}|S_0$ and $R^G_{[T,2T]} = \frac{S_{2T}}{S_T}|S_T$ and assume a multiplicative random walk with iid factors such that

$$R^G_{[0,T]} \overset{\mathbb{Q}}{\sim} R^G_{[T,2T]} \overset{\mathbb{Q}}{\sim} \text{CPD}\,(1, b(T))\,.$$

Unlike what happens in the case of log-normal asset prices, there is no such reproductive property for either sums or products of Dagum distributions, i.e.

$$R^G_{[0,2T]} = R^G_{[0,T]} \times R^G_{[T,2T]} \overset{\mathbb{Q}}{\nsim} \text{CPD}\,(1, b(2T))\,.$$

Hence

$$MP_0(S,K,2T) \neq \left(S_0^{\frac{1}{b(2T)}} + K^{\frac{1}{b(2T)}}\right)^{b(2T)}$$

but

$$MP_T(S,K,2T) = \left(S_T^{\frac{1}{b(T)}} + K^{\frac{1}{b(T)}}\right)^{b(T)}$$

Nonetheless, the two-period married put price can still be computed analytically.

Symmetric compound married put: Conjugate-power Dagum asset prices

Let $b(T) = b$. The critical $K_\star$ is defined via

$$\left(K_\star^{\frac{1}{b}} + K_2^{\frac{1}{b}}\right)^b = K_1,$$

and elementary

$$K_\star = \left(K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^b$$

The first expectation in (4) can be expressed as

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(MP_T(S, K_2, 2T)\mathbb{1}_{S_T \geq K_\star}\right) &= \int_{K_\star}^{\infty} MP_T(u, K_2, 2T) f_S^{\mathbb{Q}}(u)\, \mathrm{d}u \\
&= \int_{K_\star}^{\infty} u \left[1 + \left(\frac{u}{K_2}\right)^{-\frac{1}{b}}\right]^b \frac{1-b}{b} \frac{\left[1 + \left(\frac{u}{S_0}\right)^{-\frac{1}{b}}\right]^{b-2}}{u\left(\frac{u}{S_0}\right)^{\frac{1}{b}}} \mathrm{d}u \qquad (6) \\
&= \int_{K_\star}^{\infty} \frac{1-b}{b} \left(\frac{u}{S_0}\right)^{-\frac{1}{b}} \left[1 + \left(\frac{u}{S_0}\right)^{-\frac{1}{b}}\right]^{b-2} \left[1 + \left(\frac{u}{K_2}\right)^{-\frac{1}{b}}\right]^b \mathrm{d}u.
\end{aligned}$$

SCUOLA NORMALE SUPERIORE

Technical result

Theorem

Given random variable $X \sim CPD(\mu, b)$ with PDF f, it follows

$$I(\mu, m, b, a) = \int_a^\infty x \left[1 + \left(\frac{x}{m}\right)^{-\frac{1}{b}}\right]^b f(x; \mu, b)\, dx$$
$$= a\left(\frac{\mu}{a}\right)^{\frac{1}{b}} F_1\left(1 - b; 2 - b, -b; 2 - b; -\left(\frac{\mu}{a}\right)^{\frac{1}{b}}, -\left(\frac{m}{a}\right)^{\frac{1}{b}}\right),$$

with $m > 0$ and $F_1(a; b_1, b_2; c; x, y)$ the Appell hypergeometric function of the first kind. Moreover, if $\mu = m$

$$I(\mu, b, a) = \int_a^\infty x \left[1 + \left(\frac{x}{\mu}\right)^{-\frac{1}{b}}\right]^b f(x; \mu, b)\, dx$$
$$= a\left(\frac{\mu}{a}\right)^{\frac{1}{b}} {}_2F_1\left(1 - b; 2 - 2b; 2 - b; -\left(\frac{\mu}{a}\right)^{\frac{1}{b}}\right),$$

where ${}_2F_1(\alpha, \beta; \gamma; x)$ is the hypergeometric function. Finally, if $\mu = m$ and $b = 1/2$

$$I(\mu, a) = \int_a^\infty x \left[1 + \left(\frac{x}{\mu}\right)^{-2}\right]^{\frac{1}{2}} f(x; \mu, 1/2)\, dx = \mu \tan^{-1}\left(\frac{\mu}{a}\right).$$

Symmetric compound married put: Conjugate-power Dagum asset prices

Thus the expectation in (6) is equal to

$$\mathbb{E}^{\mathbb{Q}}\left(MP_T(S, K_2, 2T)\mathbb{1}_{S_T \geq K_\star}\right) = S_0^{\frac{1}{b}}\left(K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^{b-1}$$
$$\times F_1\left(1-b; 2-b, -b; 2-b; \frac{S_0^{\frac{1}{b}}}{K_2^{\frac{1}{b}} - K_1^{\frac{1}{b}}}, \frac{K_2^{\frac{1}{b}}}{K_2^{\frac{1}{b}} - K_1^{\frac{1}{b}}}\right)$$

and the risk-neutral probability of the $S_T < K_\star$ is

$$\mathbb{Q}(S_T < K_\star) = \left(1 + \left(\frac{K_\star}{S_0}\right)^{-\frac{1}{b}}\right)^{b-1} = \left(K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^{1-b}\left(S_0^{\frac{1}{b}} + K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^{b-1}$$

The price of the symmetric compound married put is

$$MP_0\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T \\ 2T \end{pmatrix}\right) = S_0^{\frac{1}{b}}\left(K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^{b-1} \times$$
$$\times F_1\left(1-b; 2-b, -b; 2-b; \frac{S_0^{\frac{1}{b}}}{K_2^{\frac{1}{b}} - K_1^{\frac{1}{b}}}, \frac{K_2^{\frac{1}{b}}}{K_2^{\frac{1}{b}} - K_1^{\frac{1}{b}}}\right) \tag{7}$$
$$+ K_1\left(K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^{1-b}\left(S_0^{\frac{1}{b}} + K_1^{\frac{1}{b}} - K_2^{\frac{1}{b}}\right)^{b-1}.$$

Symmetric compound married put: Conjugate-power Dagum asset prices

If the symmetric compound married put is issued at-the-money, that is $S_0 = K_2 \leq K_1$, the pricing formula in (7) simplifies as

$$MP_0\left(S, \begin{pmatrix} K_1 \\ S_0 \end{pmatrix}, \begin{pmatrix} T \\ 2T \end{pmatrix}\right) = S_0^{\frac{1}{b}}\left(K_1^{\frac{1}{b}} - S_0^{\frac{1}{b}}\right)^{b-1} \times$$

$$\times\ {}_2F_1\left(2-2b, 1-b; 2-b; \frac{S_0^{\frac{1}{b}}}{S_0^{\frac{1}{b}} - K_1^{\frac{1}{b}}}\right) + K_1^{\frac{2b-1}{b}}\left(K_1^{\frac{1}{b}} - S_0^{\frac{1}{b}}\right)^{1-b}$$

If furthermore, $b = 1/2$, its price is elementary and equal to

$$MP_0\left(S, \begin{pmatrix} K_1 \\ S_0 \end{pmatrix}, \begin{pmatrix} T \\ 2T \end{pmatrix}\right)\Bigg|_{b=1/2} = S_0 \tan^{-1}\left(\frac{S_0}{\sqrt{K_1^2 - S_0^2}}\right) + \sqrt{K_1^2 - S_0^2}$$

There is no such a parametrization for the Black-Scholes model such that the price of the compound married put (or option) is elementary.

Compound married puts and compound options

Compound married puts can be linked to compound call-on-calls and put-on-calls.

For the call-on-call

$$
\begin{aligned}
MP_{T_1}\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right) &= \max\Big\{\max\{S, K_2\}, K_1\Big\} \\
&= \max\Big\{\max\{S, K_2\} - K_1, 0\Big\} + K_1 \\
&= \underbrace{\max\Big\{\max\{S - K_2, 0\} - (K_1 - K_2)\,, 0\Big\}}_{\substack{\text{compound call struck at } K_1 - K_2 \\ \text{on call struck at } K_2}} + K_1 \\
&= c_{T_1}\Big(c(S, K_2, T_2), K_1 - K_2, T_1\Big) + K_1.
\end{aligned}
$$

Compound married puts and compound options

For the put-on-call

$$
\begin{aligned}
MP_{T_1}\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right) &= \max\Big\{\max\{S, K_2\}, K_1\Big\} \\
&= \max\Big\{K_1 - \max\{S, K_2\}, 0\Big\} + \max\{S, K_2\} \\
&= \underbrace{\max\Big\{(K_1 - K_2) - \max\{S - K_2, 0\}, 0\Big\}}_{\substack{\text{compound put struck at } K_1 - K_2 \\ \text{on call struck at } K_2}} + \max\{S, K_2\} \\
&= p_{T_1}\Big(c(S, K_2, T_2), K_1 - K_2, T_1\Big) + MP_{T_1}(S, K_2, T_2)
\end{aligned}
$$

Thus, to ensure no arbitrage

$$
\begin{aligned}
MP_0\left(S, \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right) &= c_0\Big(c(S, K_2, T_2), K_1 - K_2, T_1\Big) + K_1 \\
&= p_0\Big(c(S, K_2, T_2), K_1 - K_2, T_1\Big) + MP_0(S, K_2, T_2)
\end{aligned}
$$

must hold.

SCUOLA NORMALE SUPERIORE

Whaley's decomposition

Letting $0 < D < K_2$ representing the discrete dividend, Whaley's decomposition of the American option price is

$$C_0(S, K_2, T_2) = \underbrace{c_0(S, K_\star, T_1)}_{\text{(a)}} + \underbrace{c_0(S, K_2, T_2)}_{\text{(b)}} - \underbrace{c_0(c(S, K_\star + D - K_2, T_1), K_2, T_2)}_{\text{(c)}} \tag{8}$$

(a) long a European call option with exercise price $K_\star$ (to be determined) and maturity T_1;

(b) long a European call option with exercise price K_2 and maturity T_2;

(c) short a compound call-on-call whose mother option is a European option with exercise price K_2 and maturity T_2 which is written on a European option with exercise price $K_\star + D - K_2$ and maturity T_1.

Roll (1977) shows that the value $K_\star$ is defined implicitly via

$$c_{T_1}(K_\star, K_2, T_2) = K_\star + D - K_2,$$

American call with known dividend: Conjugate-power Dagum asset prices

Let $T = T_1 = T_2/2$ (i.e. the dividend is paid *half-way* through the option maturity) and $b = b(T)$.

The value of $K_\star$ is found solving

$$\left(K_\star^{\frac{1}{b}} + K_2^{\frac{1}{b}}\right)^b - K_2 = K_\star + D - K_2$$

that is

$$(K_\star + D)^{\frac{1}{b}} - K_\star^{\frac{1}{b}} = K_2^{\frac{1}{b}}.$$

Letting $p = 1/b \in \mathbb{N} \setminus \{1\}$ and factoring the difference of powers

$$D \sum_{j=0}^{p-1} (K_\star + D)^{p-j-1} K_\star^j = K_2^p$$

If $b \in \{1/2, 1/3, 1/4, 1/5\}$ (or, equivalently, $p \in \{2, 3, 4, 5\}$), the Abel-Ruffini theorem ensures the existence of $K_\star$ elementary obtainable as a function of polynomials. For example,

$$K_\star = \frac{K_2^2}{2D} - \frac{D}{2}, \text{ for } b = 1/2 \qquad K_\star = \sqrt{\frac{4K_2^3 - D^3}{3D}} - \frac{D}{2}, \text{ for } b = 1/3$$

American call with known dividend: Conjugate-power Dagum asset prices

Rewrite (8) in terms of married put prices, that is

$$C_0\left(S, K_2, 2T\right) = MP_0\left(S, K_\star, T\right) - K_\star + MP_0\left(S, K_2, 2T\right) - K_2 - MP_0\left(S, \begin{pmatrix} K_\star + D \\ K_\star + D - K_2 \end{pmatrix}, \begin{pmatrix} T \\ 2T \end{pmatrix}\right) + \left(K_\star + D\right)$$

If $K_2 = S_0$ (American option is at-the-money) and $b = 1/2$ ($K_\star$ is elementary as well as the function for pricing the two-period married put), then the price becomes

$$C_0\left(S, S_0, 2T\right)\Big|_{b=1/2} = \sqrt{\left(\frac{S_0^2 - D^2}{2D}\right)^2 + S_0^2} + \frac{\pi}{2}S_0 - S_0 + D - MP_0\left(S, \begin{pmatrix} \frac{S_0^2+D^2}{2D} \\ \frac{S_0^2+D^2}{2D} - S_0 \end{pmatrix}, \begin{pmatrix} T \\ 2T \end{pmatrix}\right)\Bigg|_{b=1/2}.$$

To have an elementary compound married price, we need to further impose

$$\frac{S_0^2 + D^2}{2D} - S_0 = S_0 \quad \Longleftrightarrow \quad D^2 - 4S_0 D + S_0^2 = 0. \tag{9}$$

The solution of (9) such that early exercise is possible (i.e. $D \leq K_2 = S_0$) is $D_\star = (2 - \sqrt{3})S_0 \approx 0.27 S_0$.

American call with known dividend: Conjugate-power Dagum asset prices

Hence,

$$
\begin{aligned}
C_0\left(S, S_0, 2T\right)\Big|_{b=1/2}^{D=D_\star} &= \sqrt{\left(\frac{S_0^2 - D_\star^2}{2D_\star}\right)^2 + S_0^2} + \frac{\pi}{2}S_0 - S_0 + D_\star \\
&\quad - S_0 \tan^{-1}\left(S_0 \Big/ \sqrt{\left(\frac{S_0^2 + D_\star^2}{2D_\star}\right)^2 - S_0^2}\right) \\
&\quad - \sqrt{\left(\frac{S_0^2 + D_\star^2}{2D_\star}\right)^2 - S_0^2} \\
&= \sqrt{\left(\frac{S_0^2 - D_\star^2}{2D_\star}\right)^2 + S_0^2} - \sqrt{\left(\frac{S_0^2 + D_\star^2}{2D_\star}\right)^2 - S_0^2} \\
&\quad + S_0\left(\cot^{-1}\left(S_0 \Big/ \sqrt{\left(\frac{S_0^2 + D_\star^2}{2D_\star}\right)^2 - S_0^2}\right) - 1\right) + D_\star
\end{aligned}
$$

in which we use $\cot^{-1}(x) = \pi/2 - \tan^{-1}(x)$.

American call with known dividend: Conjugate-power Dagum asset prices

As

$$\sqrt{\left(\frac{S_0^2 - D_\star^2}{2D_\star}\right)^2 + S_0^2} - \sqrt{\left(\frac{S_0^2 + D_\star^2}{2D_\star}\right)^2 - S_0^2} = \left(2 - \sqrt{3}\right) S_0 = D_\star$$

and

$$\cot^{-1}\left(S_0 \Big/ \sqrt{\left(\frac{S_0^2 + D_\star^2}{2D_\star}\right)^2 - S_0^2}\right) = \cot^{-1}\left(\sqrt{\frac{1}{3}}\right) = \frac{\pi}{3}$$

the price of such American option is stunningly simple and equal to

$$\begin{aligned} C_0\left(S, S_0, 2T\right)\Big|_{b=1/2}^{D=D_\star} &= \left(\frac{\pi}{3} - 1\right) S_0 + 2D_\star \\ &= \left(3 + \frac{\pi}{3} - 2\sqrt{3}\right) S_0 \\ &\approx 0.58 S_0 \end{aligned}$$

which is **elementary**!

SCUOLA NORMALE SUPERIORE

Conclusions

- We show how to price symmetric compound married puts, call-on-calls and put-on-calls when price ratios are iid conjugate-power Dagum random variables.
- Such distribution is consistent with risk-neutral pricing and allows for negative skewness and mild excess kurtosis.
- We show that, if the compound married put is issued at-the-money and $b = 1/2$, its price is elementary; there is no such parametrization under log-normality (or other known distribution) for which such a result is obtained.
- We show how to price an American call written on an asset that pays a known discrete dividend under the conjugate-power Dagum distribution, thus providing an alternative to the well-known Roll-Geske-Whaley formula.
- We show that the pricing of such American option also reduces to an elementary function under a given parameter combination.

Thanks for your attention

Two-period married put under conjugate-power Dagum asset prices

Let $b = b(T)$, $X = R^G_{[0,T]}$, $Y = R^G_{[T,2T]}$ and $Z = XY = R^G_{[0,2T]}$. The risk-neutral CDF of Z can be found as

$$\begin{aligned} F^{\mathbb{Q}}_Z(z) &= \mathbb{Q}(Z \le z) = \mathbb{Q}(XY \le z) \\ &= \int_0^\infty \mathbb{Q}(XY \le z | X = x) f^{\mathbb{Q}}_X(x)\,\mathrm{d}x = \int_0^\infty \mathbb{Q}\left(Y \le \frac{z}{x}\right) f^{\mathbb{Q}}_X(x)\,\mathrm{d}x \\ &= \int_0^\infty \left[1 + \left(\frac{z}{x}\right)^{-\frac{1}{b}}\right]^{b-1} \frac{1-b}{b}\left(1 + x^{-\frac{1}{b}}\right)^{b-2} x^{-\frac{1}{b}-1}\,\mathrm{d}x \\ &= \frac{1-b}{b}\int_0^\infty \left(1 + z^{-\frac{1}{b}} x^{\frac{1}{b}}\right)^{b-1}\left(1 + x^{-\frac{1}{b}}\right)^{b-2} x^{-\frac{1}{b}-1}\,\mathrm{d}x \end{aligned}$$

It can be shown that (Gradshteyn and Ryzhik 2014)

$$\begin{aligned} F^{\mathbb{Q}}_Z(z) &= (1-b)\left[z^{\frac{1-b}{b}}\int_0^1 t^{1-b}\left(1 + z^{\frac{1}{b}} t\right)^{b-1}(1+t)^{b-2}\,\mathrm{d}t \right. \\ &\qquad \left. + \int_0^1 t^{-b}\left(1 + z^{-\frac{1}{b}} t\right)^{b-1}(1+t)^{b-2}\,\mathrm{d}t\right] \\ &= F_1\left(1-b, 1-b, 2-b, 2-b; -z^{-\frac{1}{b}}, -1\right) \\ &\qquad + \frac{1-b}{2-b} z^{\frac{1-b}{b}} F_1\left(2-b, 1-b, 2-b, 3-b; -z^{\frac{1}{b}}, -1\right) \end{aligned} \tag{10}$$

Two-period married put under conjugate-power Dagum asset prices

Provided that

$$X \overset{\widehat{\mathbb{Q}}}{\sim} Y \overset{\widehat{\mathbb{Q}}}{\sim} \mathrm{CPSM}\left(1, b(T)\right)$$

the distribution of Z under the spot measure is

$$\begin{aligned}
F_Z^{\widehat{\mathbb{Q}}}(z) &= \int_0^\infty F_Y^{\widehat{\mathbb{Q}}}\left(\frac{z}{x}\right) f_X^{\widehat{\mathbb{Q}}}(x)\,\mathrm{d}x \\
&= \int_0^\infty \left\{1 - \left[1 + \left(\frac{z}{x}\right)^{\frac{1}{b}}\right]^{b-1}\right\} \frac{1-b}{b}\left(1 + x^{\frac{1}{b}}\right)^{b-2} x^{\frac{1}{b}-1}\,\mathrm{d}x \\
&= \frac{1-b}{b}\left(\int_0^\infty \left(1 + x^{\frac{1}{b}}\right)^{b-2} x^{\frac{1}{b}-1}\,\mathrm{d}x \right. \\
&\qquad \left. - \int_0^\infty \left[1 + \left(\frac{z}{x}\right)^{\frac{1}{b}}\right]^{b-1} \left(1 + x^{\frac{1}{b}}\right)^{b-2} x^{\frac{1}{b}-1}\,\mathrm{d}x\right)
\end{aligned}$$

and that the survival function is equal to

$$\begin{aligned}
\overline{F}_Z^{\widehat{\mathbb{Q}}}(z) &= F_1\left(1-b, 1-b, 2-b, 2-b; -z^{\frac{1}{b}}, -1\right) \\
&\quad + \frac{1-b}{2-b} z^{-\frac{1-b}{b}} F_1\left(2-b, 1-b, 2-b, 3-b; -z^{-\frac{1}{b}}, -1\right)
\end{aligned} \tag{11}$$

Two-period married put under conjugate-power Dagum asset prices

Given the results in (10) and (11), it is possible to price a two-period married put as

$$MP_0(S,K,2T) = S_0\widehat{\mathbb{Q}}(S_{2T}\geq K) + K\mathbb{Q}(S_{2T}<K) = S_0\overline{F}_Z^{\widehat{\mathbb{Q}}}\left(\frac{K}{S_0}\right) + KF_Z^{\mathbb{Q}}\left(\frac{K}{S_0}\right)$$

$$= S_0\left[F_1\left(1-b,1-b,2-b,2-b;-\left(\frac{K}{S_0}\right)^{\frac{1}{b}},-1\right)\right.$$

$$\left.+\frac{1-b}{2-b}\left(\frac{K}{S_0}\right)^{-\frac{1-b}{b}}F_1\left(2-b,1-b,2-b,3-b;-\left(\frac{K}{S_0}\right)^{-\frac{1}{b}},-1\right)\right]$$

$$+K\left[F_1\left(1-b,1-b,2-b,2-b;-\left(\frac{K}{S_0}\right)^{-\frac{1}{b}},-1\right)\right.$$

$$\left.+\frac{1-b}{2-b}\left(\frac{K}{S_0}\right)^{\frac{1-b}{b}}F_1\left(2-b,1-b,2-b,3-b;-\left(\frac{K}{S_0}\right)^{\frac{1}{b}},-1\right)\right]$$

A very interesting result is the price of the at-the-money two-period married put with $b=1/2$, that is

$$MP_0(S,S_0,2T)\big|_{b=1/2} = S_0\left(\overline{F}_Z^{\widehat{\mathbb{Q}}}(1)\big|_{b=1/2} + F_Z^{\mathbb{Q}}(1)\Big|_{b=1/2}\right) = \frac{\pi}{2}S_0$$

SCUOLA
NORMALE
SUPERIORE

Barone, G. (2013), 'European compound options written on perpetual american options', *Journal of Derivatives* **20**(3), 61–74.

Buraschi, A. and Dumas, B. (2001), 'The forward valuation of compound options', *Journal of Derivatives* **9**(1), 8–17.

Carr, P. (1988), 'The valuation of sequential exchange opportunities', *Journal of Finance* **43**(5), 1235–1256.

Carr, P. and Torricelli, L. (2021), 'Additive logistic processes in option pricing', *Finance and Stochastics* **25**, 689–724.

Geske, R. (1977), 'The valuation of corporate liabilities as compound options', *Journal of Financial and Quantitative Analysis* **12**(4), 541–552.

Geske, R. (1979*a*), 'A note on an analytical valuation formula for unprotected american call options on stocks with known dividends', *Journal of Financial Economics* **7**(4), 375–380.

Geske, R. (1979*b*), 'The valuation of compound options', *Journal of Financial Economics* **7**(1), 63–81.

Geske, R., Subrahmanyam, A. and Zhou, Y. (2016), 'Capital structure effects on the prices of equity call options', *Journal of Financial Economics* **36**(6), 1639–1652.

Gradshteyn, I. S. and Ryzhik, I. M. (2014), *Table of Integrals, Series, and Products*, 8th edn, Elsevier, Amsterdam.

SCUOLA
NORMALE
SUPERIORE

Hull, J. C., Nelken, I. and White, A. D. (2004), 'Merton's model, credit risk and volatility skews', *Journal of Credit Risk* **1**(1), 3–28.

Prekopa, A. and Szantai, T. (2010), 'On the analytical-numerical valuation of the bermudan and american options', *Quantitative Finance* **10**(1), 59–74.

Roll, R. (1977), 'An analytic valuation formula for unprotected american call options on stocks with known dividends', *Journal of Financial Economics* **5**(2), 251–258.

Toft, K. B. and Prucyk, B. (1997), 'Options on leveraged equity: Theory and empirical tests', *Journal of Finance* **53**(3), 1151–1180.

Whaley, R. E. (1981), 'On the valuation of american call options on stocks with known dividends', *Journal of Financial Economics* **9**(2), 207–211.