The No-arbitrage Pricing of Non-traded Assets

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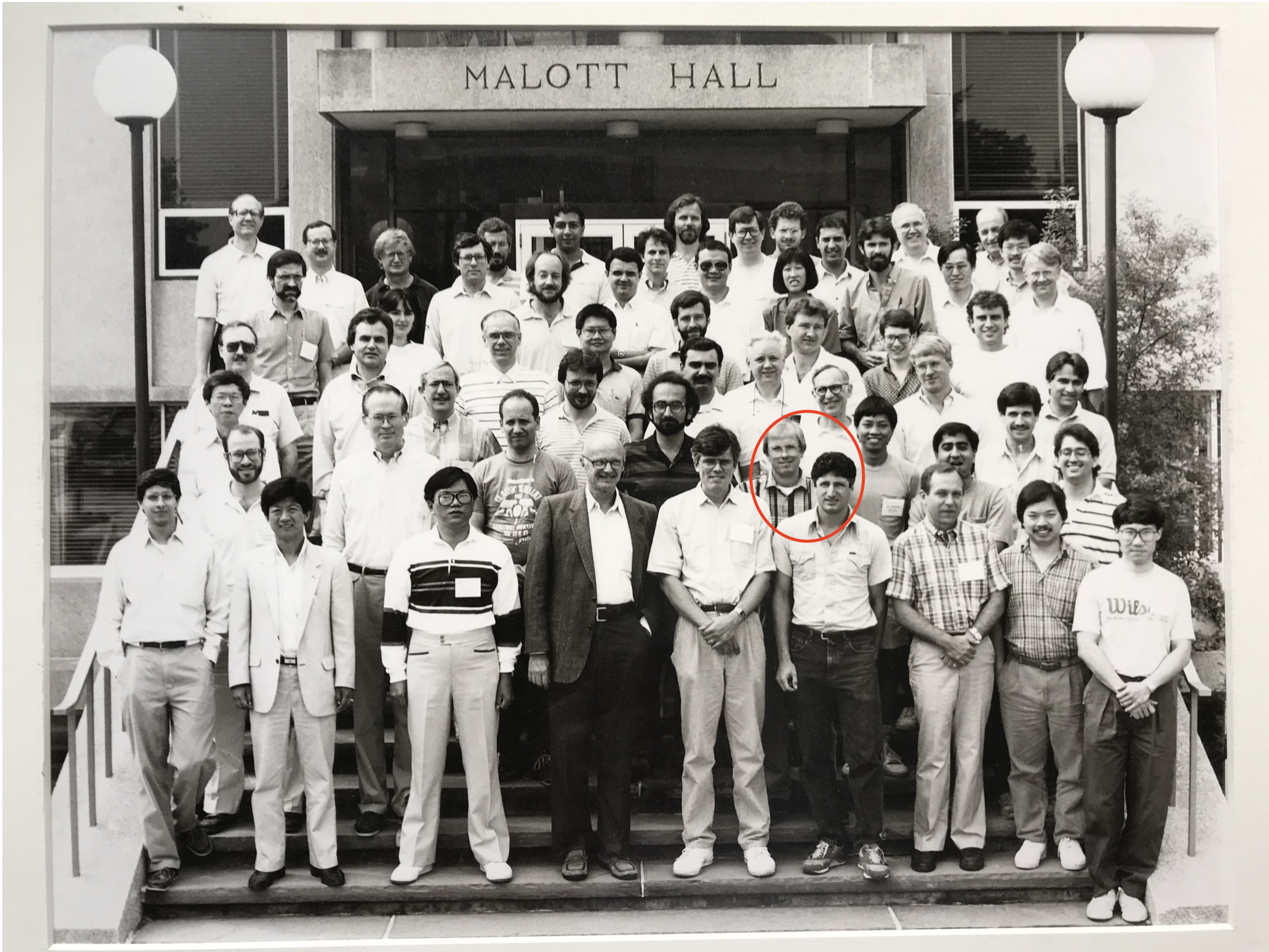
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Outline

- Introduction
- The Original Market
  - The set-up, terminology
  - The fundamental theorems
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Introduction

- The pricing of non-traded assets is an important area within finance.
  - private loans,
  - illiquid publicly traded debt,
  - insurance contacts,
  - private equity,
  - real estate, and
  - real options.
- The purpose of the paper is to provide a no-arbitrage methodology for pricing non-traded assets in an otherwise frictionless market.
- Non-traded assets are a special case of pricing derivatives in an incomplete market.

- In an incomplete market, there is no unique price for a derivative.
- Two approaches to select a unique price.
  - Use a preference function: variance and risk minimizing hedging, indifference pricing.
  - Assume certain risks are non-priced.
    - Merton 1976 jump risk
    - Hull and White 1987 volatility risk
    - Jarrow, Lando, and Yu 2005 default risk.
- This paper revisits, formalizes, and generalizes this later approach to general semimartingale price processes.

- This method
  - avoids the necessity of assuming a particular preference or objective function to determine a unique price, and
  - with the abundance of assets traded in current markets, sufficient securities exist to hedge most systematic risks.
- This implies that the remaining non-traded risks are non-priced or idiosyncratic, hence, the above methodology applies.

The Set-up

- Continuous time, continuous trading on a finite horizon $[0, T]$.
- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered complete probability space where
  - $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual hypothesis
  - $\mathcal{F}_0$ is the trivial σ algebra,
  - $\mathcal{F}_T = \mathcal{F}$, and
  - $\mathbb{P}$ is the statistical probability measure.

The Original Market

- The market is assumed to be competitive and frictionless.
  - Competitive means that traders have no quantity impact on the market price.
  - Frictionless means that there are no transaction costs and no trading constraints.
- Traded in the economy are n risky assets and a money market account (mma) whose value is unity for all times.
- Risky asset prices.
  - $S_t := (S_1(t), \ldots, S_n(t))$ for $0 \leq t \leq T$.
  - A non-negative semimartingale adapted to $\mathbb{F}$.
  - No cash flows are paid to the risky assets.
- Let $\mathbb{F}^s := (\mathcal{F}_t^s)_{0 \leq t \leq T}$ be the filtration generated by S.
- $\mathbb{F}^s \subset \mathbb{F}$ and $\mathbb{F}^s \neq \mathbb{F}$. Important to distinguish non-traded assets.

- Trading strategies are $(\alpha_0, \alpha \coloneqq (\alpha_1, \ldots, \alpha_n)) \in (\mathscr{O}, \mathscr{L}(S, \mathbb{F}))$
  - $\mathscr{O}$ the $\mathbb{F}$ - optional σ-algebra
  - $\mathscr{L}(S, \mathbb{F})$ the set of $\mathbb{F}$- predictable processes for which the stochastic integral with respect to S exists.
- To exclude doubling strategies, only consider trading strategies that are admissible (the value of the trading strategy is bounded below).
- An admissible self financing trading strategy (s.f.t.s) with initial wealth x and wealth process X is an $(\alpha_0, \alpha \coloneqq (\alpha_1, \ldots, \alpha_n)) \in (\mathscr{O}, \mathscr{L}(S, \mathbb{F}))$ such that

$$X_t = \alpha_0(t) + \alpha_t \cdot S_t = x + \int_0^t \alpha_u \cdot dS_u \geq c, \forall t \in [0, T]$$

  c a constant and $x \cdot y$ denotes the inner product.
- We denote by $\mathcal{A}(x, \mathbb{F})$ the set of admissible s.f.t.s. $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S, \mathbb{F}))$ given an initial wealth x.

- A simple arbitrage opportunity is an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathcal{A}(x, \mathbb{F})$ with initial wealth $x = 0$ and wealth process X such that

$$\mathbb{P}(X_T \geq 0) = 1, \qquad \text{and}$$
$$\mathbb{P}(X_T > 0) > 0.$$

- A Free Lunch with Vanishing Risk (FLVR) is an admissible s.f.t.s. that is an extension of a simple arbitrage opportunity that includes (the limits of) approximate simple arbitrage opportunities.

- An equivalent local martingale measure $\mathbb{Q}$ is any probability measure on $(\Omega, \mathcal{F})$ such that for $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ iff $\mathbb{P}(A) = 0$ and S is a $\mathbb{Q}$ local martingale with respect to $\mathbb{F}$.

- Define $\mathcal{M}_l(\mathbb{F})$ to be the set of equivalent local martingale measures (ELMM) with respect to $\mathbb{F}$.

- The first fundamental theorem of asset pricing states that $\mathcal{M}_l(\mathbb{F}) \neq \emptyset$ if and only if the market satisfies NFLVR.

- An admissible s.f.t.s. with wealth process X is said to be dominating for asset i if there exists an admissible s.f.t.s $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F})$ such that $x < S_i(0)$ and
$$x + \int_0^T \alpha_u \cdot dS_u = S_i(T) \quad \text{a.s.}$$
- The market is said to satisfy No Dominance (ND) if for all assets $i = 0, 1, ..., n$ there exist no such dominating s.f.t.s.
- Define $\mathcal{M}(\mathbb{F})$ to be the set of equivalent martingale measures (EMM) under which S is a $\mathbb{Q}$ martingale.
- The third fundamental theorem states that $\mathcal{M}(\mathbb{F}) \neq \emptyset$ if and only if the market satisfies NFLVR and ND.

- A market is defined to be complete with respect to some $\mathbb{Q} \in \mathcal{M}_l(\mathbb{F})$ if for any non-negative payoff $C_T \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ at time T, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F})$ such that

$$x + \int_0^T \alpha_u \cdot dS_u = C_T,$$

wealth process

$$C_t = \alpha_0(t) + \alpha_t \cdot S_t = x + \int_0^t \alpha_u \cdot dS_u$$

is a $\mathbb{Q}$ martingale with respect to $\mathbb{F}$.

- The payoff $C_T \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ can be interpreted as the cash flow to a non-traded asset or a derivative. Note $\mathcal{F}_T$ and not $\mathcal{F}_T^s$.

- By the second fundamental theorem of asset pricing, given there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the market is complete with respect to $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ if and only if the EMM is unique.

- In a complete market, $\mathbb{E}^{\mathbb{Q}}[\cdot]$ gives the unique present value operator to determine the arbitrage-free price:

$$\mathbb{E}^{\mathbb{Q}}[C_T | \mathcal{F}_t]$$

- ASSUMPTION 1: (*NFLVR, ND, and Incomplete Original Market*)
  - There exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, and
  - the original market is incomplete with respect to $\mathbb{Q}$.

- In an incomplete market satisfying NFLVR and ND, there exist payoffs $C \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})$ that cannot be replicated using the mma and the n risky assets.

- And, there are an infinite number of martingale measures $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$. Hence, there is no unique arbitrage-free price for any such payoff C.

- Define the original market as the collection $\left(S, \mathbb{F}, L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q})\right)$ for a given $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.

- PROBLEM: To determine a unique price for C.

The Restricted Market

- Given $\mathcal{M}(\mathbb{F}) \neq \emptyset$. Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ and define $\mathbb{Q}^s := \mathbb{Q} \mid_{\mathcal{F}_T^s}$ on $(\Omega, \mathcal{F}_T^s)$.
- For the given $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the restricted market is defined as the collection

$$\left(S, \mathbb{F}^s, L^1_+(\Omega, \mathcal{F}_T^s, \mathbb{Q}^s)\right).$$

- LEMMA: (*The Restricted Market also satisfies NFLVR and ND*)
  - S is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, i.e. $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$.

- ASSUMPTION 2: (*Complete Restricted Market*)
  - Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.
  - The restricted market is complete with respect to $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$.
- This assumption implies that $\mathbb{Q}^s$ is unique on $(\Omega, \mathcal{F}_T^s)$.
- By the definition of market completeness, for any $\tilde{C}_T \in L^1_+(\Omega, \mathcal{F}_T^s, \mathbb{Q}^s)$ at time T, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$ such that

$$x + \int_0^T \alpha_u \cdot dS_u = \tilde{C}_T,$$

wealth process

$$\tilde{C}_t = \alpha_0(t) + \alpha_t \cdot S_t = x + \int_0^t \alpha_u \cdot dS_u$$

is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$,

$$x = \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T] = \mathbb{E}^{\mathbb{Q}}[\tilde{C}_T]$$

The Result

- Choose an arbitrary $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.
  - Show later that the price of the non-traded asset is independent of the EMM selected.
- Fix a non-traded asset's payoff $C_T \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{Q}) \cap L^1_+(\Omega, \mathcal{F}_T, \mathbb{P})$.
- Consider the related payoff $\tilde{C}_T := \mathbb{E}^{\mathbb{P}}[C_T | \mathcal{F}^s_T]$, the "traded part" of C.
- This payoff is in the restricted market because it is $\mathcal{F}^s_T$ measurable.
- By completeness, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$ such that

$$x + \int_0^T \alpha_u \cdot dS_u = \tilde{C}_T,$$

$$x = \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T],$$

$$\tilde{C}_t = \alpha_0(t) + \alpha_t \cdot S_t = \mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T] + \int_0^t \alpha_u \cdot dS_u$$

  is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$.
- The unique risk neutral value is $\mathbb{E}^{\mathbb{Q}^s}[\tilde{C}_T]$, which can be replicated using the mma and traded risky assets.

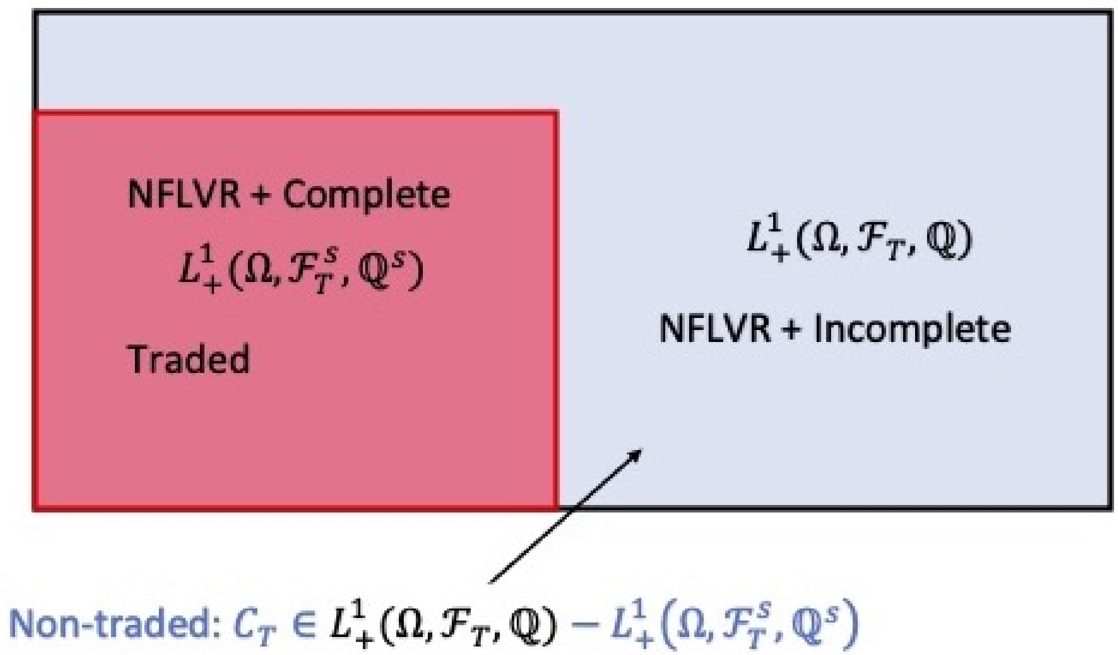
- Use this s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(x, \mathbb{F}^s)$ in the original market to construct a partial hedge for the non-traded asset's payoff.
- The hedging error ε_T is
$$\varepsilon_T = C_T - \tilde{C}_T.$$
ε_T is the "non-traded" part of the payoff C_T.
- LEMMA: (*Expected Hedging Error with respect to* $\mathbb{F}^s$)
  - $\mathbb{E}^{\mathbb{P}}\left(\varepsilon_T \,|\mathcal{F}_t^s\right) = 0$, which implies $\mathbb{E}^{\mathbb{P}}\left(\varepsilon_T\right) = 0$.
- Using the given $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, the arbitrage-free value under $\mathbb{Q}$ of the non-traded risky asset's payoff is:
$$\mathbb{E}^{\mathbb{Q}}\left(C_T\right) = \mathbb{E}^{\mathbb{Q}^s}(\tilde{C}_T) + \mathbb{E}^{\mathbb{Q}}\left(\varepsilon_T\right)$$
- To determine $\mathbb{E}^{\mathbb{Q}}\left(\varepsilon_T\right)$?

- ASSUMPTION 3: (*Non-priced Hedging Error Risk*)
  - For all $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$, $\qquad \mathbb{E}^{\mathbb{Q}}(\varepsilon_T) = \mathbb{E}^{\mathbb{P}}(\varepsilon_T)$.
- Valid if hedging error is diversifiable risk (in a large portfolio).
- By the lemma, this implies $\mathbb{E}^{\mathbb{Q}}(\varepsilon_T) = 0$.
- THEOREM: (*Arbitrage-Free Price of the Non-traded Asset*)

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}(\mathbb{E}^{\mathbb{P}}(C_T \,|\, \mathcal{F}_T^s)).$$

  - First take expectation using $\mathbb{P}$ over randomness not in $\mathbb{F}^s$.
  - Then take expectation over $\mathbb{F}^s$ with unique $\mathbb{Q}^s$.

Summary



1) Partial hedge with $\tilde{C}_T = E^{\mathbb{P}}(C_T | \mathcal{F}^S_T)$. Unique price $E^{\mathbb{Q}^s}(\tilde{C}_T)$.
2) Difference $\varepsilon_T = C_T - \tilde{C}_T$ is idiosyncratic risk, i.e. $E^{\mathbb{Q}}(\varepsilon_T) = 0$.
3) Result: $E^{\mathbb{Q}}(C_T) = E^{\mathbb{Q}^s}(\tilde{C}_T)$.

- A special case, when the randomness underlying the non-traded asset's cash flows is independent of market prices S under $\mathbb{P}$.

- COROLLARY: (*S Independent of C_T under $\mathbb{P}$*)

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{P}}(C_T).$$

- The arbitrage-free price of the non-traded asset is equal to the expected cash flow under the statistical probability $\mathbb{P}$.

- This special case is useful in the determination of arbitrage-free insurance premiums.

- To understand how to use, some examples ....

Private Debt

- mma's value is unity.
- Let a privately owned company issue a zero-coupon bond promising to pay 1 dollar at time T.
- Trading is equity for a similar company

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 + \sigma W_t}$$

  - W_t is a BM with $W_0 = 0$ under $\mathbb{P}$.
- Default indicator

$$Z_T(\omega) = \begin{cases} 1 & with \quad prob \;\; \lambda(S_T) \in (0,1) \\ 0 & with \quad prob \quad 1 - \lambda(S_T) \end{cases}$$

  where $\lambda(\cdot) : \mathbb{R} \to [0, \infty)$ is Borel measurable.

  - $\lambda(S_T)$ is the probability of default at time T
- The zero-coupon bond has time T payoff

$$C_T(\omega) = \begin{cases} \delta & if \quad Z_T(\omega) = 1 \\ 1 & if \quad Z_T(\omega) = 0 \end{cases}$$

  where $\delta \in (0, 1)$ is the recovery rate.

- $\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.
- Assume the original market (the similar company's stock and the mma) satisfies NFLVR and ND using $\mathbb{F}$, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ martingale wrt $\mathbb{F}$.
- Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.
- The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\,|_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 + \sigma \tilde{W}_t}$$

where $\tilde{W}_t = \frac{\mu}{\sigma} + W_t$ is a BM under $\mathbb{Q}^s$.
- Assumptions 1 - 2 are satisfied by construction.
- Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)$ is idiosyncratic risk, then

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)\right) = \mathbb{E}^{\mathbb{Q}^s}\left(\delta \cdot \lambda(S_T) + (1 - \lambda(S_T)\right).$$

- This is a simple case of the models contained in the credit risk literature for the pricing of credit derivatives. Applies to publicly traded debt that is very illiquid.

Insurance

- mma's value is unity.
- A term insurance contract on an event over the time period $[0, T]$.
- The contract is repriced and repurchased every T periods.
  - e.g. yearly term life insurance.
- The insurance premium of p dollars is paid at time 0 to insure the event over $[0, T]$.
- If the event occurs over $[0, T]$, K dollars is paid at time T.
  - The payoff K could be a random variable.
- It costs the insurance company c dollars to issue the insurance contract.
  - This cost is incurred at time 0.

Independent Event Risk (Life Insurance)

- The contract's payoff K is a constant.
- Death event

$$Z_T(\omega) = \begin{cases} 1 & with \quad prob \quad \lambda \in (0,1) \\ 0 & with \quad prob \quad 1-\lambda \end{cases}$$

  - λ is the actuarial probability of death in $[0, T]$.

- Let S be the market prices of the traded risky assets, and $\mathbb{F}$ the filtration generated by S_t for all $t \in [0, T)$ and $(S_T,\, Z_T)$ at time T.
- Assume that Z_T is independent of market prices S under $\mathbb{P}$.
  - Reasonable assumption for the death of an individual.
- Cash flow to the insurance policy at time T:

$$C_T(\omega) = \begin{cases} p - c - K & if \quad Z_T(\omega) = 1 \\ p - c & if \quad Z_T(\omega) = 0 \end{cases}$$

- Assume the market consisting of S and the mma satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

- The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$.
- Assumptions 1 - 2 are satisfied by construction.
- Assuming $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)$ is idiosyncratic risk, then

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{P}}\left(C_T\right) = p - c - \lambda K.$$

- The arbitrage-free insurance premium is that p such that $\mathbb{E}^{\mathbb{Q}}(C_T) = 0$, i.e.

$$p = \lambda K + c.$$

- This is the actuarial value.

Dependent Event Risk (Car Insurance)

- The car insurance payoff is $K(\omega)$ having a uniform distribution over $[0, k]$ with mean $\frac{k}{2}$ under $\mathbb{P}$.
- k is the value of the car at time 0.
- K is the damage to the car in the event of an auto accident.
- The market price of oil, a traded commodity, is

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 + \sigma W_t}$$

  where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.
- Car accident event

$$Z_T(\omega) = \begin{cases} 1 & with \quad prob \quad \lambda(S_T) \in (0,1) \\ 0 & with \quad prob \quad 1 - \lambda(S_T) \end{cases}$$

  - $\lambda(S_T)$ is the actuarial probability of car accident in $[0, T]$
  - $\lambda(S_T)$ a decreasing function of oil prices.
  - As oil prices decrease, cars are driven more frequently, and the probability of an accident increases.

- We assume that Z_T, K (the loss to the car in the event of an accident), and market prices S are independent under $\mathbb{P}$.
  - The car accident event and the damages resulting are independent of the price of oil, due to random events surrounding the accident while driving of a car.

- The cash flow to the insurance policy at time T is:

$$C_T(\omega) = \begin{cases} p - c - K(\omega) & if \quad Z_T(\omega) = 1 \\ p - c & if \quad Z_T(\omega) = 0 \end{cases}$$

- $\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, K, Z_T) at time T.

- We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

- The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q} \mid_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 + \sigma \tilde{W}_t}$$

where $\tilde{W}_t = \frac{\mu}{\sigma} + W_t$ is a Brownian motion under $\mathbb{Q}^s$.

- Assumptions 1 - 2 are satisfied by construction.
- Assuming $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)$ is idiosyncratic risk, then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(C_T) &= \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)\right) \\ &= p - c - \frac{k}{2}\mathbb{E}^{\mathbb{P}}\left(\lambda(S_T)\right).\end{aligned}$$

- The arbitrage-free insurance premium is that p such that $\mathbb{E}^{\mathbb{Q}}(C_T) = 0$, i.e.

$$p = \frac{k}{2}\mathbb{E}^{\mathbb{Q}^s}\left(\lambda(S_T)\right) + c.$$

- This is *NOT* the actuarial value of the insurance contract's payoff $(\frac{k}{2}\lambda)$ plus costs (c).

Private Equity

- mma's value is unity.
- Let a privately owned company have outstanding equity.
- Trading is equity for a similar company

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 + \sigma W_t}$$

  where W is a BM with $W_0 = 0$ under $\mathbb{P}$.

- Let $Z_T(\omega)$ be a $\mathcal{F}_T$ measurable, normally distributed $(0, 1)$ random variable under $\mathbb{P}$.
- The cash flow to the private equity at time T is

$$C_T = S_T e^{\alpha(S_T) - \frac{1}{2}\beta(S_T)^2 + \beta(S_T) Z_T}$$

  where $\alpha(\cdot) : \mathbb{R} \to [0, \infty)$ and $\beta(\cdot) : \mathbb{R} \to [0, \infty)$ are Borel measurable.

- $\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.

- We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.
- Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.
- The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\,|_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 + \sigma \tilde{W}_t}$$

where $\tilde{W}_t = \frac{\mu}{\sigma} + W_t$ is a BM under $\mathbb{Q}^s$.

- Assumptions 1 - 2 are satisfied by construction.
- Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)$ is idiosyncratic risk,

$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)\right) = \mathbb{E}^{\mathbb{Q}^s}\left(S_T e^{\alpha(S_T)}\right).$$

Real Estate

- mma's value is unity.
- Trading is a REIT (real estate investment trust) or a real estate based ETF (electronic traded fund)
$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 + \sigma W_t} \tag{1}$$
where S_0, μ, σ are strictly positive constants and W_t is a standard Brownian motion with $W_0 = 0$ under $\mathbb{P}$.
- Let $Z_T(\omega)$ be a $\mathcal{F}_T$ measurable, normally distributed $(0, 1)$ random variable under $\mathbb{P}$.
- The cash flow to selling the house at time T is
$$C_T = S_T e^{\alpha(S_T) - \frac{1}{2}\eta^2 + \eta Z_T}$$
where η is a strictly positive constant and $\alpha(\cdot) : \mathbb{R} \to [0, \infty)$ is Borel measurable.
- $\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.
- We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.

- The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\mid_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 + \sigma \tilde{W}_t}$$

where $\tilde{W}_t = \frac{\mu}{\sigma} + W_t$ is a Brownian motion under $\mathbb{Q}^s$.

- Assumptions 1 - 2 are satisfied by construction.

- Assuming $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)$ is idiosyncratic risk, then

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(C_T) &= \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)\right) \\ &= \mathbb{E}^{\mathbb{Q}^s}\left(S_T e^{\alpha(S_T)}\right). \end{aligned}$$

Real Options

- mma's value is unity.
- Consider an oil company that is deciding whether or not to extract oil from a well at time T.
- The market price of oil, a traded commodity, is
$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 + \sigma W_t}$$
where W_t is a BM with $W_0 = 0$ under $\mathbb{P}$.
- Due to the oil extraction methods, after taking into account impurities which affect the price of oil received before refinement, the cash flow from the extracted oil at time T is
$$S_T e^{-\frac{1}{2}\eta^2 + \eta Z_T}$$
where $Z_T(\omega)$ is a $\mathcal{F}_T$ measurable, normal $(0, 1)$ under $\mathbb{P}$ and $\eta > 0$.
- $\mathbb{F}$ is the filtration generated by W_t for all $t \in [0, T)$ and (W_T, Z_T) at time T.
- The (real) option to extract oil at time T has payoff
$$C_T = max\left[S_T e^{-\frac{1}{2}\eta^2 + \eta Z_T} - K, 0\right]$$
where $K > 0$ is the cost of the extraction.

- We assume the original market satisfies NFLVR and ND, so that there exists a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$ under which S is a $\mathbb{Q}$ - martingale with respect to $\mathbb{F}$.
- Fix a $\mathbb{Q} \in \mathcal{M}(\mathbb{F})$.
- The restricted market is complete, hence there exists a unique $\mathbb{Q}^s \in \mathcal{M}(\mathbb{F}^s)$ where $\mathbb{Q}^s = \mathbb{Q}\,|_{\mathcal{F}_T^s}$ such that S_t is a $\mathbb{Q}^s$ martingale with respect to $\mathbb{F}^s$, and
$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 + \sigma \tilde{W}_t}$$
where $\tilde{W}_t = \frac{\mu}{\sigma} + W_t$ is a Brownian motion under $\mathbb{Q}^s$.
- Assumptions 1 - 2 are satisfied by construction.
- Assuming assumption 3 holds, i.e. $\varepsilon_T = C_T - \mathbb{E}^{\mathbb{P}}\left(C_T \,|\mathcal{F}_T^s\right)$ is idiosyncratic risk, then
$$\mathbb{E}^{\mathbb{Q}}(C_T) = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(C_T\,|\mathcal{F}_T^s\right)\right) = \mathbb{E}^{\mathbb{Q}^s}\left(S_T N(d_1) - K N(d_2)\right)$$
where $N(\cdot)$ is the standard $(0,1)$ normal distribution function,
$$d_1 \coloneqq \frac{log(S_T/K) + \frac{1}{2}\eta^2}{\eta}, \qquad \text{and}$$
$$d_2 \coloneqq d_1 - \eta.$$

Conclusion

- This is an important application of the arbitrage-free pricing methodology because it applies to a wide range of assets in the economy, including private debt, illiquid publicly traded debt, insurance contracts, private equity, real estate, and real options.
- The methodology can be applied without assuming a particular preference or objective function.
- Its application only requires that the hedging error, properly defined, is non-priced.
- This non-priced hedging error condition is a very reasonable approximation in current markets given the plethora of traded securities.