Random tensors and propagation of randomness under the NLS flow.

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The periodic NLS

$$(i\partial_t + \Delta)\, u = \pm |u|^{p-1}\, u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d. \qquad \text{(NLS)}$$

- (NLS) is an infinite dimensional Hamiltonian system with Hamiltonian

$$H[u](t) := \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2\, dx \pm \frac{1}{p+1}\int_{\mathbb{T}^d} |u|^{p+1}\, \mathrm{d}x = H[u](0).$$

- It also conserves the mass $m(u) := \int_{\mathbb{T}^d} |u|^2 dx$
- Here p is odd; the sign only matters for the global dynamics or Gibbs measure.
- The scaling critical threshold: $\boxed{s_{cr} := \frac{d}{2} - \frac{2}{p-1}}$
- Theorem (Local-in-time well-posdedness for NLS on $\mathbb{T}^d$)

  Assume $s_{cr} \geq 0$, then NLS is LWP for data in H^s provided that $s > s_{cr}$.

  (Bourgain '93, Bourgain-Demeter '15 $\to$ Strichartz estimates on tori.)

- How about $s = s_{cr} = 0 \leftrightarrow L^2$; say cubic NLS on $\mathbb{T}^2$? Unknown!
- If $s < s_{cr}$ ill-posedness may occur (Christ-Colliander-Tao, others ...).

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Question: What happens 'generically'? That is what happens if we randomize the data and study the generic behavior of solutions?

Approach is physically meaningful; linked to the statistical ensemble point of view: instead of individual solutions, are interested in the family of solutions that are distributed according to some canonical law (e.g. Gaussian law).

Deterministic $\longrightarrow$ Nondeterministic $\rightarrow$ Random data

A key point: for random initial data $\longrightarrow$ better linear and nonlinear estimates than those for arbitrary functions of the same regularity $\longleftarrow$ (linear and multilinear) large deviation estimates and other type of random matrix estimates.

Random Data Theory of NLS

Consider NLS with the following canonical random data:

$$u^\omega(0) = f(\omega) = \sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{\langle k \rangle^\alpha} e^{ik\cdot x}, \qquad \boxed{\alpha := s + \frac{d}{2}} \qquad \text{(ID)}$$

where $\{g_k\}$ are i.i.d. complex Gaussian r.v., $\mathbb{E}g_k = 0$, $\mathbb{E}|g_k|^2 = 1$.

The law of distribution for $f(\omega)$ is formally the Gaussian measure:

$$d\rho_\alpha \sim \exp\left(-\|u\|^2_{H^\alpha}\right) \cdot \prod_{x \in \mathbb{T}^d} du(x).$$

This $d\rho_\alpha$ is supported in $H^{s-}(\mathbb{T}^d) := \bigcap_{\varepsilon>0} H^{s-\varepsilon}(\mathbb{T}^d)$.

Almost surely in ω the random initial data $f(\omega)$ belongs to H^{s-} $\;s := \alpha - \frac{d}{2}$.

Almost sure local well-posedness

By switching to almost-sure point of view, we can cross the scaling barrier and get almost sure local well-posedness for values $s < s_{cr}$.

- Historically people only know weak solutions in the supercritical case (no uniqueness, and we know little about this).
- In groundbreaking work, Bourgain '96 showed that if we consider canonical random data in some deterministically supercritical space $s < s_{cr}$ then almost surely one can get strong solutions!.
- these almost sure strong solutions $\longleftrightarrow$ classical well-posed solutions

  measurable functions $\longleftrightarrow$ continuous functions

There have been numerous follow-up works to Bourgain's, but fundamental questions remained open:

- What's the optimal value of s for almost-sure LWP to hold?
- How does a given initial random data get transported by the NLS flow?
  - If it is Gaussian initially, how does this Gaussianity propagate?
  - What's the description of the solution beyond the linear evolution?

Such questions are important in many topics such as:

— invariance of Gibbs measures (statistical mechanics; CQFT; SPDE).

— wave turbulence (density and statistics of the interacting waves).

(Weak) wave turbulence

- Consider NLS with well-prepared data; e.g. random homogeneous data

$$u(0) = f(\omega) = N^{-\alpha} \sum_k \phi(\frac{k}{N}) g_k(\omega) e^{ik\cdot x}, \quad \phi \in \mathcal{S}$$

  i.e. of a single frequency N with Fourier modes uniformly distributed on the ball $|k| \lesssim N$; and study the evolution of the ensemble average

$$\mathbb{E}|\widehat{u}(t,k)|^2, \quad k \in \mathbb{Z}^d$$

  for long times.

- It is believed that this quantity stays constant for a long time, and starts to evolve at a kinetic timescale, according to an equation called "wave kinetic equation".

Buckmaster-Germain-Hani-Shatah; Deng-Hani; Collot-Germain (2018–2020)

Invariant Gibbs measures

- If $\boxed{\alpha = 1}$ we have random data:

$$u(0) = f(\omega) = \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{\langle k\rangle} e^{ik\cdot x}, \qquad \boxed{s := 1 - \frac{d}{2}}$$

  This data represents the typical element in the support of the Gibbs measure associated to NLS.

- Gibbs measures lie nicely at the intersection of PDE and statistical physics[1].

- In 2D and 3D such Gibbs measures are supported on distributions $H^{0-}(\mathbb{T}^2)$ and $H^{-\frac{1}{2}-}(\mathbb{T}^3)$ respectively.

[1] Naturally arise in Stat Mech, CQFT, and as limits of grand canonical thermal states in many-body quantum mechanics(see Frölich-Knowles-Schlein-Sohinger; Lewin-Nam-Rougerie)

- From the Hamiltonian structure of NLS with $\mathcal{H}(u)$ (= Energy) the Gibbs measure can be formally defined as

$$d\mu \sim e^{-\mathcal{H}(u)} \cdot \prod_{x\in\mathbb{T}^d} du(x) \;=\; \underbrace{\exp\left(-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right)}_{\text{weight}} \; d\rho_1$$

- In some cases this measure can be rigorously defined as a weighted Gaussian measure.

- $d\mu$ can be formally seen to be invariant under the flow of NLS due to a formal Liouville's theorem and conservation of the (renormalized) Hamiltonian $\mathcal{H}$.

- Construction -and its properties under various dynamics- is a major problem in statistical mechanics and constructive quantum field theory (intimately related to the so-called Φ^4 model when $p = 3$).

  Glimm-Jaffe, Lebowitz-Rose-Speer, Simon, Nelson, Aizenman, Fröhlich, . . .

Construction

- $d = 1, 2$: construction can be done for any p.
  - Measure is absolutely continuous w.r.t. Gaussian measure
- $d = 3$: construction can be done for $p = 3$.
  - But measure is not absolutely continuous w.r.t. Gaussian measure!
- $d \geq 4$: it cannot be done for any p

  (Frölich, Aizenman, Aizenman- Duminil-Copin, ...).

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Invariance under NLS flow together with existence of global strong solutions on its statistical ensemble.

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Difficulty for dynamics: Gibbs msr. supported in $H^{1-d/2-}$ (rough if $d \geq 2$).

- The invariance of $\mathrm{d}\mu$ as above was justified for
  - $d = 1$, all $p \geq 3$ odd (Bourgain '94).
  - $d = 2$, $p = 3$ (Bourgain '96).
- How about $d = 2$, $p \geq 5$? Open 1996 $\rightarrow$ 2019 (Y. Deng–A.N.–H. Yue)
- How about $d = 3$, $p = 3$? Open, and very hard!

Fundamental questions

- What's the optimal value of s for almost-sure LWP to hold?
- How does a given initial random data get transported by the NLS flow?
  - If it is Gaussian initially, how does this Gaussianity propagate?
  - What's the description of the solution beyond the linear evolution?
- Invariance of Gibbs measure and a.s. existence of strong solutions to NLS on $\mathbb{T}^2$ and any odd power nonlinearity $p \geq 5$.

Question:

Why is $d = 3$, $p = 3$ ($s_{cr} = \frac{1}{2}$) so much harder than $d = 2$, $p = 5$ ($s_{cr} = \frac{1}{2}$)?

Answers

With Yu Deng and Haitian Yue, we answer these questions. We find the optimal value

$$\boxed{s_{pr} := -\frac{1}{p-1}} \leq s_{cr},$$

the critical index in probabilistic scaling, as the threshold for NLS with random data.

[2] Assume $(d, p) \neq (1, 3)$

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In particular:

- We obtain local-in-time strong solutions for random data in the full probabilistic subcritical range $s > s_{pr}$ for any dimension[2] and give a precise description of the solution in terms of multilinear gaussians.

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In particular:

- We obtain local-in-time strong solutions for random data in the full probabilistic subcritical range $s > s_{pr}$ for any dimension[2] and give a precise description of the solution in terms of multilinear gaussians.
- We prove invariance of Gibbs measure and existence of global strong solutions in its statistical ensemble on $\mathbb{T}^2$ for **any** odd power $p \geq 3$.
  - Key: Support of the Gibbs measure $\mathrm{d}\mu$ in 2D is H^{0-}, which is *probabilistically subcritical* for any such p.

[2] Assume $(d, p) \neq (1, 3)$

Probablistic scaling heuristics

The heuristics behind the probabilistic scaling critical exponent s_{pr} are based on the basic idea of the square root cancellation of sums of independent random variables.

For example in Central Limit Theorem, with high probability one has

$$\left|\sum_{k=1}^{N} X_k\right| \lesssim \sqrt{N} \quad \text{instead of } N,$$

where $\{X_k\}$ are i.i.d. mean zero random variables.

We also have the multilinear version, where we consider for example $\sum_{j<k} X_j X_k$

Let us now calculate s_{pr} the threshold for the random data problem for NLS

Probabilistic Scaling

Start with a frequency scale N and random initial data

$$u_0 = f = N^{-\alpha} \sum_{|k|\sim N} g_k(\omega) e^{ik\cdot x} \qquad \alpha = s + \frac{d}{2}$$

then f has unit size in H^s. If NLS is a.s. LWP then the second iteration

$$u^{(1)}(t) = \int_0^t e^{i(t-s)\Delta}(|e^{is\Delta} f|^{p-1} \cdot e^{is\Delta} f)\,\mathrm{d}s$$

should be bounded in H^s for fixed time t.

Fix $|t| \sim 1$, and for $|k| \sim N$ calculate the Fourier coefficients of $u^{(1)}(t)$,

$$\widehat{u^{(1)}}(t,k) \sim N^{-p\alpha} \sum_{k_1 - \cdots + k_p = k} \frac{1}{\langle \Omega \rangle} \cdot g_{k_1}(\omega)\overline{g_{k_2}(\omega)} \cdots g_{k_p}(\omega),$$

where $|k_j| \sim N$, $\;\Omega = |k|^2 - |k_1|^2 + \cdots - |k_p|^2$ is the resonance factor, and $\pm$ represent possible complex conjugates.

For simplicity we may restrict to $\Omega = 0$, reducing to the sum

$$\widehat{u^{(1)}}(t,k) \sim N^{-p\alpha} \sum_{\substack{k_1 - \cdots + k_p = k \\ |k_j| \sim N, \Omega = 0}} g_{k_1}(\omega)\overline{g_{k_2}(\omega)} \cdots g_{k_p}(\omega).$$

If we assume $k_1 \neq k_2$, $k_2 \neq k_3$ and so on, these r.v. are independent. This leads to the square root cancellation (gain). We conclude that

$$|\widehat{u^{(1)}}(t,k)| \sim N^{-p\alpha} \Bigg(\sum_{\substack{k_1 - \cdots + k_p = k \\ |k_j| \sim N, \Omega = 0}} 1 \Bigg)^{1/2} \sim N^{-p\alpha} N^{(pd-d-2)/2}.$$

again by dimension counting.

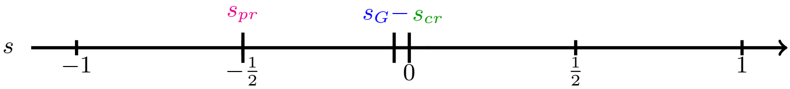
Then, $u^{(1)}(t)$ is bounded in H^s if and only if

$$-p\alpha + \frac{pd-d-2}{2} + \overbrace{s + \frac{d}{2}}^{=\alpha} \leq 0 \Leftrightarrow s \geq -\frac{1}{p-1} := s_{pr}.$$

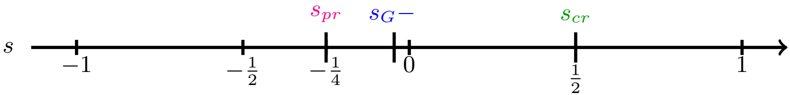
Probabilistic scaling v.s. deterministic scaling

Recall $s_{pr} = -\frac{1}{p-1}$ and let $s_G = 1 - \frac{d}{2}$ Gibbs measure supported in $H^{s_G -}(\mathbb{T}^d)$.

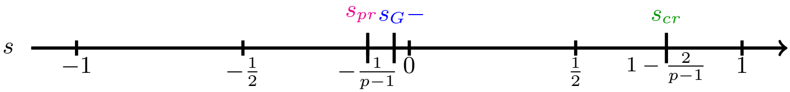
- $d = 2, p = 3$



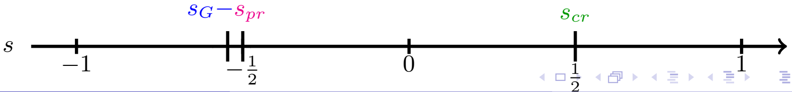
- $d = 2, p = 5$



- $d = 2, p$ large



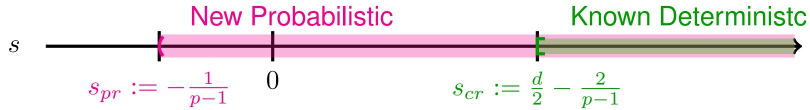
- $d = 3,\ p = 3$



Main result I

Theorem 1 (Y. Deng–A.N.–H. Yue 2020)

Let $p \geq 3$ odd and for $s > -\frac{1}{p-1} = s_{pr}$, and let $\boxed{\alpha = s + \frac{d}{2}}$. Consider (NLS) on $\mathbb{T}^d$ under suitable renormalization, with α-random initial data (ID).
Then almost surely in ω, there exists a strong local solution that is unique in a suitable sense.
Furthermore, this solution has an explicit expansion in terms of multilinear Gaussians with adapted random tensor coefficients.



Bourgain: $\mathbb{T}^2$, $p = 3$ ($s_{cr} = 0$), Prob. LWP for some $s < 0$.

Remarks

- The renormalization we need is the Wick ordering. An infinite L^2 mass implies that the potential energy is almost-surely infinte and the nonlinearity $|u|^{p-1}u$ of (NLS) does not make sense a.s. as distributions. This 'infinity' has to be removed by suitably renormalizing the nonlinearity.
- Uniqueness is in the sense that, our solution is the unique limit for all possible choices of canonical approximations (or regularizations).
- Note one barely misses a.s. LWP for the $d = 3$ cubic NLS in $H^{-\frac{1}{2}-}$ when $s_{pr} = -\frac{1}{2}$.

Main Result II. Long time.

As a byproduct of the proof Th. 1, we have that for smooth well prepared random data (e.g. random data arising in derivation of WKE in wave turbulence theory) the time of existence is longer than the deterministic one; i.e. no energy cascade until a **long time**, longer than deterministic time.

Randomization effectively extends the time of perturbative regime.

Take $p = 3$ (cubic). In the deterministic case, the first energy cascade happens at the timescale of CR equation[3] which is $N^{2(s-s_{cr})}$.

We show that in the randomized case, the first energy cascade only happens at the much later timescale $N^{2(s-s_{pr})-}$

We have:

[3] Faou-Germain-Hani and Buckmaster-Germain-Hani-Shatah

Main result II. Long time

Theorem 2 (Deng-N.–Yue 2020)

Fix (s, α) as before, let N be dyadic and ϕ be Schwartz. Let u solve (NLS) with *random homogeneous data* defined by

$$u(0) = f(\omega) = N^{-\alpha} \sum_k \phi\left(\frac{k}{N}\right) g_k(\omega) e^{ik\cdot x}.$$

Then, with high probability, there is no energy cascade between Fourier modes, i.e. $|\widehat{u}(t,k)|^2 \approx |\widehat{u}(0,k)|^2$ with negligible error for large N, up to the time $T = N^{(p-1)(s-s_{pr})-}$.

With high probability, $\|u(0)\|_{H^s} \sim 1$.

Main result III. Invariance and global strong solutions.

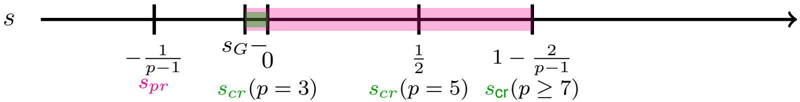
Our third main result (proved before Theorem 1) is

Theorem 3 (Deng-N.–Yue 2019)

Let $d = 2$ and $p \geq 3$ odd. Then the renormalized NLS is almost surely globally well-posed on the support of the Gibbs measure $\mathrm{d}\mu$ (which is in H^{0-}). The global nonlinear flow Φ_t maps a full measure set Σ to itself, forms a one-parameter group (i.e. $\Phi_{t+s} = \Phi_t \Phi_s$), and keeps the Gibbs measure $\mathrm{d}\mu$ invariant under the flow:

$$\mu(E) = \mu(\Phi_t(E))$$

for any Borel set $E \subset \Sigma$.



Remarks

- The renormalization involved is also the Wick ordering, and the uniqueness of the solution is in the sense of Theorem 1.
- Weak solutions (with no uniqueness) that preserve $\mathrm{d}\mu$ (in some sense) were previously obtained by Oh-Thomann.
- The only case where the measure is constructed but not known to be invariant is $d = 3$, $p = 3$. We expect this to be much harder than $d = 2$, as this is **critical** under the probabilistic scaling

Bourgain's method ('96)

Before describing our new methods let us review the existing approach thus far for studying random data local well-posedness due to Bourgain.

Bourgain considered the cubic (Wick ordered) NLS equation on $\mathbb{T}^2$

$$iu_t + \Delta u = \underbrace{|u|^2 u - 2u\left(\int |u|^2 dx\right)}_{\mathcal{C}(u)}$$

with random initial data:

$$u_0^\omega(x) := \sum_{k \in \mathbb{Z}^2} \frac{g_k(\omega)}{\langle k \rangle} e^{ik\cdot x}, \qquad x \in \mathbb{T}^2.$$

The initial data (in support of the Gibbs measure) is in H^{0-}, and is thus is deterministically supercritical but probabilistic subcritical:

$$s_{pr} = -\frac{1}{4} < 0- < 0 = s_{cr}.$$

Then

- Bourgain's main idea is to make a linear-nonlinear decomposition, where the linear part is **rough** and **random**, and the nonlinear part is smoother.
- He constructed solutions of form $\boxed{u = e^{it\Delta} f(\omega) + v}$ and showed v has **positive** regularity.
- Idea: Solve the difference initial value problem via a Banach fixed point argument on a ball in **a smoother space**:

$$\begin{cases} iv_t + \Delta v = \mathcal{C}(\underbrace{e^{it\Delta} f(\omega)}_{\text{R:=rough-random}} + \underbrace{v}_{\text{smoother-deterministic=:D}}) \\ v(x,0) = 0, \qquad x \in \mathbb{T}^2 \end{cases}$$

Tools:

- multilinear large deviation estimates
- integer lattice counting estimates $\leftrightarrow$ analytic number theory
- TT^* arguments $\leftrightarrow$ random matrix estimates (correct way to exploit randomness in absence of gain of regularity).

The theory of random tensors

- We now focus on the proof of Theorem I which relies on the theory of random tensors. For simplicity here we assume $p = 3$.

- Theorem II is a special case.

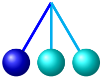
- To prove Theorem III -which precedes Theorems I, II– we introduced the method of random averaging operators (RAO) which lays out the foundation for the more general random tensors' theory (RTT)

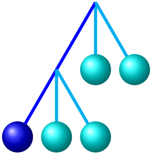
The theory of random tensors

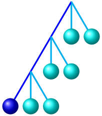
- We now focus on the proof of Theorem I which relies on the theory of random tensors. For simplicity here we assume $p = 3$.
- Theorem II is a special case.
- To prove Theorem III -which precedes Theorems I, II– we introduced the method of random averaging operators (RAO) which lays out the foundation for the more general random tensors' theory (RTT)
- RAO has a much simpler form than random tensors and less notation-heavy, and suffices in many cases where one is not too close to probabilistic criticality.
- We don't discuss the RAO in detail, but they come in the RTT as they are the simplest and basic cases of random tensors $\rightarrow$ the $(1, 1)$-tensors.

Simple example (p=3): (1,1) tensor terms

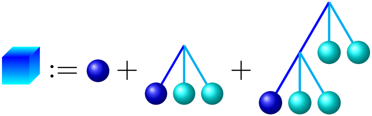
Let $\mathcal{I}$ = Duhamel operator. Denote  $:= e^{it\Delta} f_N(\omega)$ and  $:= u_{N^\delta}$. Naturally we can also define

 $:= \mathcal{I}\mathcal{C}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}),$

 $:= \mathcal{I}\mathcal{C}(\mathcal{I}\mathcal{C}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}), u_{N^\delta}, u_{N^\delta}),$

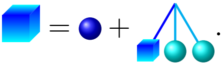
and so on  $, \cdots$.

The sum of these trees forms on an infinite series of trees:

$\Psi_{N,N^\delta} =$  $+ \cdots .$

which is equivalent to the para-linearized equation:

$$\begin{cases} (i\partial_t + \Delta)\Psi_{N,N^\delta} = \mathcal{C}(\Psi_{N,N^\delta}, u_{N^\delta}, u_{N^\delta}); \\ \Psi_{N,N^\delta}(0) = f_N(\omega). \end{cases} \iff$$



By solving this equation, we have that the k-th Fourier mode of Ψ_{N,N^δ} is in the following form:

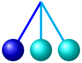
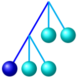
$\mathcal{F}($  $)(k) = \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha}$

where h_{kk_1} is the $(1,1)$ random tensor (matrix); indep. of $g_{k_1}(\omega)$.

Random averaging operators $\to$ Gibbs measure

Ansatz for the solution is

$$u = \sum_N \Psi_{N,N^\delta} + \text{remainder} = \sum_N$$  $$+ \text{ remainder}$$

$$= u_{\text{lin}} + \Big(\sum_N$$  $$+$$  $$+ \cdots \Big) + \text{remainder}$$

$$= u_{\text{lin}} + \mathcal{P}(u_{\text{lin}}) + \text{remainder}$$

Idea: We para-linearize NLS and view the key high-low interactions where the high frequencies come from u_{lin} as a linear operator $\mathcal{P}$ applied to u_{lin}. We expand the solution u in Fourier space, where $u_k(t) := \widehat{u}(t,k)$, as

$$u_k(t) = \frac{g_k(\omega)}{\langle k \rangle^\alpha} + \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha} + (\text{remainder})_k \qquad \text{(RAO)}$$

where h_{kk_1} is the $(1,1)$ random tensor (matrix) independent from g_{k_1} and containing all the randomness information of the low frequency components of the solution u and prove suitable operator norm estimates for h_{kk_1}.

Higher order random tensors: how they arise.

To prove Result 1 we need arbitrary high order expansions $\leftrightarrow$ multilinear expressions $\leftrightarrow$ random $(q, 1)$ tensors $h = h_{kk_1\cdots k_q}$, which depend on the low frequency components of the solution.

Ansatz: expand the solution u to NLS in Fourier space; and write $u_k(t) := \widehat{u}(t, k)$ as

$$\boxed{u_k(t) = \sum_q \sum_{k_1,\cdots,k_q} h_{kk_1\cdots k_q}(t) \prod_{j=1}^{q} \frac{g_{k_j}^{\pm}(\omega)}{\langle k_j \rangle^{\alpha}} + (\text{remainder})_k} \qquad \text{(EXP)}$$

where $(g^+, g^-) := (g, \overline{g})$ and assume there is no *pairing*, i.e. $k_{j'} \neq k_j$ if the corresp. $\pm$ signs are the opposite in the given q-tuple.

Higher order random tensors: how they arise.

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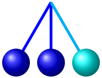
- These quantities $h_{kk_1\cdots k_q}$, where t is viewed as a parameter, are the random tensors which will be the main subject of study
- The convergence of the expansion (EXP) is completely determined by the properties and estimates of these tensors.
- The high-order tensors $h_{kk_1\cdots k_q}$ are from the high-order iteration trees, as in the following examples:

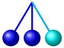
Simple example (p=3): (2,1) tensor term

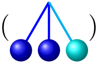
- The $(2,1)$ tensors are from the multilinear expressions satisfying:
  - There are 2 terminal leaves which are Gaussian term $e^{it\Delta} f_N(\omega)$;
  - The other terminal leaves are low frequency components u_{N^δ}.
- Denote



$$:= \mathcal{I}\mathcal{C}(e^{it\Delta} f_{N_1}(\omega),\, e^{it\Delta} f_{N_2}(\omega),\, u_{N^\delta})$$

where $\mathcal{I}$ is the Duhamel operator, $N = \max(N_1, N_2)$ and $N_1, N_2 > N^\delta$.
Then (modulo details about the temporal frequency):

The k-th Fourier mode of  is

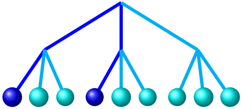
$\mathcal{F}($  $)(k) \sim \sum_{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2}} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha} \frac{\overline{g_{k_2}(\omega)}}{\langle k_2 \rangle^\alpha} \widehat{u}(k_3)$

$$= \sum_{k_1,k_2} \underbrace{\left(\sum_{|k_3| \leq N^\delta} \mathbf{1}_{\left\{ \substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2} \right\}} \widehat{u}(k_3) \right)}_{h_{kk_1k_2}} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha} \frac{\overline{g_{k_2}(\omega)}}{\langle k_2 \rangle^\alpha}$$

where $|k_1| \sim N_1$, $|k_2| \sim N_2$ and $|k_3| \leq N^\delta$. Note that here $h_{kk_1k_2}$ is a (2,1) random tensor -say- maps $k_1, k_2 \to k$.

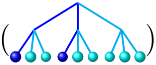
One more example for (2,1) tensor term

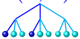
$$\mathcal{IC}\big(\mathcal{IC}(e^{it\Delta}f_{N_a}, u_{N^\delta}, u_{N^\delta}), \mathcal{IC}(e^{it\Delta}f_{N_b}, u_{N^\delta}, u_{N^\delta}), \mathcal{IC}(u_{N^\delta}, u_{N^\delta}, u_{N^\delta})\big)$$

$:=$ 

where $N = \max(N_a, N_b)$ and $N_a, N_b > N^\delta$.

- Similarly

$\mathcal{F}\big($  $\big)(k) = \sum_{\substack{|a|\sim N_a \\ |b|\sim N_b}} h_{kab} \cdot \frac{g_a(\omega)}{\langle a\rangle^\alpha} \frac{\overline{g_b(\omega)}}{\langle b\rangle^\alpha}$

- Note that here h_{kab} is a $(2,1)$ random tensor (which maps $a,\, b \to k$) associated to the term .

In our previous slide we denoted by $a = k_{11}$ and $b = k_{21}$. Then

$$h_{kab} = h_{kk_{11}k_{21}} \;=\; \Big(\sum_{\substack{k_{12},k_{13},k_{22},k_{23},\\ k_{31},k_{32},k_{33}}} \mathbf{1}_{(\star)_{kk_{11}k_{21}}} \prod_{\substack{\mathfrak{l}\in\{12,13,22,\\ 23,31,32,33\}}} \widehat{u}(k_{\mathfrak{l}}) \Big)$$

where

$$\begin{aligned}(\star)_{kk_{11}k_{21}} := \{(k_{12},k_{13},k_{22},k_{23},k_{31},k_{32},k_{33}) : |k_{\mathfrak{l}}| \le N^{\delta}, \mathfrak{l} \in \{12,13,22,23,31,32,33\}\\ k = (k_{11}-k_{12}+k_{13}) - (k_{21}-k_{22}+k_{23}) + (k_{31}-k_{32}+k_{33})\\ |k|^2 = (|k_{11}|^2 - |k_{12}|^2 + |k_{13}|^2) - (|k_{21}|^2 - |k_{22}|^2 + |k_{23}|^2) + (|k_{31}|^2 - |k_{32}|^2 + |k_{33}|^2)\}\end{aligned}$$

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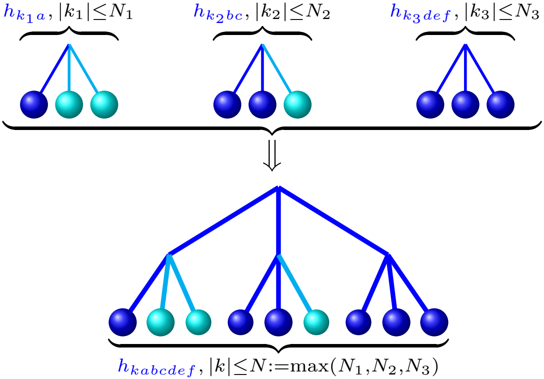
- We also develop the analytic theory, which entails choosing suitable norms for the tensors $h = h_{kk_1\ldots k_q}$ which behave well with our algebraic theory. We prove several multilinear estimates to provide suitable bounds for the (merged and trimmed) tensors and remainder in (EXP).

Algebraic Theory: Merging I

- No pairing case: $\{a, c, d, f\} \cap \{b, e\} = \varnothing$



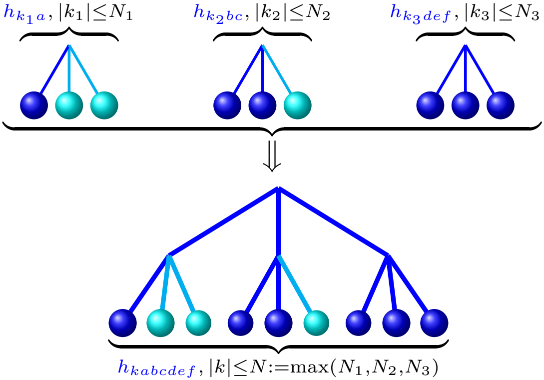
- $\mathcal{IC}(\text{“}(1,1)\text{”}, \text{“}(2,1)\text{”}, \text{“}(3,1)\text{”})$ yields the $(6,1)$ tensor:

$$h_{kabcdef} = \sum_{k_1, k_2, k_3} \underbrace{(h_{\mathrm{b}})_{kk_1k_2k_3}}_{\text{base tensor}=} \cdot h_{k_1 a}\, h_{k_2 bc}\, h_{k_3 def}.$$

NLS nonlinearity is a multilinear form thus represented by an explicit constant (in ω)tensor.

Algebraic Theory: Merging II

- Pairing case: $\{a, c, d, f\} \cap \{b, e\} \neq \varnothing,\ a = b.$

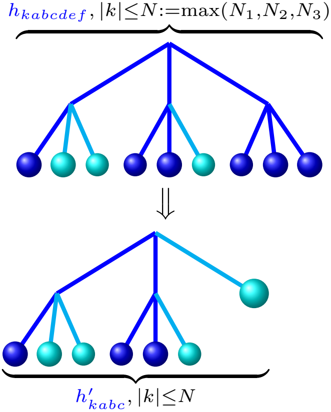


- The corresponding $(4, 1)$ tensor in this case is:

$$h_{kcdef} = \sum_{a=b} h_{kabcdef} \frac{|g_a(\omega)|^2}{\langle a \rangle^{2\alpha}}.$$

Algebraic Theory: Trimming

For example, in the no pairing case if -say- after merging $N_3 < N^\delta$ and $N_1, N_2 \geq N^\delta$, we need to trim the tree to guarantee independence.



where

$$h'_{kabc} := \sum_{d,e,f} h_{kabcdef} \frac{g_d(\omega)}{\langle d \rangle^\alpha} \frac{\overline{g_e(\omega)}}{\langle e \rangle^\alpha} \frac{g_f(\omega)}{\langle f \rangle^\alpha}$$

Analytic theory

- A crucial component of the RTT is the choice of norms for the tensors $h = h_{kk_1 \dots k_q}$.
- It turns out that the suitable norms here are the operator norms when the tensors are viewed as linear mappings from a function of part of the variables to a function of the remaining variables.
- For example for a tensor $h = h_{kxyz}$ we define

$$\|h\|^2_{kx \to yz} := \sup \left\{ \sum_{y,z} \left| \sum_{k,x} h_{kxyz} \cdot z_{kx} \right|^2 : \sum_{k,x} |z_{kx}|^2 = 1 \right\}$$

- In some instances, we just use the ℓ^2 norm of h in all its variables (eg. Hilbert-Schmidt norm), for example for $h = h_{ab}$, we have

$$\|h\|^2_{ab} = \sum_{a,b} |h_{ab}|^2.$$

Analytic theory

- Only such norms are needed in the proof. Importantly, they behave well with the algebraic process of merging and trimming.
- For example, for merged tensors

$$\mathfrak{h}_{bczw} = \sum_{a,e,f} (h^1)_{abc} (h^2)_{aef} (h^3)_{efzw}$$

we have the following multilinear estimate

$$\|\mathfrak{h}\|_{bz \to cw} \leq \|h^1\|_{ab \to c} \|h^2\|_{ef \to a} \|h^3\|_{z \to wef}$$

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So we actually get a set of inequalities by reordering the tensors, from which we may choose at our disposal.

- Similarly we have general estimates for trimmed tensors assuming suitable independence. For example:

Random Matrices Estimates

Suppose that

$$h'_{kxz} = \sum_{yw} h_{kxyzw} \cdot g_y(\omega)\overline{g_w(\omega)},$$

where the random tensor $h = h_{kxyzw}$ is independent with g_y and g_w.
Then with high probability we have that

$$\|h'\|_{kx \to z} \lesssim N^{\varepsilon} \max\big(\|h\|_{kxyw \to z}, \|h\|_{kxy \to zw}, \|h\|_{kxw \to zy}, \|h\|_{kx \to zyw}\big)$$

where N is the max size of $kxyzw$, and $\varepsilon > 0$ is arbitrarily small.

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Even in the simplest of cases it seems nontrivial to find a direct proof! We think these elegant inequalities are of independent interest in the study of random matrices with general Gaussian entries.

To conclude

Armed with both the algebraic and analytic theory of the RTT we can go back to NLS and analyze the norms of the tensors appearing in

$$u_k(t) = \sum_q \sum_{k_1,\cdots,k_q} h_{kk_1\cdots k_q}(t) \prod_{j=1}^{q} \frac{g_{k_j}^{\pm}(\omega)}{\langle k_j \rangle^{\alpha}} + (\text{remainder})_k \qquad \text{(EXP)}$$

These bounds are in fact quite simple. We aim at proving essentially that

$$\|h_{kk_1\ldots k_q}\|_{kk_1\ldots k_r \to k_{r+1}\ldots k_q} \lesssim \prod_{j=1}^{q} N_j^{\beta} \left(\max_{r+1 \le j \le q} N_j \right)^{-\beta} \qquad \text{(TB)}$$

for any r, where $\langle k_j \rangle \sim N_j$ and $\beta \equiv \alpha -$.

Moreover we prove a Fourier weighted estimate that localizes h as a multilinear Fourier multiplier; i.e. in the support of $h_{kk_1\ldots k_q}$ we have that

$$k \approx \pm k_1 \cdots \pm k_q$$

The proof of (TB) is done by induction. The first main ingredient is of course the random tensor framework descried above. The other ingredients are:

- The base tensor estimate: to bound its norm which appears in all merging estimates we simply apply Schur's test and reduce matters to integer lattice *counting estimates* which follow from elementary number theory.

- The selection algorithm: this is needed in the inductive step where we need to control the norms of the merged tensors. Here a delicate *selection algorithm* is needed in order to exploit the flexibility (order in which we estimate) in the multilinear merging estimates.

Localization hyperplanes of $(n, 1)$ tensors

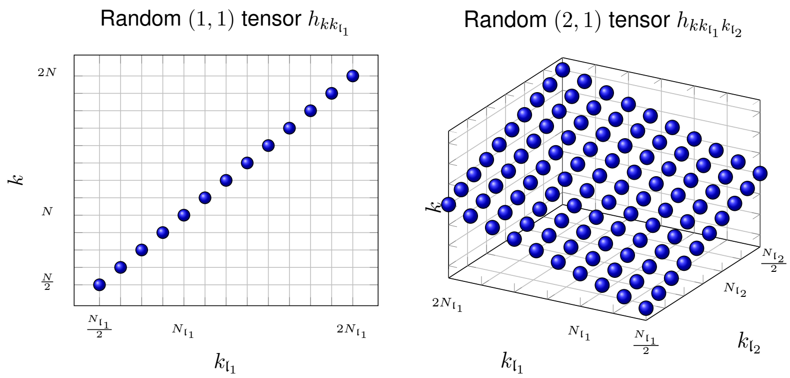

Figure: Localization hyperplanes of random $(1, 1)$ and $(2, 1)$ tensors

Many thanks for your attention!!