W-shaped Smiles and the Gaussian Mixture Model

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(joint work with Paul Glasserman)

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Outline

- Motivation: Event-driven implied volatility shapes
- Constraints on the implied volatility shapes through level crossings
  - Upper bound from risk-neutral density crossings with a log-normal
  - Lower bound from moment equalities
- $N-$Gaussian mixture model
- Summary and conclusions

P. Glasserman, D. Pirjol, W-shaped Implied Volatility Curves and the Gaussian Mixture Model, 2021

Options and implied volatility

Option prices are parameterized in terms of an implied *volatility surface* $\Sigma(K, T; t) \to$ volatility which reproduces the market option price when substituted into the Black-Scholes option formula

$$C_{\rm mkt} = C_{\rm BS}(K, T; \Sigma(K, T))$$

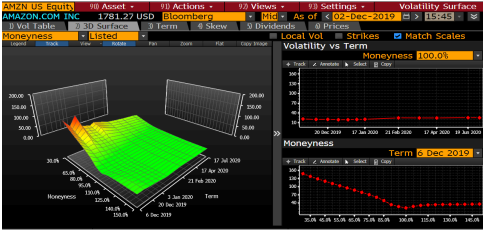

Figure: AMZN volatility surface as of 2 Dec 2019.

Source: *Bloomberg.com*

ATM concave smiles

Typically the implied volatility curve is U-shaped and convex at the at-the-money point. Under special circumstances it may become concave.

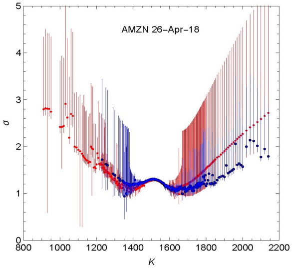

Figure: The implied volatility for AMZN options with maturity 27-Apr-2018 as of 26-Apr-2018.

1-day maturity options just before earnings announcement.

Source: *OptionMetrics*

Examples from *VolaDynamics*

Concave smiles appear for many names.

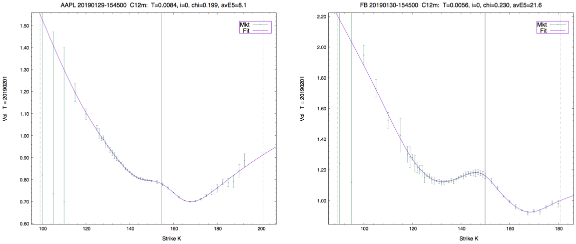

Figure: AAPL and FB just before earnings release.

Source: *VolaDynamics.com*

Prevalence of ATM concavity near earnings announcements (Alexiou et al 2021)

Figure 3: Fraction of concave IV curves around EAD

This figure shows the fraction of firms exhibiting a concave IV curve on each trading day from d-5 to d+5, where d is the quarterly EAD. The definition of a concave IV curve is provided in Section 2.2. IV curves are computed for the 100 firms with the highest option trading activity per year during the period 2013-2019.

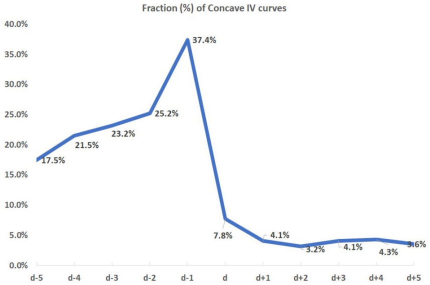

Figure: "... implied volatility curves [...] frequently become concave prior to earnings announcement day, reflecting a *bimodal risk-neutral distribution* for the underlying price. "

"What do index options teach us about COVID-19?" (Jackwerth 2020)

Argues that S& P 500 risk-neutral density became bimodal on 16-Mar-2020.

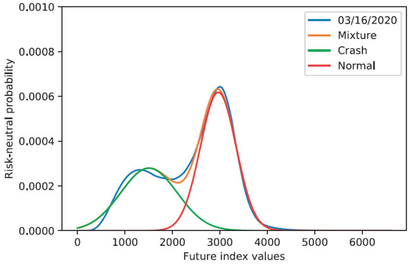

Figure 5
Risk-neutral distribution and mixture of S&P 500 index options with December 18, 2020, maturities
The figure shows the risk-neutral distribution on March 16, 2020, and a fitted mixture of normal distributions. Also depicted are the two components of that mixture, Crash and Normal.

Why W-shaped smiles?

- Anecdotally, W-shaped smile $\Rightarrow$ bimodal risk-neutral density (RND)
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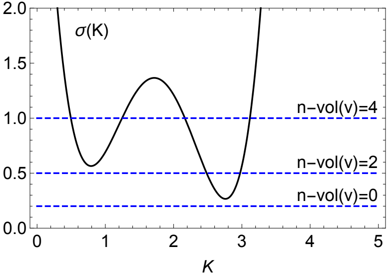
  - More generally: what can be said about the shape of the implied volatility in the Gaussian mixture model?
    This is the simplest model which produces RND with multiple modes.

The shape of the smile through its level crossings

We propose to study the shape of the smile through its level crossings.

Definition

Denote $n_{vol}(v)$ the number of times the implied volatility $\sigma(K)$ crosses the level v.



We derive upper and lower bounds on $n_{vol}(v)$ in terms of the risk-neutral distribution of the asset price S.

Risk neutral distribution (RND) and the implied volatility

- Fix maturity T

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- Log-normal random variable with volatility v and same mean F

$$X_v = F e^{vN(0,1) - \frac{1}{2}v^2}$$

- $n_{vol}(v)$ = number of times the implied vol for S crosses v
- $n_{pdf}(v)$ = number of times the density of S crosses the density of X_v

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Proposition (Upper bound on $n_{vol}(v)$)

Assume that the risk-neutral density of S differs from that of X_v on an interval. Then $n_{vol}(v) \leq n_{pdf}(v) - 2$

Simple consequences

- Call implied volatility function $\sigma_{BS}(\cdot)$ U-shaped if σ'_{BS} has a single sign change (-,+) or (+,-)
- Call $\sigma_{BS}(\cdot)$ W-shaped if it has 3 sign changes (-,+,-,+)

Corollary

- *If $n_{pdf}(v) = 2$, then the implied volatility does not cross the level v*
- *If $n_{pdf}(v) = 3$, then the implied volatility crosses v at at most one strike.*
- *If $\sigma_{BS}(K)$ is U-shaped, then $n_{pdf}(v) \geq 4$ for all v in some interval.*
- *If σ_{BS} is W-shaped, and the limits of $\sigma_{BS}(K)$ agree as $K \to 0$ and $K \to \infty$, then $n_{pdf}(v) \geq 6$ for all v in some interval.*

Lower bound on implied volatility crossings

Proposition

Suppose S and X_v have m matching moments

$$\mathbb{E}[S^{\alpha_i}] = \mathbb{E}[X_v^{\alpha_i}]\,, \quad i = 1, \dots, m$$

$1 < \alpha_1 < \cdots < \alpha_m$, *and the densities of S and X_v differ on some interval. Then the implied volatility $\sigma_{BS}(K)$ under S crosses level v at least m times: $n_{vol}(v) \geq m$.*

- Could replace moments with any strictly convex, linearly independent functions
- Argument uses the Carr-Madan formula, and the Karlin-Studden (1965) theory of Tchebycheff systems

Proof for $m = 1$

Assume that $\mathbb{E}[\phi(S)] = \mathbb{E}[\phi(X_v)]$ for some convex payoff ϕ, e.g. $\phi(x) = x^2$. The lower bound implies that $\sigma_{BS}(K)$ crosses v at least once.

Proof.

The function $h_v(K) := \mathbb{E}[(S-K)^+] - \mathbb{E}[(X_v - K)^+]$ has zeros at the points where the smile crosses the level v.
By Carr-Madan decomposition

$$\mathbb{E}[\phi(S)] = \phi(F) + \int_0^F \phi''(K)\mathbb{E}[(K-S)^+]dK + \int_F^\infty \phi''(K)\mathbb{E}[(S-K)^+]dK$$

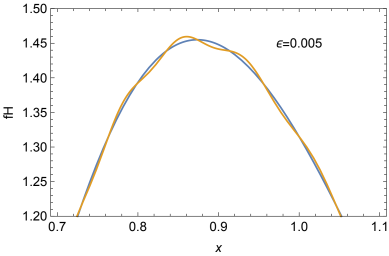
and similar for $\mathbb{E}[\phi(X_v)]$. Taking differences gives

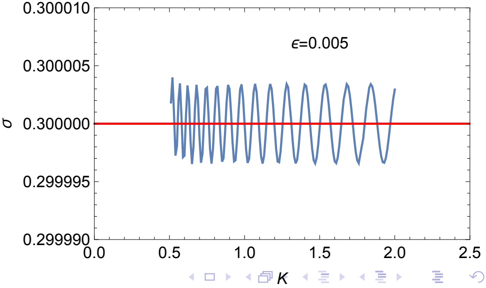
$$\mathbb{E}[\phi(S)] - \mathbb{E}[\phi(X_v)] = \int_0^\infty \phi''(K)h_v(K)dK = 0$$

Since $\phi''(K) > 0$, this can vanish only if $h_v(K)$ has at least one zero. □

WWW...W-shaped smile, unimodal distribution

- The log-normal distribution is not determined by its moments
- Heyde (1963) constructed a family of densities that have all the same integer moments as a log-normal distribution
- Assuming that S has a Heyde-type RND with same moments as X_v, the lower bound implies $n_{vol}(v) = \infty$
- Density can be chosen to be unimodal.





Summary of results so far

- The number of crossings of the smile with the level v is bounded from above by the number of density crossings of the RND of S with a lognormal X_v
- W-smiles require a risk-neutral density which crosses a log-normal at least 6 times.
- The number of crossings of the smile with the level v is bounded from below by the number of independent convex payoffs (e.g. moments) which are priced identically by S and X_v
- The modality of the RND is not a good predictor of smile shape. In particular, a unimodal RND can produce a smile crossing a level infinitely many times.

Some control over the number of the density crossings of S and X_v is possible in parametric models for the RND $\Rightarrow$ Gaussian mixture model

Gaussian Mixture Model (GMM)

- General properties of the smile in the N-components GMM
  - Upper bound on n_{vol} depends only on N
  - Lower bound on n_{vol} gives predictions for the range of the implied vol
- Tail asymptotics of the GMM
  - The extreme strike asymptotics of the implied vol depends only on v_{max}
  - The approach to the asymptotic limit depends on the location of the mixture component with maximal vol
- Dependence on mixture parameters
- Application: Classification of smile shapes in the $N = 3$ GMM

Gaussian Mixture Model

- $\log S$ is distributed as a mixture of N Gaussians

$$\log S \sim N(\log \mu_i - \frac{1}{2} v_i^2, v_i^2) \quad i = 1, \dots, N$$

  with weights $w_1, w_2, \dots, w_N$

- Forward price constraint $\sum_{i=1}^N w_i \mu_i = F$
- Option prices are convex combinations of Black-Scholes prices

$$\mathbb{E}[(S-K)^+] = \sum_{i=1}^N w_i C_{BS}(K; \mu_i, v_i)$$

Popular in financial practice, both as a static model for vol surfaces, and as the basis for a local vol model - Brigo, Mercurio (2002)

What shapes can the implied volatility take? How are they related to the mixture parameters?

Gaussian Mixture Model

Proposition

In the N-component Gaussian mixture model with distinct volatilities $v_i \neq v_j$ *for* $i \neq j$, $n_{vol}(v) \leq 2N-2$, *for all* $v > 0$.

The proof combines the model-independent upper bound $n_{vol}(v) \leq n_{pdf} - 2$ with results of Hummel, Gidas (1984) and Kalai, Moitra, Valiant (2010) on zeros of linear combinations of Gaussians.

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Corollary

A 2-component Gaussian mixture cannot produce a W-shaped smile. (The implied volatility crosses any level at most twice.)

In order to produce a W-shaped smile, the risk-neutral distribution must allow at least three components: up/middle/high

What about $N = 3$ components?

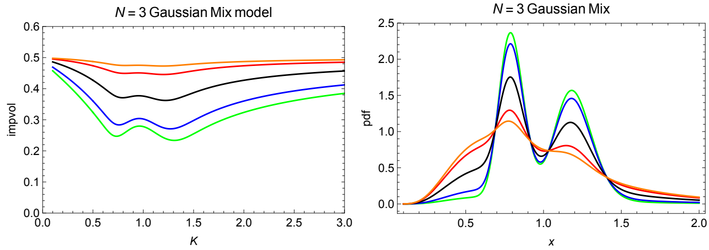


Figure: Implied volatility (left) and S density (right) in the $N = 3$ Gaussian mixture model for location parameters $\mu_i = (0.8, 1.0, 1.2)$ and volatilities $v_i = (0.1, 0.5, 0.1)$. The 5 curves shown are obtained by varying the weight of the middle component $w_2 = 0.1$ (green), $w_2 = 0.2$ (blue), $w_2 = 0.5$ (black), $w_2 = 0.8$ (red) and $w_2 = 0.9$ (orange).

Unimodal density with W-shaped volatility

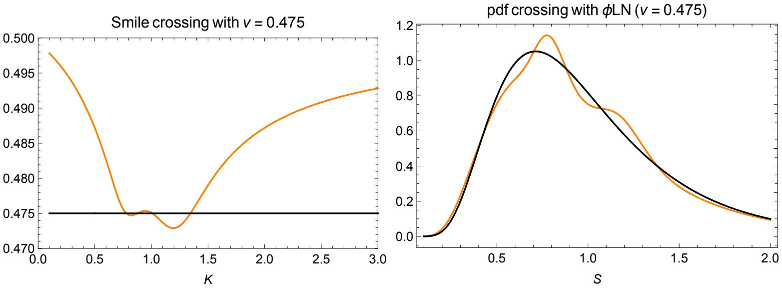


Figure: Unimodal S distribution producing a W-shaped smile in the $N = 3$ Gaussian mixture model. Model parameters: $\mu_i = (0.8, 1.0, 1.2), v_i = (0.1, 0.5, 0.1), w_i = (0.05, 0.9, 0.05)$. The smile crosses the line $v = 0.475$ four times (left). The pdfs of S and X_v cross 6 times (right).

Tail asymptotics for the Gaussian mixture model

Proposition

In the N-Gaussian mixture model, the implied volatility has the small/large strike asymptotics

$$\lim_{K\to 0} \sigma_{BS}(K) = \lim_{K\to 0} \sigma_{BS}(K) = v_{max} := \max_i v_i$$

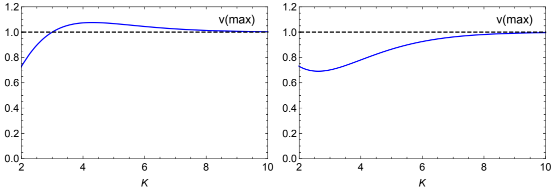
- Largest volatility component dominates in the tails...
- ... regardless of weight on that component

Approach to the asymptotic limit

Proposition ($K \to \infty$ asymptotics)

Overshoot: *If there is a v_{max} component with location parameter greater than F, then $\sigma_{BS}(K)$ approaches v_{max} from above as $K \to \infty$*

No overshoot: *If all v_{max} components have location parameters smaller than F, then $\sigma_{BS}(K)$ approaches v_{max} from below as $K \to \infty$.*

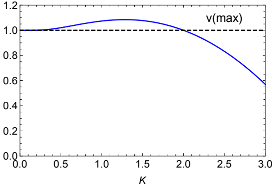


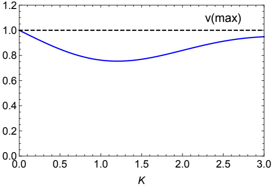
Approach to the asymptotic limit

Proposition ($K \to 0$ asymptotics)

Overshoot: *If there is a v_{max} component with location parameter smaller than F, then $\sigma_{BS}(K)$ approaches v_{max} from above as $K \to 0$*

No overshoot: *If all v_{max} components have location parameters greater than F, then $\sigma_{BS}(K)$ approaches v_{max} from below as $K \to 0$.*





Approach to the asymptotic limit: No limiting overshoot

Proposition

If the only v_{max} component is located at F, then $\sigma_{BS}(K)$ approaches v_{max} from below as $K \to 0$ and $K \to \infty$.

Proof.

The proof of all asymptotic results follows from:

- The option price representation in the GMM as convex combinations of BS prices
- the extreme strikes asymptotics of the Black-Scholes formula from Gulisashvili (2010) and Gao and Lee (2014)

□

Wings asymptotics

- It may seem that the GMM cannot capture the steep left wing of the smile required by the asymptotics $\sigma^2_{BS}(K) \simeq c_L |\log K/F|$.
- In fact this is possible by taking $\mu_{min} \to 0$.
- Follows from the lower bound on $n_{vol}(v)$.

Corollary

The implied vol crosses $v_c(\kappa)$ at least once, where $v_c(\kappa)$ is the solution of $\mathbb{E}[S^{-\kappa}] = \mathbb{E}[X_v^{-\kappa}]$.

$$\lim_{\kappa\to 0} v_c^2(\kappa) = \sum_{i=1}^N w_i v_i^2 + 2\sum_{i=1}^N w_i \log \frac{F}{\mu_i}$$

This grows to ∞ as $\mu_i \to 0$.

Wings asymptotics - Example

- $N = 3$ Gaussian mixture with means $\mu_1 < 1 < \mu_2$ (forward $F = 1$).
- Non-central components have equal volatilities (v_0, v, v_0) with $v_0 < v$ - ensures no-overshoot as $K \to 0$ and $K \to \infty$.
- Fix $\mu_2 = 1, \mu_3 = 1.2$ and $v_i = (0.2, 0.4, 0.2)$ and take $\mu_1 \to 0$.

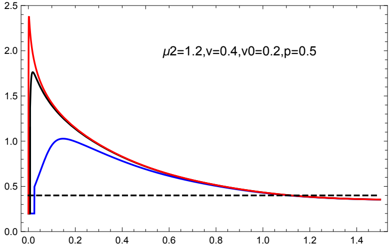


Figure: $N = 3$ GMM with $\mu_1 = 0.1$ (blue), 0.01 (black) and 0.001 (red).

Smiles ordering vs mixture parameters ordering

The N-component GMM has parameters

$$\text{(mean, vol): } (\mu_1, v_1), \cdots, (\mu_N, v_n)$$

with weights

$$w_1, \cdots, w_n$$

Two particular mixtures:

- Volatility mixture: $\mu_1 = \cdots = \mu_N = F$, different v_i

  Appears naturally when considering uncorrelated stochastic volatility models

$$dS_t = S_t\sqrt{V_t}dW_t + rS_t dt \Rightarrow S_T = F(T)e^{v_T N(0,1) - \frac{1}{2}v_T^2}, \quad v_T^2 = \int_0^T V_t^2 dt$$

- Location mixture: $v_1 = \cdots = v_N$, different μ_i

Stochastic comparisons

For any two random variables X, Y

- $X \leq_{st} Y$ means $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing f
- $X \leq_{icx} Y$ means $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing convex f
- $X \leq_{cx} Y$ means $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex f

Comparing Location Mixtures

- A location mixture has components with a common volatility $v_1 = \cdots = v_N$, and different locations μ_i
- Represent a location mixture through a random variable $M = \{\mu_1, \mu_2, \cdots, \mu_N\}$ with probabilities $w_1, \cdots, w_N$.

Proposition

For any two location mixtures with associated random variables M_1, M_2,

$$M_1 \leq_{icx} M_2 \Rightarrow \sigma_{BS}(K; M_1) \leq \sigma_{BS}(K; M_2)\,, \text{ for all } K \geq 0$$

For any mixture M, the underlying asset is bounded in convex order as

$$X_{F,v} \leq_{cx} S \leq \bar{w}_1 X_{\mu_{min},v} + \bar{w}_2 X_{\mu_{max},v}$$

with $\bar{w}_1 \mu_{min} + \bar{w}_n \mu_{max} = F$.

Comparing Location Mixtures - Example

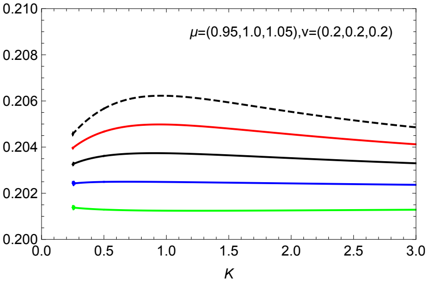


Figure: Implied volatility curves in the $N = 3$ Gaussian mixture model with uniform volatility $v_i = v = 0.2$ are above the level v and bounded from above by the $N = 2$ mixture with extreme location parameters $\{\mu_1, \mu_3\}$ (dashed curve). Location parameters $\mu_i : (0.95, 1, 1.05)$. From bottom to top: $w_1 = w_3 = 0.1 - 0.4$, with $w_2 = 1 - 2w_1$.

Symmetric mixtures

- Suppose we have $N = 2n$ components with parameters of the form

$$(F\mu_1, v_1), (F/\mu_1, v_1), \cdots, (F\mu_n, v_n), (F/\mu_n, v_n)$$

- Weights

$$p_1 w_1, p_1(1 - w_1), \cdots, p_n w_n, p_n(1 - w_n)$$

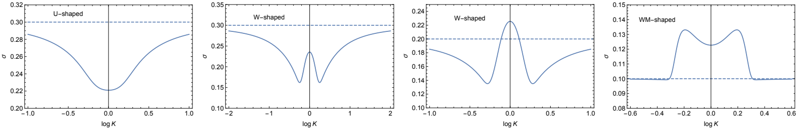
- Within each pair, $w_i\mu_i + (1 - w_i)\frac{1}{\mu_i} = 1$

Proposition

In the symmetric Gaussian mixture model, the implied volatility is symmetric in log-strike, i.e. the implied volatilities at K and F^2/K coincide, for all $K > 0$.

Symmetric mixture: $N = 3$

- Symmetric mixture, with components $(F/\mu, v_0), (F, v), (F\mu, v_0)$ with $v > v_0$
- Weights $\{p\frac{1}{\mu+1}, 1 - p, p\frac{\mu}{\mu+1}\}$
- The combined constraints $n_{vol} \leq 4$, symmetry in log-strike and extreme strikes asymptotics allow only four distinct smile shapes



They are mapped explicitly to the model parameters (μ, v_0, v, p).

Empirical example: AMZN smile

Can we reproduce the observed AMZN smile on 26-Apr-2018 with a $N = 3$ Gaussian mixture model?

- Start with a symmetric mixture with four parameters μ, v, v_0, p
- Minimize a target function = weighted sum of squared pricing errors

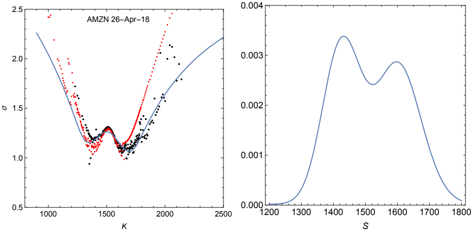
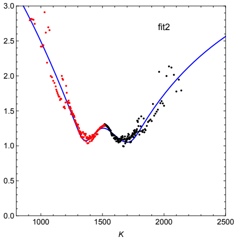


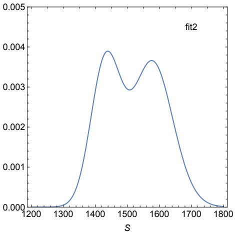
Figure: Fit parameters: $\mu = 1.06, v = 0.192, v_0 = 0.043, p = 0.982$

Reasonable fit in the central region. Increase v_1 for steeper left wing.

Empirical example: AMZN smile

Best fit to $N = 3$ GMM matching the left wing of the smile.





$$
\begin{aligned}
w_i &: \quad 0.003, 0.440, 0.557 \\
\mu_i/F &: \quad 0.944, 0.946, 1.043 \\
v_i &: \quad 0.25, 0.033, 0.039\,.
\end{aligned}
$$

Concluding remarks

- A bimodal density does not imply a W-shaped smile
- A W-shaped smile does not imply a bimodal density
- In fact, a unimodal density can produce an arbitrary number of volatility level crossings
- A two-component Gaussian mixture cannot produce a W-shaped smile...
- ... but a three-component mixture provides a lot of flexibility
- Stochastic comparison results predict how implied volatility in the mixture model changes with the mixture parameters
- Combining constraints from level crossings and extreme strike asymptotics restricts the number of allowed smile shapes in the GMM. E.g. for $N = 3$ there are only 4 distinct smile shapes.

Call for papers