When to Sell an Asset?

A Distribution Builder Approach

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Joint work with Peter Carr

The Peter Carr Memorial Conference
June 2–4, 2022 @ NYU Tandon

In Memoriam

Lake Arrowhead, CA and Brooklyn, NY

Introduction

Question

- Assume you own an asset with current price x. You think about selling this asset.
- Knowing the (stochastic) asset price dynamics (X_t), what is the best time to sell the asset?

Solution Ideas

Two main approaches:

- Universal: Be as close to the maximal asset price as possible:

$$\inf_{0\leqslant\tau\leqslant T}\mathbb{E}\Big[\frac{M_T - X_\tau}{M_T}\Big], \qquad M_T = \sup_{0\leqslant t\leqslant T} X_t$$

(Shiryaev, Xu & Zhou [QF '08], du Toit & Peskir [AAP '09])

- Subjective: Maximize the utility of the seller:

$$\sup_{\tau}\mathbb{E}\big[U\big(X_\tau\big)\big]$$

(Leung & Wang [AF '18], Pedersen & Peskir [MFE '16; Mean–Variance], Henderson [MS '12; Prospect theory], ...)

Distribution Builder

Utility Functions

- We are interested in incorporating personal preferences in the optimal selling strategy
- Classical: The preferences are given by a utility functions (or, simplified, by risk aversion coefficients)
- Goes Back to D. Bernoulli (1738), axiomatization by Von Neumann–Morgenstern, long strain of financial mathematics literature starting with Merton and Samuelson: portfolio optimization, optimal selling, . . .

Utility Functions

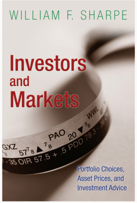
- Criticism:
  - Rational utility functions do not take into account the actual preferences of people (Kahnemann–Tversky, Quiggin,...): Utility functions are not concave, rare extreme events are not adequately considered, . . .
  - Utility functions (or risk aversion) is very hard to measure for practical purposes: How to estimate? Are estimates consistent?

Distribution Builder

Distribution Builder method developed by Sharpe, Goldstein e.a. for portfolio optimization (Sharpe [Book '06], Goldstein, Sharpe & Blythe [TechRep '07]; Goldstein, Johnson & Sharpe [JCR '08]):

- Investors are notoriously bad in estimating their utility function
- Try instead to get more direct information from the agent
- Specifically, for terminal time portfolio optimization let the agent directly choose the desired distribution of terminal wealth that is reachable with given initial capital

Distribution Builder



future consumption. In a complete market setting such an investor's decision process can be summarized as follows:

Budget + Prices + Preferences → Distribution

Given a budget, a set of state prices, and his or her preferences, the investor will choose the most desirable distribution—formally, the one that maximizes his or her expected utility.

Assume that an investor has chosen a distribution and that an outsider can observe the budget, state prices, and the selected distribution. From this information it may be possible to infer the investor's preferences:

Budget + Prices + Distribution → Preferences

Figure 1: Philosophy of the Distribution Builder (Source: Sharpe)

Distribution Builder

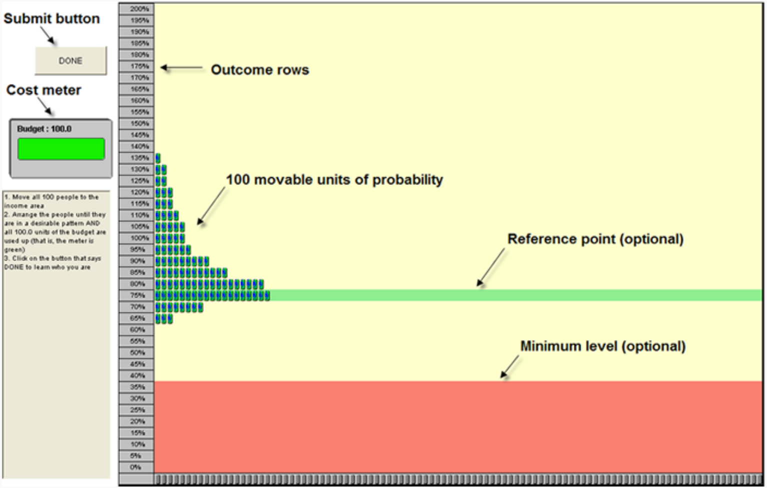


Figure 2: An implementation of the Distribution Builder (Source: Sharpe)

Method

Method

We adapt the distribution builder approach for optimal selling:

- The potential seller should specify its desired target distribution F of the asset value at the time of the sale (a stopping time)
- The distribution builder should provide feedback if the desired distribution F indeed attainable, i.e., there exists a stopping time τ a.s. finite that

$$X_\tau \sim F$$

Method

- More generally, we ask if the distribution is super–attainable,

$$X_\tau \succ F$$

(where $\succ$ stands for first order stochastic dominance, $\mathbb{P}[X_\tau \leqslant z] \leqslant F(z)$ for all z)

- Also: Is the distributional optimal? In which sense?
- Is the resulting stopping strategy easily practically implementable?

Skorokhod Embedding

- The first question is intimately tied to the Skorokhod Embedding Problem
- Classical version: Given a Brownian motion W and a distribution F, does there exist a stopping time τ such that

$$W_\tau \sim F$$

- Many solutions... For our purposes we need a generalized form for diffusions (Azéma & Yor [SdP XIII '79], Grandits & Falkner [SPA '00], Pedersen & Peskir [SPA '01], Cox & Hobson [SPA '04])

Example – GBM

Geometric Brownian motion

- Assume the asset price follows a geometric Brownian motion

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \qquad X_0 = x$$

- The investor discounts using the rate r. This rate is in general subjective indicating the investor's time preference. It can be a market interest rate, but doesn't have to (we have no hedging in the model...)

- The discounted asset price follows

$$dX_t = (\mu - r) X_t \, dt + \sigma X_t \, dW_t, \qquad X_0 = x$$

Theorem

i) *If* $\mu - r \geqslant \sigma^2/2$ *then any distribution* F *is super-attainable.*

ii) *If* $\mu - r < \sigma^2/2$, *a distribution* F *is super-attainable if and only if*

$$\int \left(\frac{z}{x}\right)^A F(dz) \leqslant 1, \qquad A = 1 - \frac{2(\mu - r)}{\sigma^2}.$$

In this case, moreover,

a) *If* $0 < \mu - r < \sigma^2/2$ *there exist super-attainable distributions* F *with* $m = \int z\, F(dz) > x$.

b) *If* $\mu \leqslant r$, *then* $m \leqslant x$ *always.*

Geometric Brownian motion

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Theorem

i) *If* $\mu - r \geqslant \sigma^2/2$ *then any distribution* F *is super-attainable.* *DO NOT SELL!*

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In this case, moreover,

a) *If* $0 < \mu - r < \sigma^2/2$ *there exist super-attainable distributions* F *with* $m = \int z\, F(dz) > x$. *INTERESTING ... tell me more!*

b) *If* $\mu \leqslant r$, *then* $m \leqslant x$ *always.* *SELL IMMEDIATELY!*

Geometric Brownian motion

- But which distribution(s) is/are the best ones too chose for the investor? Of course they want that the distribution F is "as large as possible"...

- We say a distribution F is optimal if

$$\int \left(\frac{z}{x}\right)^A F(dz) \leqslant 1, \qquad A = 1 - \frac{2(\mu - r)}{\sigma^2}.$$

holds with equality.

- Note that all optimal distributions are attainable (but not all attainable distributions are optimal...)

Geometric Brownian motion

The stopping time τ can be chosen for optimal distributions as a first hitting time of GBM and its running minimum:

$$\begin{aligned}\tau &= \inf\{t > 0 \,:\, m_t \leqslant \Psi_F^{\mu,\sigma}(X_t)\} \\ &= \inf\{t > 0 \,:\, (m_t, X_t) \in \mathcal{D}_F\}, \quad \mathcal{D}_F := \{(m,x) \in \mathbb{R}^2 \,:\, m \leqslant \Psi_F^{\mu,\sigma}(x)\}\end{aligned}$$

where $m_t = \inf_{s \leqslant t} X_s$ and

$$\Psi_F^{\mu,\sigma}(z) = x\left(1 - \overline{\psi}_F\left(1 - \left(\frac{x}{z}\right)^A\right)\right)^{-\frac{1}{A}}$$

for the function,

$$\overline{\psi}_F(y) = \begin{cases} \frac{1}{F(y)} \int_{(-\infty),y]} \frac{1}{\sigma} \log \frac{z}{x} \, F(dz) & \text{if } F(y) > 0; \\ y & \text{else.} \end{cases}$$

Azéma–Yor stopping

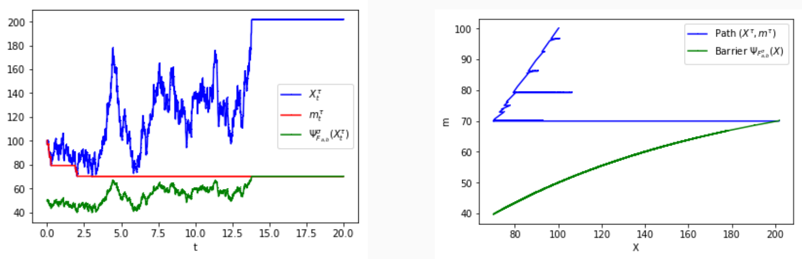

Figure 3: Azéma–Yor barrier function $\Psi^{\varsigma}_{F_{a,b}}(X_t)$ for a log-normal target.

Examples

- Assume the investor is restricted to choose from a specified parametrized family of distributions (e.g. Log-normal, Pareto, Weibull, Gamma, . . .)
- Their choice will be optimal for specific parameter constellations
- Specifically, the problem can be rephrased as a mean-variance trade–off, similar to the efficient frontier in Markowitz's model

Examples

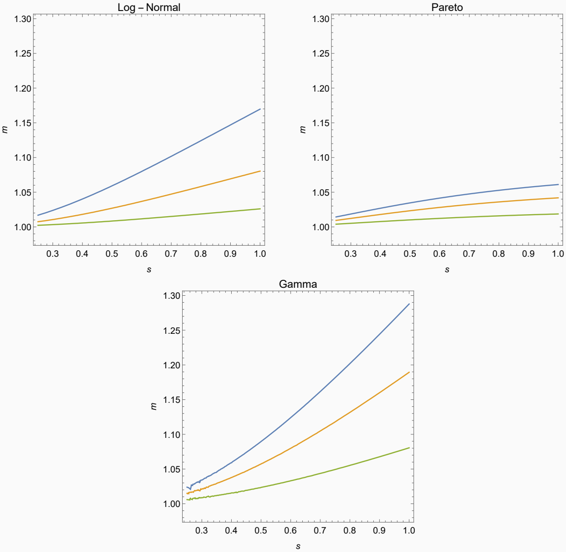

Figure 4: Attainable frontiers for different distributions and $A = 0.2$ (blue), $A = 0.5$ (orange), and $A = 0.8$ (green): Lognormal, Pareto, and Gamma

Example – Log-normal

- Log-normal target distribution, $F_{a,b} \sim \mathcal{LN}(b, a^2)$
- Attainable if $b \leqslant \log(x) - A\frac{a^2}{2}$
- Azéma–Yor barrier for optimal F

$$\Psi^{\mu,\sigma}_{F_{a,b}}(z) = \left(\frac{\Phi\left(\frac{\log(z)-b}{a}\right)}{\Phi\left(\frac{\log(z)-b}{a} + aA\right)} \right)^{\frac{1}{A}} e^{b-\frac{a^2}{2}A}$$

where Φ standard-normal cdf

Conclusion

- We have shown that the distribution builder methodology can successfully harnessed to provide an accessible and implementable preference–based optimization of the timing of an asset sale
- The analysis is based on solutions to the Skorokhod embedding problem for diffusions
- The results seem sensible and in line with economic intuition

Thank You

Thank you, Peter, for all your support and advice. And foremost for all the fun exploring together new and exciting ideas!