REVISITING BLACK-SCHOLES

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OUTLINE

Peter loved everything Black-Scholes….

1) A simple derivation of Black-Scholes
2) Arbitraging Black-Scholes
3) High strikes implied variance cannot rise

A SIMPLER PROOF OF BLACK-SCHOLES-MERTON PDE

It provides another proof of the Black-Scholes-Merton result

- Without hedging argument
- Without arbitrage argument
- Without change of measure
- Without local time/Tanaka

- Simply Itô’s lemma and basic accounting

COST OF EXTENDING THE MATURITY

$\frac{dS}{S} = \mu dt + \sigma dW$ Assume no carry European payoff *f*

Itô to $f(S)$: $df = f'(S)dS + \frac{1}{2}\sigma^2 S^2 f''(S)dt$

Cost of extending the maturity of the f payoff:

$\frac{\sigma^2 S_T^2}{2} f''(S_T)$ (no Itô anymore now)

It is a payoff at time T that can be decomposed with the Carr-Madan formula

2 MIRACLES

$$\frac{\sigma^2 S_T^2}{2} f''(S_T)$$

For general payoff it leads to an integral over strikes

For the Call payoff there are 2 miracles, at least good surprises:

- Integral reduces to 1 point evaluation
- The Dirac function can be evaluated as limit of Call options

FORWARD EQUATION FOR CALL OPTIONS

For $f(S) = (S - K)^+ : \frac{\sigma^2 S_T^2}{2} f''(S_T) = \frac{\sigma^2 K^2}{2} \delta_K(S_T)$

It has a value at time 0 of $\frac{\sigma^2 K^2}{2} \frac{\partial^2 C}{\partial K^2}$ (from Breeden-Litzenberger)

So $$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2} \frac{\partial^2 C}{\partial K^2}$$

ANOTHER BEAUTY OF FORWARD EQUATIONS

More good surprises in the presence of carry:

- Interest rates (if constant or function of time) lead to a constant term ITM, which can be expressed as a first derivative wrt K

- Dividend yields or foreign rate for a currency option (if constant or function of time) lead to Calls themselves

- So the forward equation is a forward PDE, which allows for fast computation

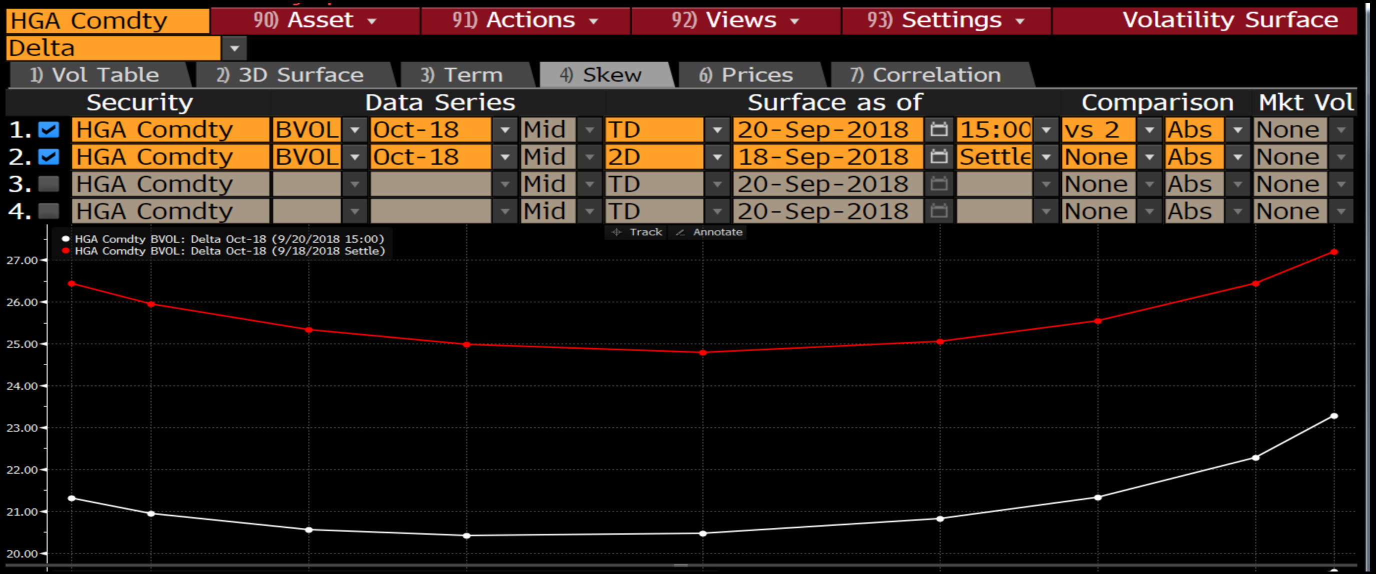
$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2} \frac{\partial^2 C}{\partial K^2} - (r - d) K \frac{\partial C}{\partial K} - d \cdot C$$

- When interest rates or dividend yields are a function of S it requires an integral and we obtain a PIDE instead of a PDE

ARBITRAGING BLACK-SCHOLES

COPPER VOLATILITY DYNAMICS

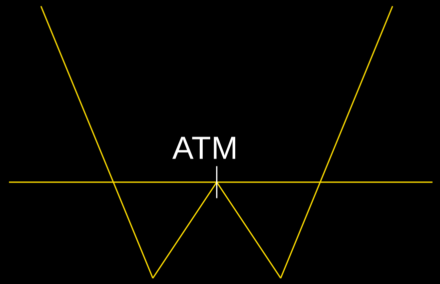
- Skew close to flat everyday
- Level changes from one day to the next



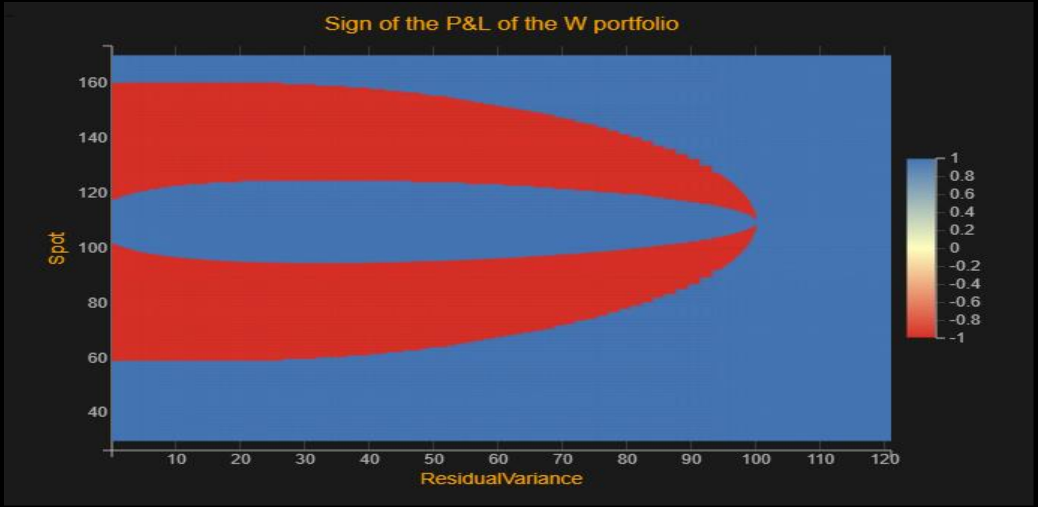
FLAT CASE: CAN WE ARBITRAGE PARALLEL MOVES?

- OTM options have more convexity in volatility
- Symmetric Strangle has more Volga than an ATM Straddle
- Strangle – Gamma ratio Straddle (W portfolio) has all derivatives of order 1 and 2 cancelled except the volga
-

	Straddle	Strangle	$Strangle - \frac{\Gamma_2}{\Gamma_1} Straddle$
s	0	0	0
v	x	x	0
t	x	x	0
ss	x	x	0
sv	0	0	0
vv	x	x	x



REALLY?



- Maximum (Minimum) principle applied to the heat equation in (Spot, Residual variance) shows there is no arbitrage with options of just 1 maturity
- Always a red (negative) region in any neighborhoods of the start point

VARIANCE PF

- Price of log in BS:

$$L^{1,T}(S,t,\sigma) = E[\ln S_T \,|S_t = S, \sigma_t = \sigma] = \ln S - \frac{\sigma^2(T-t)}{2}$$

- Log Calendar Spread captures σ^2:

$$LCS(S,t,\sigma) = \frac{L^{1,T_2}(S,t,\sigma) - L^{1,T_1}(S,t,\sigma)}{T_2 - T_1} = -\frac{\sigma^2}{2}$$

VARIANCE SQUARE PF

- Price of a log square:

$$L^{2,T}(S,t,\sigma) = E[\ln^2 S_T \,|S_t = S, \sigma_t = \sigma] = (\ln S \, - \frac{\sigma^2(T-t)}{2})^2 + \sigma^2(T-t)$$

- Butterfly of log square at equidistant times (T_1, T_2, T_3) captures σ^4:

$$BL^2(S,t,\sigma) = \frac{L^{2,T_3}(S,t,\sigma) - 2L^{2,T_2}(S,t,\sigma) + L^{2,T_1}(S,t,\sigma)}{T_3^2 - 2T_2^2 + T_1^2} = \frac{\sigma^4}{4}$$

WRAPPING UP

- Trading σ^2: $-2\,LCS(S,t,\sigma)$
- Trading σ^4: $4\,BL^2(S,t,\sigma)$
- Portfolio:

$$4BL^2(S,t,\sigma) + 4\sigma_0^2 LCS(S,t,\sigma) + \sigma_0^4 = \sigma^4 - 2\sigma_0^2\sigma^2 + \sigma_0^4 = (\sigma^2 - \sigma_0^2)^2$$

- Captures all volatility moves at any time for any S

DEEP OTM IMPLIED VARIANCE CAN NEVER RISE

LONG TERM RATE CAN NEVER FALL

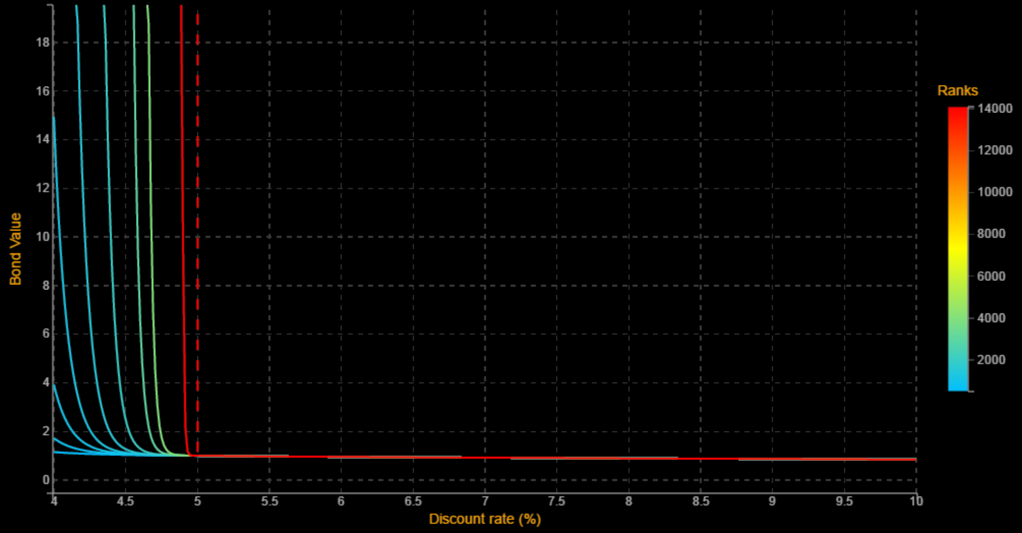
- **Dybvig, Ingersoll and Ross** (1996): In an **arbitrage-free** world without transaction costs, Long Forward and Zero Rates can never fall!

- Long Zero-Coupon rate at time t

$$z_L(t) = \lim_{\{T \to +\infty\}} z(t,T) \quad \text{(if it exists)}$$

- Illustrative Example: The Perpetual Bond
  - Today's yield curve is **flat** at r_0
  - **Infinite** stream of zero-coupon bonds with face value $\frac{\exp(r_0 T)}{T(T+1)}$

PERPETUAL BOND



- Value today:

$$V_0(r_0) = \sum_{T=1}^{+\infty} \frac{1}{T(T+1)} = 1$$

- Value tomorrow:

$$V_{\delta t}(r_{\delta t}) = \exp(r_{\delta t}\delta t) \sum_{T=1}^{+\infty} \frac{\exp((r_0 - r_{\delta t})T)}{T(T+1)}$$

$$\begin{cases} +\infty \text{ if } r_{\delta t} < r_0 \\ \text{Finite if } r_{\delta t} \geq r_0 \end{cases}$$

- Arbitrage-Free assumption:

$$P[r_{\delta t} < r_0] = 0$$

WHAT ABOUT LONG TERM VARIANCE SWAP?

- If the instantaneous variance v_t is a martingale.

For instance, $$\begin{cases} \dfrac{dS_t}{S_t} = \sqrt{v_t}\, dW_t \\ \dfrac{dv_t}{v_t} = \alpha\, dZ_t \end{cases}$$

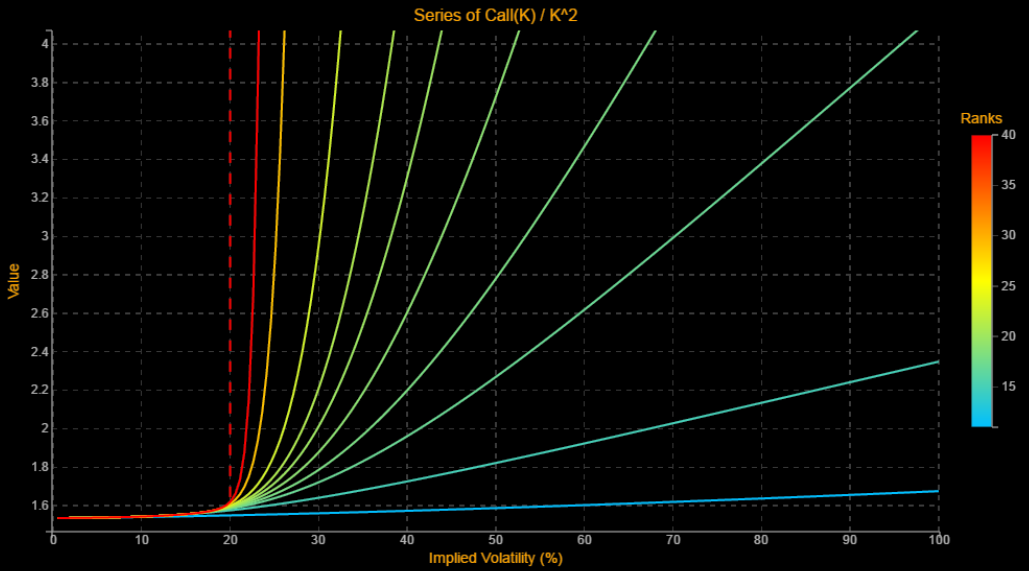
- Then the Variance Swap Term Structure at t:

$$VS_{t,T}^2 = \frac{1}{T-t} E_t\left[\int_t^T v_u du\right] = v_t$$

- Independent of T → Flat! No constraint on its evolution.

FAR OTM IMPLIED VARIANCE CAN NEVER RISE

For fixed T, Implied Variance at t: $IV_t \equiv \sigma_t^2(T-t)$



- Considering a Portfolio of $\frac{1}{K^2}\frac{1}{C_{K,T}(\sigma_0)}$ Calls(K)
- Value today:

$$V_0(\sigma_0) = \sum_{K=1}^{+\infty} \frac{1}{K^2}$$

- Value tomorrow:

$$V_{\delta t}(\sigma_{\delta t}) = \sum_{K=1}^{+\infty} \frac{1}{K^2} \frac{C_{K,T-\delta t}(\sigma_{\delta t})}{C_{K,T}(\sigma_0)}$$

$$\begin{cases} +\infty \text{ if } IV_{\delta t} > IV_0 \\ \text{Finite if } IV_{\delta t} \leq IV_0 \end{cases}$$

$$\rightarrow P[IV_{\delta t} > IV_0] = 0$$

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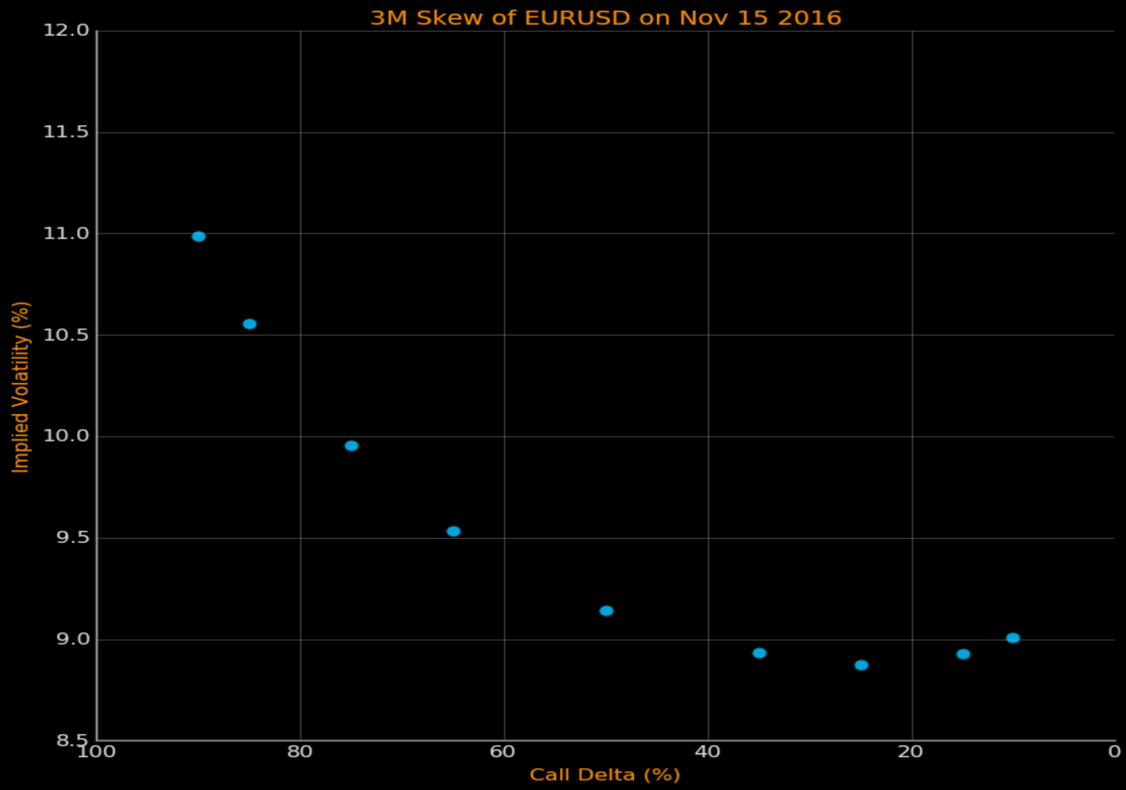
FX OPTIONS ARE QUOTED IN DELTA



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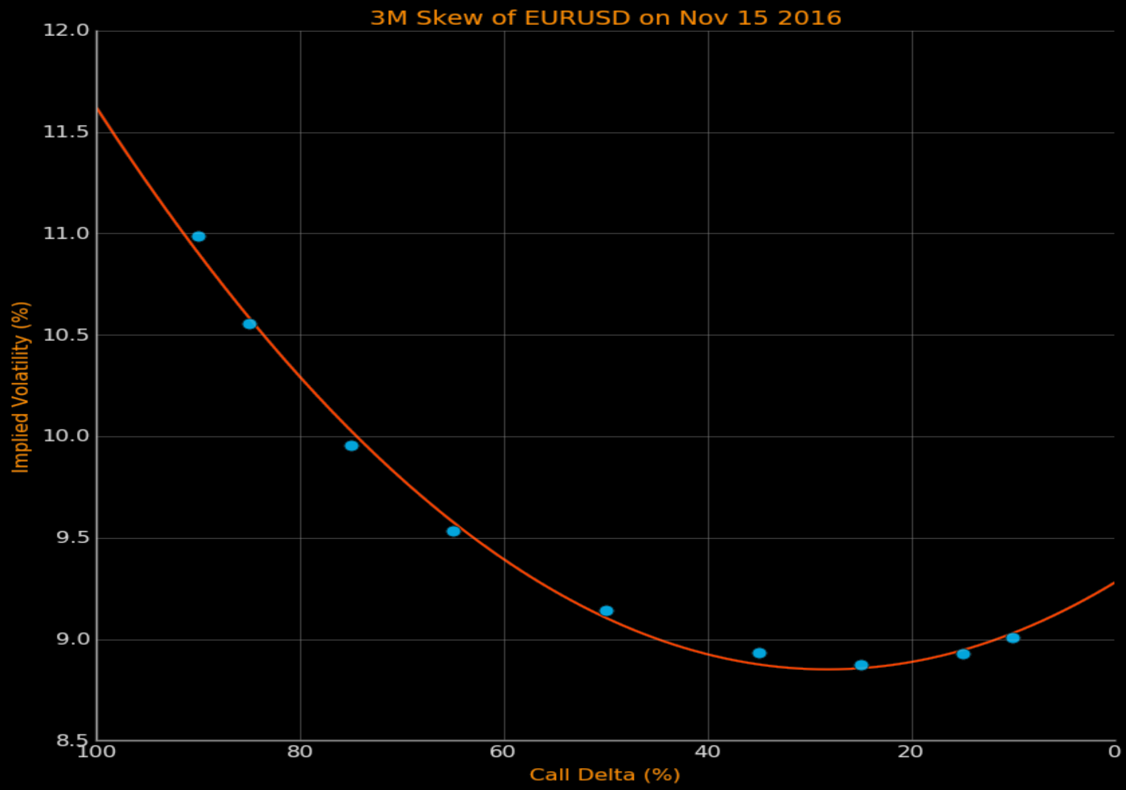
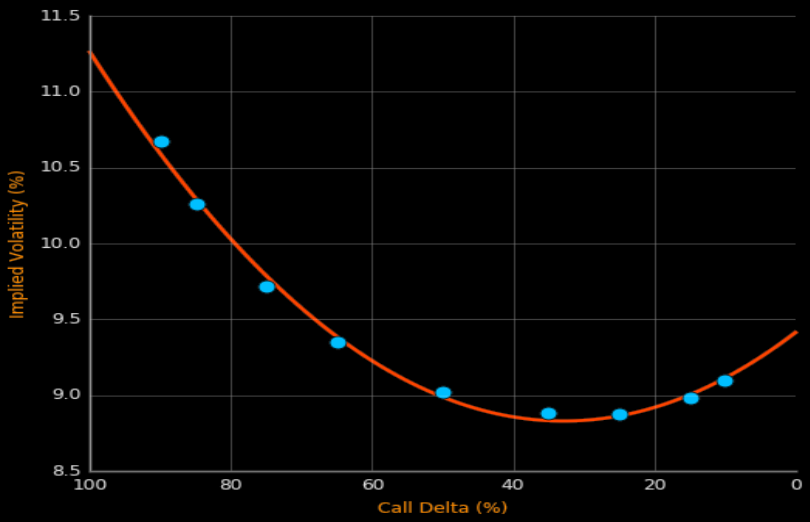
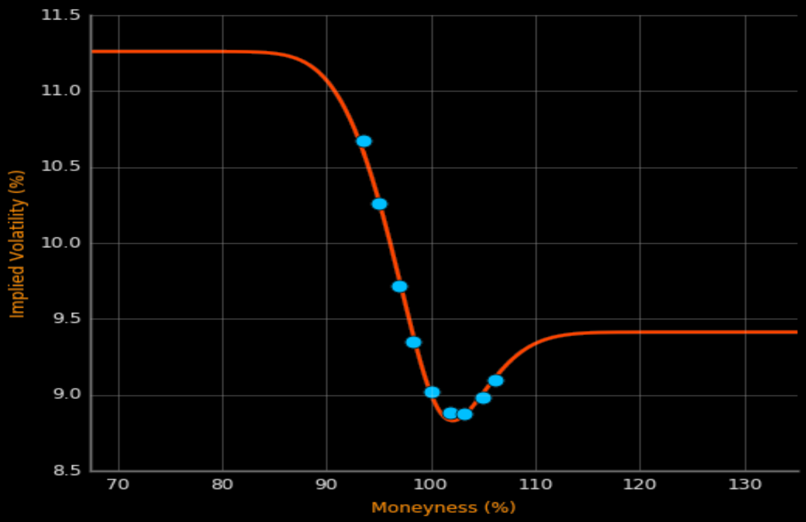


FLAT FAR OTM VOLATILITY

3M Skew of EURUSD on Nov 11 2016





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HIGH STRIKE VOLATILITY THROUGH TIME

CONCLUSION

- Long Term rate can never fall.
- Long Term VS can fall or rise.
- Deep OTM implied variance can never rise.

Thank You