Landau damping and Plasma echoes

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Of interest in Plasma Physics:

- The dynamics of charged particles (ions / electrons)

- The classical Vlasov-Poisson system[1] on $\mathbb{T}^d \times \mathbb{R}^d$, $d \geq 1$:

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \qquad \nabla_x \cdot E = \rho - 1$$

for density distribution $f(t,x,v) \geq 0$ and charge density $\rho = \int f \, dv$.

- The neglecting of
  - Boltzmann/Landau collision (short range) $\Longrightarrow$ Vlasov
  - Magnetic effect (Maxwell at $c \to \infty$) $\Longrightarrow$ Poisson
  - The dynamics of ions (mass ratio $m_e/m_i \to 0$) $\Longrightarrow$ Electrons

[1] that is, transport along $\dot{x} = v$, $\dot{v} = E(t,x)$

(in great similarity to 2D Euler....)

- Hamiltonian:

$$\mathcal{H}[f] = \frac{1}{2}\iint |v|^2 f \, dxdv + \frac{1}{2}\int |E|^2 \, dx$$

- Invariant Casimir's:

$$\mathcal{C}[f] = \iint \Phi(f) \, dxdv.$$

(e.g., $\int \rho \, dx = 1$ for all times; perfect for $\nabla_x \cdot E = \rho - 1$ on $\mathbb{T}^d$)

- The global Cauchy problem is classical: e.g., Glassey (SIAM, '96)

• Large time behavior of spatially homogenous equilibria

$$f = \mu(v), \qquad E = 0, \qquad \rho = 1$$

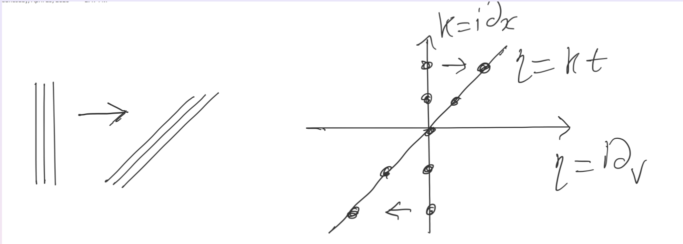
• In the large times,

- Landau damping: decay of the electric field
- Plasma echoes: new waves are excited at later times....

• Linear damping: Landau '46.

• Progress on Nonlinear damping: Mouhot-Villani ('11), Bedrossian-Masmoudi-Mouhot ('16), Bedrossian ('16), Lin Zeng ('11).

- Phase mixing: $\partial_t f + v\partial_x f = 0$.



Explicitly,

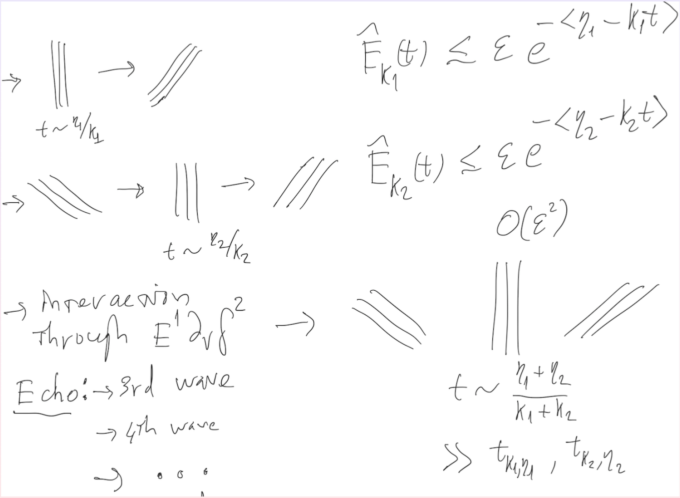
$$f_0(x,v) = g_0(v)e^{ikx+i\eta v} \quad \Longrightarrow \quad f(t,x,v) = g_0(v)e^{ikx+i(\eta-kt)v}$$

$$\rho_0(x) = \widehat{g}_0(\eta)e^{ikx} \quad \Longrightarrow \quad \rho(t,x) = \widehat{g}_0(\eta-kt)e^{ikx}$$

Regularity in v $\Longrightarrow$ Decay of the electric field $E = \frac{1}{ik}\rho$.

(analyticity) $\Longrightarrow$ (exponentially localized near $t = \frac{\eta}{k}$)

- Plasma echoes: $\partial_t f + v\partial_x f + E\partial_v f = 0, \qquad \partial_x E = \rho - 1$



what's going on with 1?

A cascade of echoes?

Actually, data is large in any Sobolev:

$$f^0(x,v) = \mu(v) + f_1^0(v)e^{iK_1x+iL_1v} + f_2^0(v)e^{iK_2x+iL_2v}$$

Observations:

- Electric field is localized near critical times (exponentially, if analytic).
- Echoes "take time" to appear, and so not seen in analytic (or Gevrey) classes studied by Mouhot-Villani.

- Linear Landau damping:

$$\partial_t f + v\partial_x f + E\partial_v \mu = 0, \qquad \partial_x E = \rho,$$

yielding a *closed* equation for density:

$$\widehat{\rho}_k(t) + \int_0^t \widehat{E}_k(s)\widehat{\partial_v \mu}(k(t-s))ds \; = \widehat{f}^0_{k,kt}$$

$$\widehat{\rho}_k(t) + [t\widehat{\mu}(kt)] \star_t \widehat{\rho}_k(t) = \widehat{f}^0_{k,kt} \quad \Longrightarrow \quad \mathcal{L}[\widehat{\rho}_k(t)] = \frac{\mathcal{L}[\widehat{f}^0_{k,kt}]}{1+\mathcal{L}[t\widehat{\mu}(kt)]}$$

assuming Penrose. Further, writing the resolvent kernel:

$$\frac{1}{1+\mathcal{L}[t\widehat{\mu}(kt)]} \quad = \quad 1 - \frac{\mathcal{L}[t\widehat{\mu}(kt)]}{1+\mathcal{L}[t\widehat{\mu}(kt)]} \qquad (1 = \text{phase mixing})$$

Theorem (Linear Landau damping: Grenier-Toan-Rodnianski)

Assume Penrose: $|1 + \mathcal{L}[t\widehat{\mu}(kt)](\lambda)| \gtrsim 1$. *Then,*

$$\widehat{\rho}_k(t) = \widehat{S}_k(t) + \int_0^t \widehat{K}_k(t-s)\widehat{S}_k(s)\,ds, \qquad |\widehat{K}_k(t)| \lesssim e^{-\theta_0|kt|},$$

with $\widehat{S}_k(t) = \widehat{f}^0_{k,kt}$. *Namely, Landau ≈ phase mixing (under Penrose).*

- Linear damping:

$$\widehat{\rho}_k(t) \quad \lesssim \quad \widehat{f}^0_{k,kt} + \cdots \quad \lesssim \begin{cases} \langle kt\rangle^{-\sigma}, & \text{Sobolev data} \\ e^{-\langle kt\rangle^{\gamma}}, & \text{Analytic or Gevrey-}\gamma\text{ data} \end{cases}$$

- Penrose holds for monotone equilibria (or even small bumps in tail), including Gaussians and any radial positive 3D or higher-D equilibria.

- Generator functions:

$$G[g](z) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{z\langle k,\eta\rangle} \Big[|\widehat{g}_{k,\eta}| + |\partial_\eta \widehat{g}_{k,\eta}| \Big] \langle k,\eta\rangle^\sigma \, d\eta$$

for analyticity radius $z \ge 0$ (eventually $z = \lambda(t)$):

- $G[\partial_x g] \le \partial_z G[g]$ and $G[\partial_v g] \le \partial_z G[g]$
- Derving simple transport inequality:

$$\partial_t G[g] \le A[g] + B[g] \partial_z G[g].$$

- As for Gevrey-γ: use L^2 with weight $e^{z\langle k,\eta\rangle^\gamma}$ and integrate by parts.

- Application: Instability of boundary layers, Inviscid limit,...

- Transport equation for $g(t,x,v) = f(t,x+vt,v)$:

$$\partial_t g = -E(t,x+vt)\partial_v \mu(v) - E(t,x+vt)\,(\partial_v - t\partial_x) g.$$

- Transport inequality for norm $G[g](t,z)$:

$$\partial_t G[g] \quad \lesssim \quad F[E] + (1+t)F[E]\;\partial_z G[g]$$

where

$$F[E](t,z) := \sum_{k\in\mathbb{Z}} e^{z\langle k,kt\rangle} |\widehat{E}_k(t)| \langle k,kt\rangle^\sigma.$$

$$\partial_t G[g] \quad \lesssim \quad F[E] + (1+t)F[E]\partial_z G[g]$$

• Extra fast decay on $F[E]$ controls the shrinking of analyticity radius:

$$\lambda'(t)+C_0(1+t)F[E](t,\lambda(t)) \le 0 \quad \Longrightarrow \quad \partial_t G[g](t,\lambda(t)) \lesssim F[E](t,\lambda(t)).$$

Theorem (Nonlinear Landau damping: Mouhot-Villani)

For small Gevrey-3^- initial data near Penrose stable equilibria:

$$E(t,x) \to 0, \qquad f(t,x+vt,v) \to f_\infty(x,v),$$

exponentially fast in $\langle t \rangle^\gamma$.

- Again, a "closed" equation for density,

$$\widehat{E}_k(t) + [t\widehat{\mu}(kt)] \star_t \widehat{E}_k(t) = \widehat{S}_k(t)$$

$$\begin{aligned}
\widehat{S}_k(t) &:= \frac{1}{ik}\widehat{f}^0_{k,kt} - \sum_{l\neq 0}\int_0^t (t-s)\widehat{g}_{k-l,kt-ls}(s)\widehat{E}_l(s)\, ds \\
&\lesssim k^{-1}\widehat{f}^0_{k,kt} + \sum_{l\neq 0}\int_0^t (t-s)\frac{e^{-\lambda(s)\langle k-l,kt-ls\rangle^\gamma}}{\langle kt-ls\rangle^\sigma}\frac{e^{-\lambda(s)\langle l,ls\rangle^\gamma}}{\langle ls\rangle^\sigma} e^{-\langle s\rangle^{\gamma-\delta}}\, ds \\
&\lesssim e^{-\lambda(t)\langle k,kt\rangle^\gamma}\langle kt\rangle^{-\sigma} e^{-\langle t\rangle^{\gamma-\delta}} \qquad \text{(extra decay for } E\text{)}
\end{aligned}$$

Try analyticity radius $\lambda(t) = \lambda_0 + t^{-\delta}$. For s away from t,

$$e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma} \le e^{-\theta_0|t-s|\langle k,kt\rangle^\gamma/t^{1+\delta}} \le e^{-\theta_0'\langle t\rangle^{\gamma-\delta}}.$$

Suppression of echoes: For $|kt - ls| \ll t$,

$$k(t-s) = kt - ls + (l-k)s \quad \gtrsim \quad |l-k|t \quad \gtrsim \quad t$$

and so

$$e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma} \quad \le \quad e^{-\theta_0|k|(t-s)\langle kt\rangle^\gamma/|k|t^{1+\delta}} \le e^{-\theta_0'\langle kt\rangle^\gamma/|k|t^\delta}$$

$$\int_0^t (t-s)\langle kt - ls\rangle^{-\sigma}\, ds \quad \lesssim \quad \frac{t}{k^2} = \frac{|kt|}{k^3} \quad \lesssim \quad \frac{|k|^{\frac{1-\delta}{\gamma-\delta}}}{|k|^3} \lesssim 1$$

provided $3\gamma > 1$, giving Mouhot-Villani's for Gevrey-3^-.

An infinite cascade of echoes:

- Highly oscillatory data:

$$f^0(x,v) = \mu(v) + \sum_{(k,\eta)\in\mathbb{Z}\setminus\{0\}\times\mathbb{Z}} \epsilon f^0_{k,\eta}(v) e^{iKkx+iL\eta v}$$

for large K, L and small ϵ (hence, arbitrarily large in any Sobolev).

- Search for a *complete* echo cascade Ansatz:

$$f(t,x,v) = \mu(v) + \sum_{p=1}^{\infty} \epsilon^p \sum_{(k,\eta)\in\mathbb{Z}\times\mathbb{Z}} f_{k,\eta,p}(t,v) e^{iKkx+i(L\eta-Kkt)v}$$

where $f_{k,\eta,p}(t,v)$ solves the linearized Vlasov-Poisson, inductively in $p \geq 1$.

- Application: a complete instability of boundary layers.

- Assumptions:
  - Echoes times are uniformly bounded:

$$L \lesssim K.$$

  - Coefficients are analytic:

$$|\widehat{f}^0_{k,\eta}(\eta')| \le e^{-2\lambda_0\langle k,\eta,\eta'\rangle}, \qquad \forall\ k, \eta, \eta'.$$

Proposition (Estimates for echoes)

There is some universal constant C_0 so that

$$|\widehat{f}_{k,\eta,p}(t,\eta')| \le C_0^p e^{-\lambda_p(t)\langle k,\eta,p,\eta'\rangle},$$
$$|\widehat{E}_{k,\eta,p}(t)| \le C_0^p e^{-\lambda_p(t)\langle k,\eta,p,L\eta-Kkt\rangle}\langle t\rangle^{-\sigma},$$

where "analyticity" radius $\lambda_p(t)$ is defined by

$$\lambda_p(t) = \lambda_0 + \langle t\rangle^{-\delta} + p^{-\delta}, \qquad \delta \ll 1.$$

Proof.

It follows from Landau damping and "localized" interaction. □

Theorem (An infinite cascade of echoes: Grenier-Toan-Rodnianski)

There exists a complete echo solution to the Vlasov-Poisson system:

- *initial data are arbitrarily large in $W^{s,\infty}$ Sobolev, for any $s > 0$*
- *Landau damping holds for echoes:*

$$E(t,x) = \sum_{(k,\eta,p)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}^\star} \epsilon^p \widehat{f}_{k,\eta,p}(t, Kkt - L\eta)\frac{e^{iKkx}}{iKk} \longrightarrow 0$$

exponentially fast in any Sobolev spaces $W^{s,q}$, $s \geq 0, q \geq 1$.