Simpler Option Pricing

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“Thank you, Peter Carr.”

ברוך דיין האמת.

- “While working on my class and the book last summer, I discovered it was very difficult to teach option pricing to master's students who had never seen stochastic calculus before, and developed a new derivation that requires only college-level calculus.
- “In disbelief, Peter asked me to prove it to him. I stood at the whiteboard outside his office and explained it... When I finished, he asked me to stop and held up his hand. After about a half-minute, he broke out in laughter, with a tear in his eye. How can it have been so easy? Why didn't we see this before?
- “The proof remained unerased from his whiteboard for several weeks. He offered guidance and urged me to publish it immediately.”
- The paper will appear this summer (2022) in the **Journal of Derivatives**.

Peter Carr (left) with David Shimko and Maggie Copeland, 1987.

Review: Difficult Option Pricing

Black-Scholes-Merton Proof	Comments
$dS = \alpha S\, dt + \sigma S\, dW$	Stochastic differential eq
$C = C(S,\tau)$	
$dC = (\alpha C_s S + 0.5\sigma^2 C_{SS} S^2 - C_\tau)\, dt + \sigma C_S S\, dW$	Itô's Lemma
$P = C - hS$	
$dP = dC - h\, dS$	Self-financing
$dP = (\alpha C_s S + 0.5\sigma^2 C_{SS} S^2 - C_\tau - h\alpha S)\, dt + (\sigma C_S S - h\sigma S)\, dW$	
$h = C_S$	Hedge ratio eliminates risk
$rP\, dt = (\alpha C_s S + 0.5\sigma^2 C_{SS} S^2 - C_\tau - \alpha C_S S)\, dt$	Risk premium vanishes
$r(C - C_S S) = 0.5\sigma^2 C_{SS} S^2 - C_\tau$	No arbitrage
$C(S,0) = Max(S - X, 0)$	Solve PDE with boundary
$C(S,\tau) = SN(d_1) - Xe^{-rT} N(d_2)$	European call value

$h = h(S,\tau)$

Preamble

European call options can be priced in equilibrium *without*

- stochastic calculus, complete markets, the self-financing condition, differential equations, tailored utility functions or explicit integration
- Only need the Market Model or the CAPM.

Academics tried this in the past, with disastrous results

- Negative and non-convex option prices
- Free lunch for traders! (arbitrage profits)
- This occurred because the asset valuation formulae were applied directly to derivatives without any adjustment

The problem is solved by imposing a no-arbitrage condition in an equilibrium model

- *...Before computing the equilibrium*

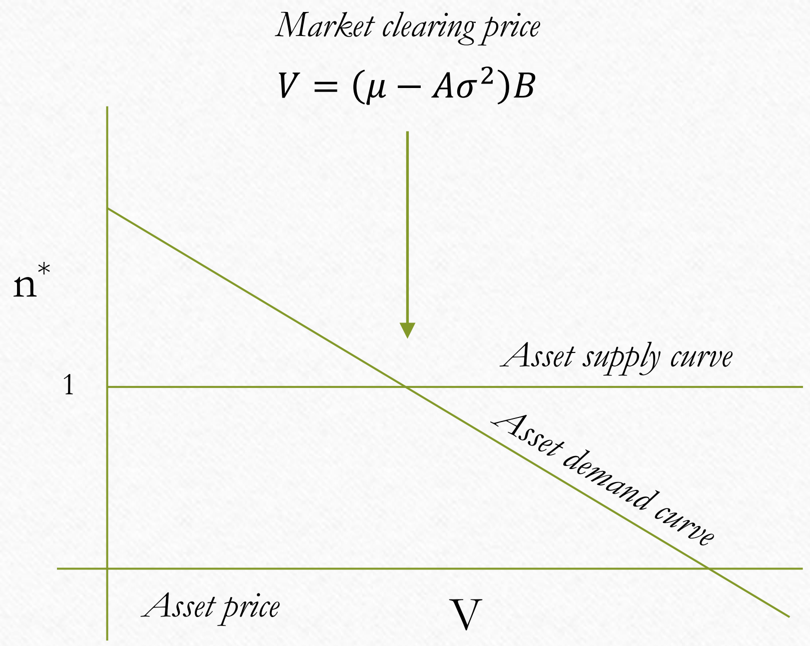
Question:
"So, is this an equilibrium model or an arbitrage model?"

Response:
"*Every* equilibrium model should preclude arbitrage, since its presence would necessarily upend any purported equilibrium."

One asset version

Flashback: The Lintner-Mossin CAPM (1965-66)

- The value of a single risky asset in 1 period is an exogenous random variable "S"
  - Mean μ, Variance σ^2
- The market must fully finance 100% of the asset
  - n = 1, the "supply curve",
  - i.e. 100% of the asset offering
- Unconstrained investors maximize risk-adjusted added value for any given asset price, i.e. maximize over choice of n
  - Return from risk-free asset
  - + Expected risk-adjusted return from risky asset



L-M notation and single-asset derivation

- W = Initial wealth
- V = Asset price today
- B = PV of riskless ZCB paying $1
- A = Risk aversion parameter,
  - or the cost of variance risk

$$\max_{n} \frac{W - nV}{B} + n\mu - 0.5An^2\sigma^2$$

(RF inv + exp risky asset – risk charge)

$$n^* = \frac{\mu - V/B}{A\sigma^2} = 1$$

$$V = (\mu - A\sigma^2)B$$

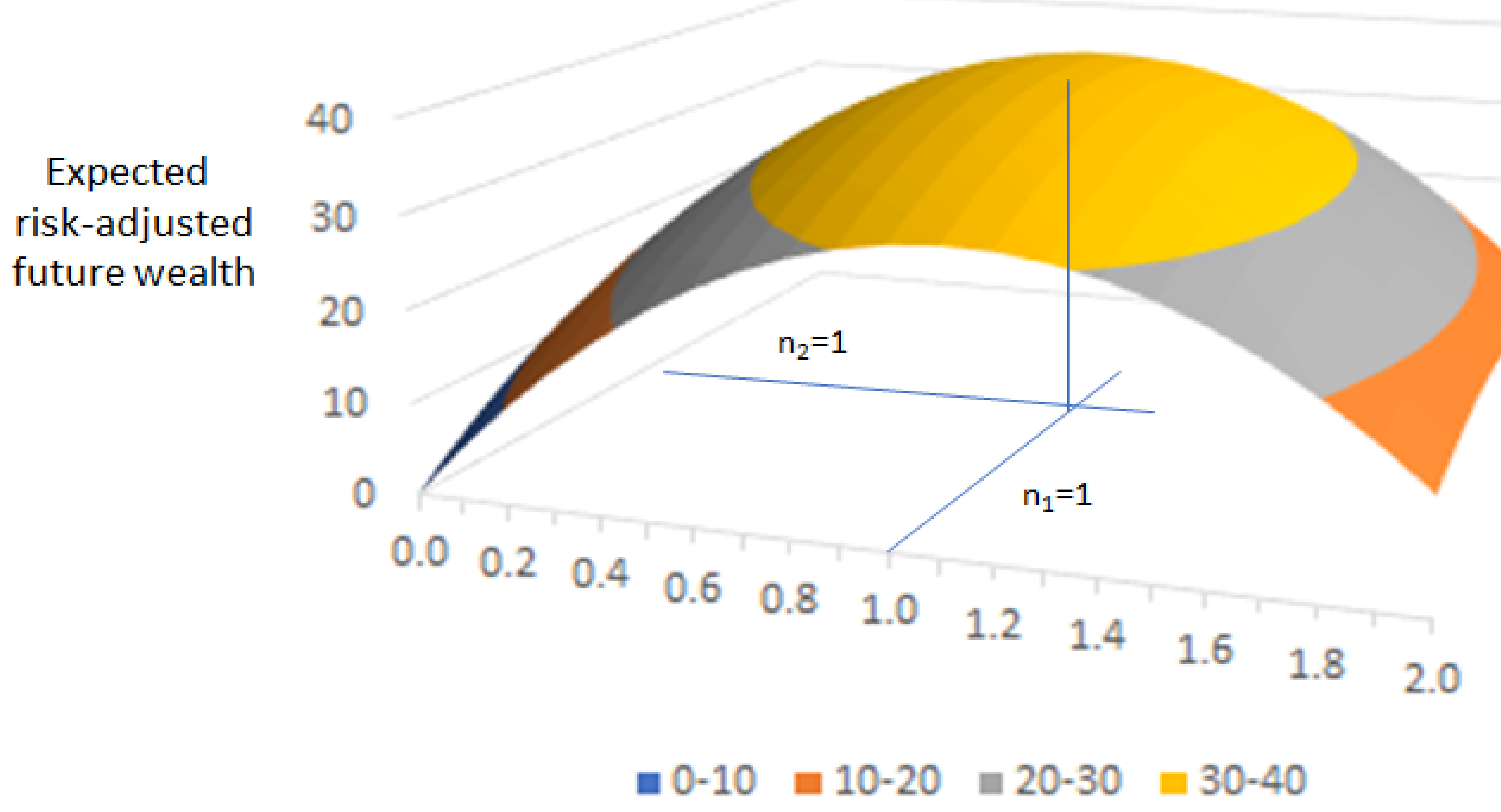
Two asset version

Lintner-Mossin with two risky assets

- The parameters for the second asset are
  - Future value C in one period
  - Mean μ_C , Variance σ_C^2
  - Covariance σ_{SC} , Dollar Beta $\beta = \sigma_{SC}/\sigma^2$
  - Present Value V_C , Position n_C
- The market must fully finance 100% of each asset
  - $n = n_C = 1$, the “supply curves”,
- Unconstrained investors maximize risk-adjusted added value for any given asset price, i.e. maximize over choice of n and n_C

CAPM Equilibrium - 2 assets

Select asset prices so that unconstrained investors choose $n_1 = n_2 = 1$



Elliptic paraboloid
Constant Hessian matrix
First term unambiguously negative
Positive determinant
→ Unique global maximum

L-M two-asset equilibrium valuations

- Investors perform an unconstrained maximization
- Pricing is set so that markets clear
- This produces equilibrium valuations *and a relative pricing formula for asset #2*
- *Note:*
  - *$n = n_C = 1$ in the two-asset case*

$$\max_{n,n_C} \; n\mu + n_C\mu_C + (W - nV - n_C V_C)/B - \tfrac{1}{2}A(n^2\sigma^2 + 2nn_C\sigma_{SC} + n_C^2\sigma_C^2)$$

Solution

$$V/B = \mu - A(n\sigma^2 + n_C\sigma_{SC})$$

$$V_C/B = \mu_C - A(n\sigma_{SC} + n_C\sigma_C^2)$$

$$\boldsymbol{V_C = V\beta + B(\mu_C - \beta\mu)}$$

$$\boldsymbol{\beta = \frac{n\sigma_{SC} + n_C\sigma_C^2}{n\sigma^2 + n_C\sigma_{SC}}}$$

Transitional summary

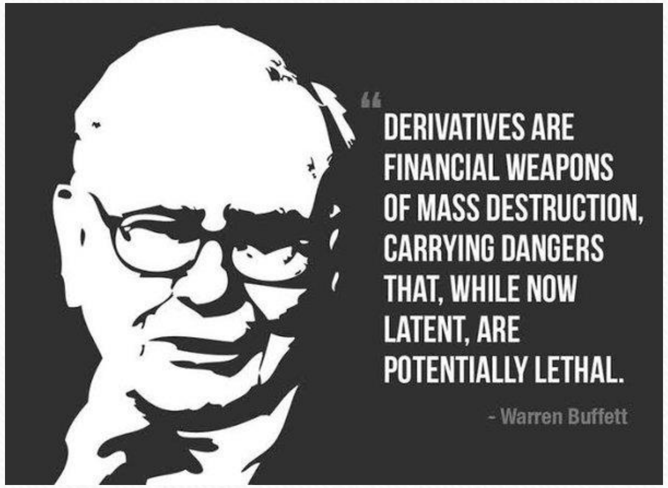
- So far we have only applied the Lintner-Mossin CAPM
    - Nothing new to this paper except for emphasis and interpretation
- We have a relative pricing model of asset "C" relative to asset "S" with value "V" that eliminates the risk aversion parameter:

$$V_C = V\beta + B(\mu_C - \beta\mu) \qquad \beta = \frac{n\sigma_{SC} + n_C\sigma_C^2}{n\sigma^2 + n_C\sigma_{SC}} \qquad n = n_C = 1$$

- This is a risk-neutral model (for assets) since it does not depend on A.
- Now we would like to know if we can price derivatives with the asset-based CAPM.

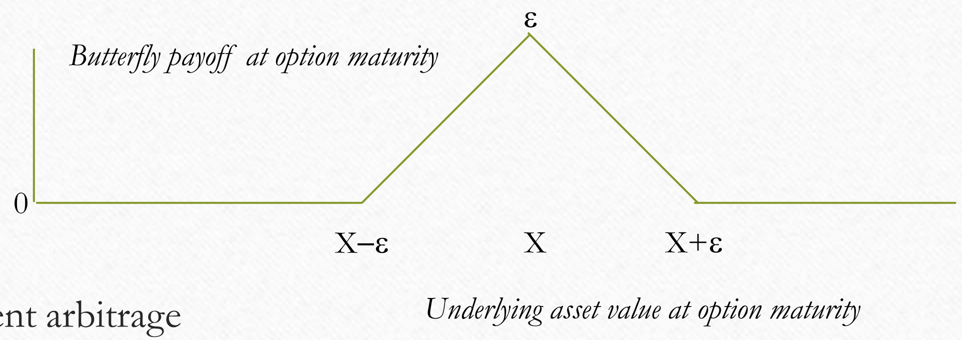
What if asset C were a call option on asset S?

- In this case, asset C is a *derivative*
- It exists in zero net supply, since there must be a buyer for every seller
- Therefore, n=1 and $n_C=0$ in equilibrium.
- The CAPM pricing therefore simplifies to
  - $V_C = V\beta + B(\mu_C - \beta\mu)$ $\beta = \frac{\sigma_{SC}}{\sigma^2}$
  - β can be called the static hedge ratio



What is option *convexity*?

- This means convexity of the call option pricing function with respect to the option strike price X
- Computed as the second derivative, or $\frac{\partial^2 V_C}{\partial X^2} = \lim_{\varepsilon \to 0} \frac{V(X+\varepsilon) - 2V(X) + V(X-\varepsilon)}{\varepsilon^2}$
- The numerator of the limit term also describes an option "butterfly" position composed of options on the same underlying and maturity
  - Buy a call option at strike X+ε
  - Sell two call options at strike X
  - Buy a call option at strike X–ε
- This strategy always has a non-negative payoff
- Therefore, its current value must be non-negative for any X to prevent arbitrage



Special Case

The Bachelier European Call

One period, discrete discounting, no dividends, normally distributed asset price

These are properties of the truncated normal distribution (X is the strike price, S is the future asset price):

$$\mu_C = E[\max(S - X, 0)] = (\mu - X)N(d) + \sigma\, n(d) \qquad d = \frac{\mu - X}{\sigma}$$

$$cov(S, \max(S - X, 0)) = \sigma^2 N(d) \qquad \beta = N(d)$$

Stein's Lemma

$$V_C = (V - XB)N(d) + \sigma\, n(d)\, B$$

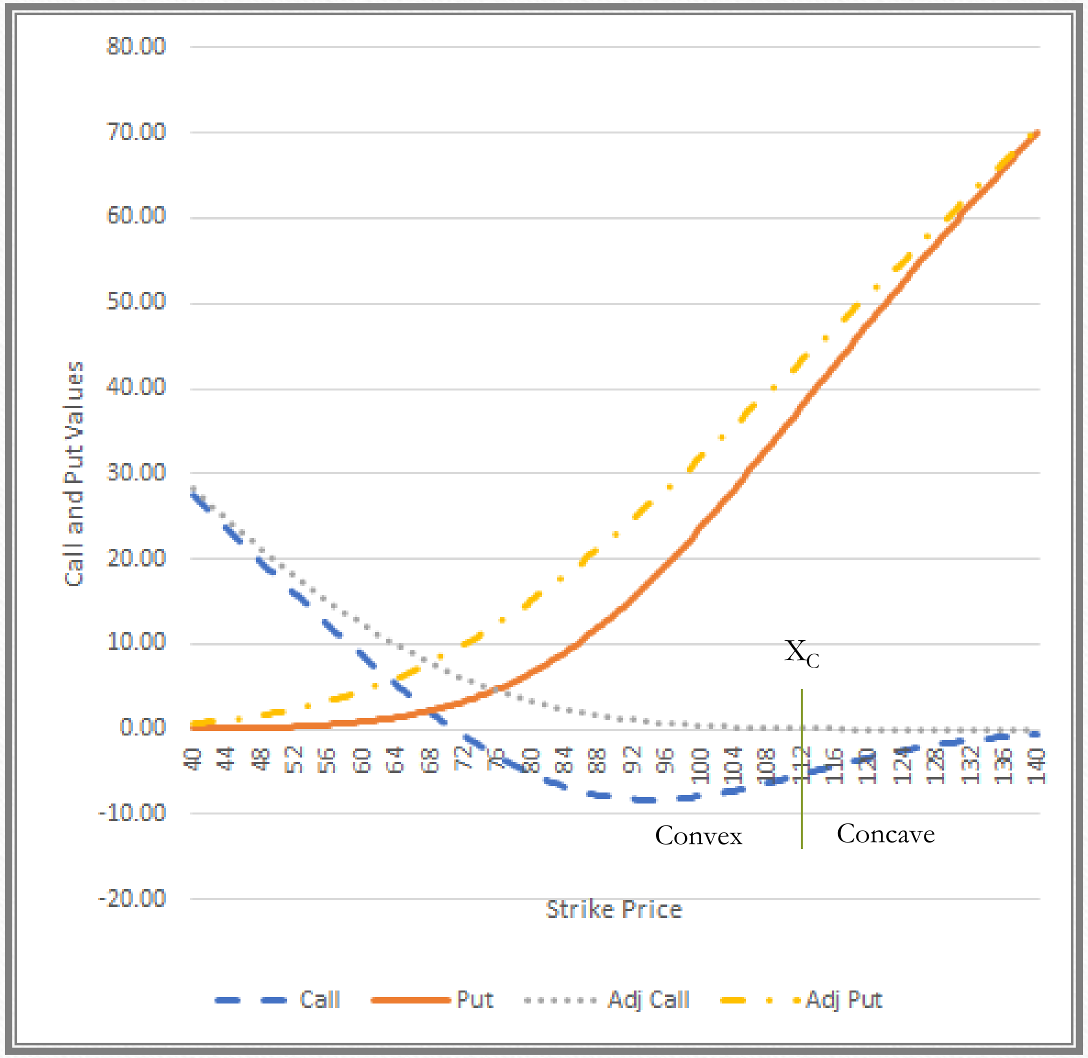
This resembles Bachelier except that d is a function of μ

NOT CONVEX

First Try Fails

- Options have negative values at some strike prices
- Option prices are not convex with respect to the exercise price for $X>X_C$
- This implies butterfly spreads have negative value when $X>X_C$

Bachelier Parameters	
Mean	100
Sigma	20
r	0.02
A	0.08



Back to the General Case

How to fix this

- **Add a convexity constraint to the unconstrained CAPM optimization**
- Use the Lagrangean method to solve

$$\max_{n,n_C,\gamma} \mathcal{L} = n\mu + n_C\mu_C + (W - nV - n_C V_C)/B - \frac{1}{2}A(n^2\sigma^2 + 2nn_C\sigma_{SC} + n_C^2\sigma_C^2) - \gamma g$$

$$s.t.\ \frac{\partial^2 V_C}{\partial X^2} \equiv g(n, n_C|X) \geq 0 \quad \forall X$$

$$\partial\mathcal{L}/\partial n = 0 \quad \rightarrow \quad \mu - V/B - A(n\sigma^2 + n_C\sigma_{SC}) - \gamma g_1' = 0$$

$$\partial\mathcal{L}/\partial n_C = 0 \quad \rightarrow \quad \mu_C - V_C/B - A(n\sigma_{SC} + n_C\sigma_C^2) - \gamma g_2' = 0$$

$$\partial\mathcal{L}/\partial\gamma = 0 \quad \rightarrow \quad g(n, n_C) > 0 \quad \forall X \ \ with\ \gamma = 0$$

$$or \quad g(n, n_C) = 0 \quad \forall X \ \ with\ \gamma > 0$$

The second derivative "g" must be strictly positive on probability distributions with compact support.

Interpreting the Lagrangean solution

$$V_C = V\beta + B(\mu_C - \beta\mu) = B\mu_C + \beta(V - B\mu)\ ; \quad \beta = \frac{\sigma_{SC}}{\sigma^2}$$

$$\frac{\partial^2 V_C}{\partial X^2} = (V - B\mu)\frac{\partial^2 \beta}{\partial X^2} + B\frac{\partial^2 \mu_C}{\partial X^2}$$

$$\frac{\partial^2 V_C}{\partial X^2} = \left[\frac{1}{\sigma^2}(V - B\mu)(X - \mu) + B\right] f(S|S = X)$$

- There is almost always a strike price X_C such that the second derivative is negative for $X>X_C$.

$$X_C = \mu + \frac{B\sigma^2}{\mu B - V}$$

- Arbitrage exists if X_C exists.
- X_C is undefined **only if** the denominator is zero.
- $\mu B - V$ must equal zero in the option pricing formula V_C.

Finally. Risk-neutral pricing for options.

- To find the price of a European option knowing its terminal probability distribution…
  - Compute the natural mean of its payoff as a function of the mean of the underlying asset
  - Whenever the mean of the underlying asset appears, replace it with the forward price of the underlying asset in the option pricing formula
  - Discount the result at the riskless rate

$$V_C = B\mu_C|_{\mu \leftarrow V/B} \equiv B\mu_C^*$$

The "self-financing" condition in discrete time?

- Let's try the "self-financing" condition even though it seems not to apply
  - $\partial V_C = \beta\, \partial V - (\mu_C - \beta\mu)\, \partial B$
- This means the remaining terms in the full derivative must be 0
  - $V\, \partial\beta + B\, \partial(\mu_C - \beta\mu) = 0$
- Adding dμ to the mix, we can derive
  - $\frac{\partial V_C}{\partial\mu} = V\frac{\partial\beta}{\partial\mu} + B\left(\frac{\partial\mu_C}{\partial\mu} - \mu\frac{\partial\beta}{\partial\mu} - \beta\right) = 0$
- For this to be zero, we need
  - $V_C = B\mu_C^*$
  - $\beta^* = \frac{\partial V_C}{\partial V} = \frac{\partial\mu_C}{\partial\mu}|_{\mu=\mu^*}$
- The "self-financing" condition can therefore be used to derive risk-neutral pricing in this model
  - Done by forcing $\partial V_C/\partial\mu = 0$
  - No convexity constraint needed if self-financing is imposed
  - Self-financing immediately yields an additional beta-delta equivalence property

Conclusions Proved

- All of the following imply risk-neutral static option pricing
  - The Market Model or CAPM re-optimized with a no-arbitrage convexity constraint
  - The self-financing condition
    - Must be reinterpreted for static models
  - Setting the derivative of the incorrect option pricing formula with respect to μ equal to zero.

Contributions of this Paper

- Pedagogical
  - Allows undergraduates to learn option derivations
  - Shows the link between no-arbitrage and option pricing in a static setting.
  - Establishes a path to "convergence" of asset pricing and derivative pricing
  - Reconciles equilibrium and arbitrage models in a broadly applicable way
  - Illustrates the power of the self-financing condition in the BSM model
  - Fills a gap in the classical finance literature.
- Option applications
  - Eliminates need to specify stochastic processes for European options
  - Broadens applicability of option pricing formulae, e.g. illiquid options and underlying assets
- Portfolio Theory: Future Research
  - Brings co-determined options into portfolio analysis, leading to better hedge ratios
  - Allows explicit incorporation of corporate limited liability and bankruptcy options into portfolio analysis

Simplified Option Price Derivations

David C. Shimko, **Journal of Derivatives**, Forthcoming Summer 2022

ACADEMIC ABSTRACT

Prior academic research reveals that mean-variance asset pricing (MVAP) models such as the single-period CAPM fail to produce rational European option prices. We show two adaptations of MVAP models that may be used to value derivatives with nonlinear payouts. The first removes static option arbitrage in investors' optimized aggregate portfolio selection. The second linearizes the pricing kernel using a static version of the self-financing condition applied in dynamic option modeling. Both adaptations produce risk-neutral derivative prices in equilibrium for all finite-moment probability distributions of underlying asset prices with compact support. The derivation does not require stochastic calculus, frictionless continuous trading assumptions or the solution of differential equations. The resulting model is a hybrid of equilibrium and arbitrage techniques that values assets and derivatives rationally.

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