Wellposedness and decay rates for the Cauchy problem of the Jordan–Moore–Gibson–Thompson equation

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Acoustic is concerned with the generation and space-time evolution of small mechanical perturbation in fluid (sound waves) or in solid (elastic waves). One of the important equations in acoustics is the Kuznetsov equation

$$u_{tt} - a^2 \Delta u - b \Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right). \tag{1}$$

The derivation of equation (1) can be obtained from the general equations of fluid mechanics. The equation of conservation of mass (continuity):

$$\varrho_t + \nabla \cdot (\varrho v) = 0. \tag{2}$$

The equation of conservation of momentum (Newton's second law)

$$\varrho(v_t + (v \cdot \nabla)v) = \nabla \cdot \mathbb{T}. \tag{3}$$

Conservation of energy (first law of thermodynamics)

$$\varrho\theta(\eta_t + (v \cdot \nabla)\eta) = -\nabla \cdot q + \mathbb{T} : \mathbb{D}. \tag{4}$$

- $\mathbb{D}$ is the deformation tensor given by $\mathbb{D} = \frac{1}{2}(\nabla v + (\nabla v)^T)$,
- $\mathbb{T}$ is the Cauchy–Poisson stress tensor given by $\mathbb{T} = (-p + \lambda(\nabla \cdot v))\mathbb{I} + 2\mu\mathbb{D}$, where $\mathbb{I}$ is the identity matrix.
- μ is the shear viscosity (the first coefficient of viscosity) $\lambda = \zeta - \frac{2}{3}\mu$, where ζ is the second coefficient of viscosity (the bulk viscosity).
- $\mathbb{T} : \mathbb{D} = \sum_{ij} T_{ij} D_{ij}$ where T_{ij} are the components of $\mathbb{T}$ and D_{ij} are the components of $\mathbb{D}$.

Equations (3) and (4) can be rewritten as

$$\varrho(v_t + (u \cdot \nabla)u) = -\nabla p + \mu\Delta u + (\zeta + \mu/3)\nabla(\nabla \cdot u). \tag{5}$$

and

$$\varrho\theta(\eta_t + (v \cdot \nabla)\eta) = 2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot v)^2 - \nabla \cdot q, \tag{6}$$

respectively.

The equation of state is

$$p = p(\varrho, \eta). \tag{7}$$

First, we assume that the deviation of ϱ, p, η and θ from their equilibrium values ϱ_0, p_0, η_0 and θ_0 are assumed to be small.

By taking the Taylor series expansion of (7) around values at rest ϱ_0 and η_0 and ignoring the higher order terms, we get

$$p(\varrho,\eta) = p(\varrho_0,\eta_0) + \left(\frac{\partial p}{\partial \varrho}(\varrho_0,\eta_0)\right)(\varrho - \varrho_0) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \varrho^2}(\varrho_0,\eta_0)\right)(\varrho - \varrho_0)^2 + \left(\frac{\partial p}{\partial \eta}(\varrho_0,\eta_0)\right)(\eta - \eta_0).$$

We put

$$p_0 = p(\varrho_0,\eta_0), \quad A = \varrho_0\frac{\partial p}{\partial \varrho}(\varrho_0,\eta_0) = \varrho_0 c^2, \quad B = \varrho_0^2\frac{\partial^2 p}{\partial \varrho^2}(\varrho_0,\eta_0), \quad \varrho_0\frac{\gamma - 1}{\chi} = \frac{\partial p}{\partial \eta}(\varrho_0,\eta_0),$$

then, the pressure p is given by

$$p(\varrho,\eta) = p_0 + \varrho_0 c^2\left[\frac{\varrho - \varrho_0}{\varrho_0} + \frac{B}{2A}\left(\frac{\varrho - \varrho_0}{\varrho_0}\right)^2 + \frac{\gamma - 1}{\chi c^2}(\eta - \eta_0)\right], \tag{8}$$

where $\nabla p_0 = 0$.

In the above equations,

- v is the acoustic particle velocity,
- p is the acoustic pressure,
- ϱ is the mass density,
- η is the specific entropy,
- q is the heat flux,
- θ is the absolute temperature,
- K is the thermal conductivity,
- c is the speed of sound,
- χ the coefficient of volume expansion,
- $\gamma = c_p/c_v$ is the ratio of specific heat,
- c_p and c_v are the specific heat capacities at constant pressure and constant volume

By assuming that the flow is rotation free, $\nabla \times v = 0$ and introducing the acoustic velocity potential $v = -\nabla u$, then it has been shown by (Kuznetsov, 1971) and (Coulouvrat, 1992), that equation (1), can be derived from the above set of equations by assuming the Fourier law of heat conduction

$$q = -K\nabla\theta. \tag{9}$$

Some results on the Kuznetsov equation

We consider Kuznetsov equation

$$u_{tt} - a^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right).$$

- $b > 0$ corresponds to the viscous case.
  - K. Mizohata and S. Ukai (1993) considered the Kuznetsov equation in $\mathbb{R}^N$ and showed the global existence of small amplitude solution.

    No decay rate was given.
  - B. Kaltenbacher and I. Lasiecka, (2012) proved the global existence and exponential decay of the solution in bounded domain.
- $b = 0$ corresponds to the inviscid case.
  - A. Dekkers and A. Rozanova-Pierrat (2019).

It is known that by modeling heat conduction with the Fourier law, leads to the paradox of infinite heat propagation speed. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. To overcome this drawback, we may replace (9) with the Maxwell–Cattaneo heat conduction law:

$$\tau q_t + q + K\nabla\theta = 0, \qquad (\tau > 0,\ \text{relatively small}) \tag{10}$$

By considering (10), instead of (9) and combining it with the equations of fluid mechanics, we get, instead of (1), the equation **(P.M. Jordan, 2014)** the Jordan–Moore–Gibson–Thompson equation

$$\tau u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right) \tag{11}$$

where $b = \delta + \tau c^2$, with δ is the diffusivity of sound.
The linearized version of equation (58a), known as the Moore–Gibson–Thompson equation in the acoustics theory:

$$\tau u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = 0. \tag{12}$$

- The linearized equation
  - B. Kaltenbacher, I. Lasiecka, and R. Marchand (2011)
  - B. Kaltenbacher, I. Lasiecka, and M. K Pospieszalska (2012)).
    1. $b = 0$ there arises a lack of existence of a semigroup associated with the linear dynamics.
    2. $b > 0$ the linear dynamics are described by a strongly continuous semigroup, which is exponentially stable provided that $\gamma = \alpha - \tau c^2/b > 0$.
    3. If $\gamma = 0$ the energy is conserved.
- The nonlinear problem
  - B. Kaltenbacher, I. Lasiecka, and M. K Pospieszalska (2012)) considered the nonlinear model (known as the Westervelt equation):

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\left(1 + \frac{1}{c^2}\right)\frac{B}{2A}(u_t)^2\right) \tag{13}$$

    and showed the global existence of small data solution with an expential decay rate

  - B. Kaltenbacher and V. Nikolić, (2019) They rigorously justified the singular limit problem when $\tau \to 0$ and showed that the limit of JMGT equation as $\tau \to 0$ leads to the Kuznetsov equation.

The Cauchy problem–The linearized equation

In bounded domain Poincaré's inequality is applicable and the derivation of the estimates is much easier.

So, our goal is to show the well-posedness and investigate the decay rate of the solutions of the Moore–Gibson–Thompson equation in an unbounded domain. Namely, we consider the equation

$$\tau u_{ttt} + \underbrace{u_{tt} - c^2\Delta u - c^2\beta\Delta u_t}_{\text{Strongly damped wave equation}} = 0 \quad \text{in } \mathbb{R}^N, \quad t > 0, \tag{14}$$

with the following initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x). \tag{15}$$

For the linear strongly damed wave equation:

- D. Hoff and K. Zumbrum (1997)
- Y. Shibata, 2000 investigate the $L^p - L^q$ decay estimate of solutions to the Cauchy problem.

Well-posedness of problem for $0 < \tau < \beta$

First, we write problem (14)-(15) as a first-order evolution equation. By taking $v = u_t$ and $w = u_{tt}$, this problem can be reduced to

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}U(t), \quad t \in [0, +\infty) \\ U(0) = U_0 \end{cases} \tag{16}$$

where $U(t) = (u, v, w)^T$, $U_0 = (u_0, u_1, u_2)^T$ and $\mathcal{A} : \mathbb{D}(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the following linear operator

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ \frac{1}{\tau}\Delta(u + \beta v) - \frac{1}{\tau}w \end{pmatrix}$$

(we recall that we have taken $c = 1$ in (14)).

We introduce the energy space $\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with the following inner product

$$\begin{aligned}
&\langle (u, v, w), \, (u_1, v_1, w_1) \rangle_{\mathcal{H}} \\
&= \tau(\beta - \tau) \int_{\mathbb{R}^N} \nabla v \cdot \nabla \overline{v}_1 \, dx + \int_{\mathbb{R}^N} \nabla(u + \tau v) \cdot \nabla(\overline{u}_1 + \tau \overline{v}_1) \, dx \\
&+ \int_{\mathbb{R}^N} (v + \tau w) \cdot (\overline{v}_1 + \tau \overline{w}_1) \, dx + \int_{\mathbb{R}^N} (u + \tau v) \cdot (\overline{u}_1 + \tau \overline{v}_1) \, dx \\
&+ \int_{\mathbb{R}^N} v \cdot \overline{v}_1 \, dx
\end{aligned}$$

and the corresponding norm
We consider (16) in the Hilbert space $\mathcal{H}$, with the following domain

$$\mathbb{D}(\mathcal{A}) = \left\{ (u, v, w) \in \mathcal{H}; w \in H^1(\mathbb{R}^N), u + \beta v \in H^2(\mathbb{R}^N) \right\}. \tag{17}$$

Theorem

Under the dissipative condition $0 < \tau < \beta$, *the operator* $\mathcal{A}$ *generates a* C_0*-semigroup on* $\mathcal{H}$. *In particular, for any* $U_0 \in \mathbb{D}(\mathcal{A})$, *there exists a unique function*

$$U \in C^1([0, +\infty); \mathcal{H}) \cap C([0, +\infty); \mathbb{D}(\mathcal{A}))$$

satisfying (16).

Proof.

Instead of considering our problem (16), we now consider the perturbed problem

$$\begin{cases} \dfrac{d}{dt} U(t) = \mathcal{A}_B U(t), \quad t \in [0, +\infty) \\ U(0) = U_0 \end{cases} \tag{18}$$

where

$$\mathcal{A}_B \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (\mathcal{A} + B) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ \frac{1}{\tau}\Delta(u + \beta v) - \frac{1}{\tau} w - \frac{1}{\tau} u - v - \frac{1}{\tau^2} v \end{pmatrix}.$$

If $\mathcal{A}_B$ generates a C_0-semigroup on $\mathcal{H}$. As $\mathcal{A}_B$ is a bounded perturbation of $\mathcal{A}$, standard semigroup theory allows us to say that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$ □

Decay of the solution

Theorem (M. Pellicer & B. Said-Houari, 2019)

Let u be the solution of (14)-(15), with initial condition $U_0 = (u_0, u_1, u_2) \in \mathbb{D}(\mathcal{A}^s)$ for $s \geq 1$. Assume that $0 < \tau < \beta$. Let $V = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$ and assume in addition that $V_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Then, for all $0 \leq j \leq s$, we have

$$\|\nabla^j V(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-N/4-j/2}\|V_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^j V_0\|_{L^2(\mathbb{R}^N)}. \tag{19}$$

- No decay rate is given for u. In fact we proved that

$$\|u(t)\|_{L^2} \lesssim (1+t)^{1-N/4}. \tag{20}$$

- Slow decay rate of $\|\nabla u_t\|_{L^2} \lesssim (1+t)^{-\frac{N}{4}-\cancel{\frac{3}{2}}}$

Ideas of the proof

- The energy method in the Fourier space,
- The Hausdorff–Young inequality:

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}, \qquad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

In particular $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ (Plancherel's identity)

Taking the Fourier transform of equation (14) and the initial data (15). We then obtain the following ODE initial value problem:

$$\tau \hat{u}_{ttt} + \hat{u}_{tt} + |\xi|^2 \hat{u} + \beta|\xi|^2 \hat{u}_t = 0 \tag{21}$$

with $\xi \in \mathbb{R}^N$. As in (16), after introducing the new variables $\hat{v} = \hat{u}_t$ and $\hat{w} = \hat{u}_{tt}$ the previous ODE can be rewritten as the following first order system

$$\begin{cases} \hat{u}_t = \hat{v}, \\ \hat{v}_t = \hat{w}, \\ \hat{w}_t = -\dfrac{|\xi|^2}{\tau}\hat{u} - \dfrac{\beta|\xi|^2}{\tau}\hat{v} - \dfrac{1}{\tau}\hat{w}. \end{cases} \tag{22}$$

We can write the previous system in a matrix form as

$$\hat{U}_t(\xi, t) = \Phi(\xi)\hat{U}(\xi, t), \tag{23}$$

with the initial data

$$\hat{U}_0(\xi) = \hat{U}(\xi, 0),$$

where $\hat{U}(\xi, t) = (\hat{u}(\xi, t), \hat{v}(\xi, t), \hat{w}(\xi, t))^T$ and

$$\Phi(\xi) = L + |\xi|^2 A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\dfrac{1}{\tau} \end{pmatrix} + |\xi|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\dfrac{1}{\tau} & -\dfrac{\beta}{\tau} & 0 \end{pmatrix}. \tag{24}$$

Now we define the vector $V = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$. Thus, the pointwise estimate of the Fourier image of V reads as follows

Proposition

Let $\hat{u}$ be the solution of (21)*. Assume that* $0 < \tau < \beta$*. Then, the Fourier image of the above vector* V *satisfies the estimate*

$$|\hat{V}(\xi, t)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{V}(\xi, 0)|^2, \tag{25}$$

for all $t \geq 0$ *and certain* $c, C > 0$*, where*

$$\rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}. \tag{26}$$

Proof of Theorem 2

First, observe that

$$\rho(\xi) \geq \begin{cases} c|\xi|^2, & \text{if } |\xi| \leq 1, \\ c, & \text{if } |\xi| \geq 1. \end{cases} \tag{27}$$

Applying the Plancherel theorem, we obtain

$$\begin{aligned} \left\| \nabla_x^j V(t) \right\|_{L^2(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{V}(\xi,t)|^2 d\xi \\ &\leq C \int_{\mathbb{R}^N} |\xi|^{2j} e^{-c\rho(\xi)t} |\hat{V}(\xi,0)|^2 d\xi \\ &= C \int_{|\xi|\leq 1} |\xi|^{2j} e^{-c\rho(\xi)t} |\hat{V}(\xi,0)|^2 d\xi + C \int_{|\xi|\geq 1} |\xi|^{2j} e^{-c\rho(\xi)t} |\hat{V}(\xi,0)|^2 d\xi \\ &=: I_1 + I_2. \end{aligned} \tag{28}$$

Exploiting (27), we infer that

$$I_1 \leq C \|\hat{V}_0\|_{L^\infty(\mathbb{R}^N)}^2 \int_{|\xi|\leq 1} |\xi|^{2j} e^{-c|\xi|^2 t} d\xi \leq C(1+t)^{-N/2-j} \|V_0\|_{L^1(\mathbb{R}^N)}^2, \tag{29}$$

In the high-frequency region ($|\xi| \geq 1$), we have

$$I_2 \leq Ce^{-ct} \int_{|\xi|\geq 1} |\xi|^{2j} |\hat{V}(\xi,0)|^2 d\xi \leq Ce^{-ct} \|\nabla^j V_0\|_{L^2(\mathbb{R}^N)}^2.$$

Collecting the above two estimates, we obtain (19). This finishes the proof of Theorem 2.

First, we may rewrite system (59) as

$$\begin{cases} \hat{u}_t = \hat{v}, \\ \hat{v}_t = \hat{w}, \\ \tau \hat{w}_t = -|\xi|^2 \hat{u} - \beta |\xi|^2 \hat{v} - \hat{w}. \end{cases} \tag{30}$$

Lemma

The energy functional associated to system (30) *is*

$$\hat{E}(\xi, t) = \frac{1}{2} \left\{ |\hat{v} + \tau \hat{w}|^2 + \tau(\beta - \tau)|\xi|^2 |\hat{v}|^2 + |\xi|^2 |\hat{u} + \tau \hat{v}|^2 \right\} \tag{31}$$

and satisfies, for all $t \geq 0$, *the identity*

$$\frac{d}{dt} \hat{E}(\xi, t) = -(\beta - \tau)|\xi|^2 |\hat{v}|^2. \tag{32}$$

The main goal is to establish a Lyapunov-type inequality in time that satisfies:

$$\frac{d}{dt} L(\xi, t) + \frac{|\xi|^2}{1 + |\xi|^2} \hat{E}(\xi, t) \leq 0, \quad \text{and} \quad L(\xi, t) \sim \hat{E}(\xi, t) \tag{33}$$

Now, we define the functional $F_1(\xi, t)$ as

$$F_1(\xi, t) = \mathrm{Re}\left\{(\bar{\hat{u}} + \tau\bar{\hat{v}})(\hat{v} + \tau\hat{w})\right\}. \quad (34)$$

Then, we have the following lemma.

Lemma

For any $\epsilon_0 > 0$, we have

$$\frac{d}{dt}F_1(\xi, t) + (1 - \epsilon_0)|\xi|^2|\hat{u} + \tau\hat{v}|^2 \leq |\hat{v} + \tau\hat{w}|^2 + C(\epsilon_0)|\xi|^2|\hat{v}|^2. \quad (35)$$

Next, we define the functional $F_2(\xi, t)$ as

$$F_2(\xi, t) = -\tau\,\mathrm{Re}(\bar{\hat{v}}(\hat{v} + \tau\hat{w})). \quad (36)$$

Lemma

For any $\epsilon_1, \epsilon_2 > 0$, we have

$$\frac{d}{dt}F_2(\xi, t) + (1 - \epsilon_1)|\hat{v} + \tau\hat{w}|^2 \leq C(\epsilon_1, \epsilon_2)(1 + |\xi|^2)|\hat{v}|^2 + \epsilon_2|\xi|^2|\hat{u} + \tau\hat{v}|^2. \quad (37)$$

We define the Lyapunov functional $L(\xi, t)$ as

$$L(\xi,t) = \gamma_0 \hat{E}(\xi,t) + \frac{|\xi|^2}{1+|\xi|^2} F_1(\xi,t) + \gamma_1 \frac{|\xi|^2}{1+|\xi|^2} F_2(\xi,t), \tag{38}$$

where γ_0 and γ_1 are positive numbers that will be fixed later on.
Taking the derivative of (77) with respect to t and making use of (69), (73) and (75), we obtain

$$\begin{aligned} &\frac{d}{dt} L(\xi,t) + \Big(\gamma_1(1-\epsilon_1) - 1\Big) \frac{|\xi|^2}{1+|\xi|^2} |\hat{v} + \tau \hat{w}|^2 \\ &+ \Big((1-\epsilon_0) - \gamma_1 \epsilon_2\Big) \frac{|\xi|^2}{1+|\xi|^2} (|\xi|^2 |\hat{u} + \tau \hat{v}|^2) \\ &+ \Big(\gamma_0(\beta - \tau) - C(\epsilon_0) - \gamma_1 C(\epsilon_1, \epsilon_2)\Big) |\xi|^2 |\hat{v}|^2 \leq 0. \end{aligned} \tag{39}$$

Consequently, we deduce that there exists a positive constant γ_2 such that for all $t \geq 0$,

$$\frac{d}{dt}L(\xi,t) + \gamma_2 \frac{|\xi|^2}{1+|\xi|^2}\hat{E}(\xi,t) \leq 0. \tag{40}$$

On the other hand, it is not difficult to see that from (77), (68), (72) and (74) and for γ_0, large enough, that there exists two positive constants γ_3 and γ_4 such that

$$\gamma_3\hat{E}(\xi,t) \leq L(\xi,t) \leq \gamma_4\hat{E}(\xi,t). \tag{41}$$

Combining (79) and (41), we deduce that there exists a positive constant γ_5 such that for all $t \geq 0$,

$$\frac{d}{dt}L(\xi,t) + \gamma_5 \frac{|\xi|^2}{1+|\xi|^2}L(\xi,t) \leq 0. \tag{42}$$

A simple application of Gronwall's lemma, leads to the estimate (25), as L and the norm of $\hat{V}$ are equivalent.

Optimality of the result: Eigenvalues expansion

The characteritic equation associated to (23) is

$$\det(L + |\xi|^2 A - \lambda I) = \tau\lambda^3 + \lambda^2 + \beta|\xi|^2\lambda + |\xi|^2 = 0. \tag{43}$$

The solutions λ_j, $i = 1, 2, 3$ of the previous equation are the eigenvalues of $\Phi(\xi)$. We write for $\zeta = i|\xi|$,

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \dots, \qquad j = 1, 2, 3. \tag{44}$$

or, equivalently,

$$\lambda_j(|\xi|) = \lambda_j^{(0)} + \lambda_j^{(1)}\, i\, |\xi| - \lambda_j^{(2)}|\xi|^2 + \dots, \qquad j = 1, 2, 3.$$

Consequently, we have for $|\xi| \to 0$ that

$$\mathrm{Re}(\lambda_j(|\xi|)) = \begin{cases} -\dfrac{1}{\tau} + O(|\xi|), & \text{for} \quad j = 1, \\ -\dfrac{1}{2}(\beta - \tau)|\xi|^2 + O(|\xi|^3), & \text{for} \quad j = 2, 3. \end{cases} \tag{45}$$

for $|\xi| \to \infty$ we have

$$\mathrm{Re}(\lambda_j(|\xi|)) = \begin{cases} -\dfrac{1}{\beta} + O(|\xi|^{-1}), & \text{for} \quad j = 1, \\ -\dfrac{1}{2}\left(\dfrac{1}{\tau} - \dfrac{1}{\beta}\right) + O(|\xi|^{-1}), & \text{for} \quad j = 2, 3, \end{cases} \tag{46}$$

To obtain the estimate of $\|u(t)\|_{L^2}$, Let us now divide the frequency space into three regions: low frequency, high frequency and middle frequency region, that is

$$\begin{aligned}
\Upsilon_L &= \left\{\xi \in \mathbb{R}^N; |\xi| < \nu_1 \ll 1\right\}, \\
\Upsilon_H &= \left\{\xi \in \mathbb{R}^N; |\xi| > \nu_2 \gg 1\right\}, \\
\Upsilon_M &= \left\{\xi \in \mathbb{R}^N; \nu_1 \leq |\xi| \leq \nu_2\right\}
\end{aligned}$$

and write the estimate of $|\hat{u}(\xi, t)|$ in each region.

Proposition

If $0 < \tau < \beta$, *the solution* $\hat{U}(\xi, t)$ *of* (23) *satisfies, for all* $\xi \in \Upsilon_L$ *with* $|\xi| \neq 0$, *the estimates:*

$$\begin{aligned} |\hat{u}(\xi,t)| \leq\ & C_L\left(|\xi|^2|\hat{u}_0| + |\xi|^2|\hat{u}_1| + |\hat{u}_2|\right)e^{-c_1 t} \\ & + C_L\left(|\hat{u}_0| + |\xi|^2|\hat{u}_1| + |\hat{u}_2|\right)e^{-c_2|\xi|^2 t}\cos(|\xi|t) \\ & + C_L\left(|\xi||\hat{u}_0| + \frac{1}{|\xi|}|\hat{u}_1| + \frac{1}{|\xi|}|\hat{u}_2|\right)e^{-c_2|\xi|^2 t}\sin(|\xi|t). \end{aligned} \tag{47}$$

Moreover, if $\int_{\mathbb{R}^N} u_1(x)dx = \int_{\mathbb{R}^N} u_2(x)dx = 0$ *we have*

$$\begin{aligned} |\hat{u}(\xi,t)| \leq\ & C_L\left(|\xi|^2|\hat{u}_0| + |\xi|^2|\hat{u}_1| + |\hat{u}_2|\right)e^{-c_1 t} \\ & + C_L\left(|\hat{u}_0| + |\xi|^2|\hat{u}_1| + |\hat{u}_2|\right)e^{-c_2|\xi|^2 t}\cos(|\xi|t) \\ & + C_L\left(|\xi||\hat{u}_0| + \|u_1\|_{L^{1,1}} + \|u_2\|_{L^{1,1}}\right)e^{-c_2|\xi|^2 t}\sin(|\xi|t), \end{aligned} \tag{48}$$

where $L^{1,1}$ *is the* L^1*-weighted space defined by*

$$L^{1,1}(\mathbb{R}^N) = \left\{u \in L^1\left(\mathbb{R}^N\right); \|u_1\|_{L^{1,1}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}(1+|x|)|u(x)|dx < \infty\right\}. \tag{49}$$

Proposition

If $0 < \tau < \beta$*, the solution* $\hat{u}(\xi, t)$ *of* (21) *satisfies in* Υ_H *the estimate:*

$$|\hat{u}(\xi,t)| \leq C_H \left(\left(1 + \frac{1}{|\xi|} + \frac{1}{|\xi|^2}\right) |\hat{u}_0(\xi)| + \left(\frac{1}{|\xi|} + \frac{1}{|\xi|^2}\right) |\hat{u}_1(\xi)| + \left(\frac{1}{|\xi|^2} + \frac{1}{|\xi|^3}\right) |\hat{u}_2(\xi)| \right) e^{-c_3 t},$$

for all $t \geq 0$*, where* $c_3 = \min\left\{ \frac{1}{\beta}, \frac{1}{2}\left(\frac{1}{\tau} - \frac{1}{\beta}\right) \right\}$ *and* $C_H = C_H(\beta, \tau) > 0$ *(all positive constants).*

Proposition

There exists two positive constants C_M *and* c_4 *such that the solution* $\hat{u}(\xi, t)$ *of* (21) *satisfies in* Υ_M *one of the following estimates:*

$$|\hat{u}(\xi,t)| \leq C_M \left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| + |\hat{u}_2(\xi)|\right) e^{-c_4 t}, \qquad \text{if } |\xi| \neq 0, \sqrt{m_1}, \sqrt{m_2}. \tag{51}$$

or

$$|\hat{u}(\xi,t)| \leq C_M (1+t) \left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| + |\hat{u}_2(\xi)|\right) e^{-c_4 t}, \quad \text{if } |\xi| = 0, \text{ or } \frac{\tau}{\beta} \neq \frac{1}{9} \text{ and } |\xi| = \sqrt{m_1}, \sqrt{m_2} \tag{52}$$

or

$$|\hat{u}(\xi,t)| \leq C_M (1+t+t^2) \left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| + |\hat{u}_2(\xi)|\right) e^{-\frac{3}{\beta} t}, \qquad \text{if } \frac{\tau}{\beta} = \frac{1}{9} \text{ and } |\xi| = \sqrt{m_1} = \sqrt{m_2},$$

Theorem (L^1–initial data)

Let $(u_0, u_1, u_2) \in \mathbb{D}(\mathcal{A}^s)$, $s \geq 1$, with $u_0, u_1, u_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$, and $0 < \tau < \beta$. Then for any $t \geq 0$ the following decay estimates hold for all $0 \leq j \leq s$ and certain constants $C, c > 0$ independent of t and of the initial data:

$$\begin{aligned} \left\|\nabla^j u(t)\right\|_{L^2(\mathbb{R}^N)} \leq\ & C(\|u_0\|_{L^1(\mathbb{R}^N)} + \|u_1\|_{L^1(\mathbb{R}^N)} + \|u_2\|_{L^1(\mathbb{R}^N)})(1+t)^{1-N/4-j/2} \\ & + C(\|\nabla^j u_0\|_{L^2(\mathbb{R}^N)} + \|\nabla^j u_1\|_{L^2(\mathbb{R}^N)} + \|\nabla^j u_2\|_{L^2(\mathbb{R}^N)})e^{-ct}. \end{aligned} \tag{54}$$

Theorem (Improved decay estimates for $N + j \geq 3$)

Let $(u_0, u_1, u_2) \in \mathbb{D}(\mathcal{A}^s)$, $s \geq 1$, with $u_0, u_1, u_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$, and $0 < \tau < \beta$. Assume that $N + j \geq 3$. Then, for $t \geq 0$, the following decay estimates hold for all $0 \leq j \leq s$ and certain constants $C, c > 0$ independent of t and of the initial data:

$$\begin{aligned} \left\|\nabla^j u(t)\right\|_{L^2(\mathbb{R}^N)} \leq\ & C(\|u_0\|_{L^1(\mathbb{R}^N)} + \|u_1\|_{L^1(\mathbb{R}^N)} + \|u_2\|_{L^1(\mathbb{R}^N)})(1+t)^{-(N-2)/4-j/2} \\ & + C(\|\nabla^j u_0\|_{L^2(\mathbb{R}^N)} + \|\nabla^j u_1\|_{L^2(\mathbb{R}^N)} + \|\nabla^j u_2\|_{L^2(\mathbb{R}^N)})e^{-ct}. \end{aligned} \tag{55}$$

where $c = \min\left\{\frac{1}{\beta}, |\operatorname{Re}(\lambda_{2,3}(\xi_{\nu_1}))|\right\}$.

Theorem ($L^{1,1}$–initial data)

Let $0 < \tau < \beta$ *and let* $u_0, u_1, u_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. *Also, let* $(u_1, u_2) \in L^{1,1}(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N} u_i(x)dx = 0$, $i = 1, 2$. *Then, for* $0 \leq j \leq s$, *the following decay estimate holds:*

$$\begin{aligned} \left\|\nabla^j u(t)\right\|_{L^2(\mathbb{R}^N)} &\leq C(\|u_0\|_{L^1(\mathbb{R}^N)} + \|u_1\|_{L^{1,1}(\mathbb{R}^N)} + \|u_2\|_{L^{1,1}(\mathbb{R}^N)})(1+t)^{-N/4-j/2} \\ &+ C(\|\nabla^j u_0\|_{L^2(\mathbb{R}^N)} + \|\nabla^j u_1\|_{L^2(\mathbb{R}^N)} + \|\nabla^j u_2\|_{L^2(\mathbb{R}^N)})e^{-ct}. \quad (56) \end{aligned}$$

Hence, we first write the characteristic polynomial of the matrix $\Phi = (L + \xi^2 A)$:

$$p_0(\lambda) = a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 \tag{57}$$

where

$$a_0 = \tau, \quad a_1 = 1 + \tau|\xi|^2, \quad a_2 = (\beta + 1)\,|\xi|^2, \quad a_3 = \left(\beta|\xi|^2 + 1\right)|\xi|^2, \quad a_4 = |\xi|^4$$

Now, as all $a_i > 0$ for $i = 0, \ldots, 4$, we apply the Routh–Hurwitz theorem that ensures that all the roots of the polynomial $p_0(\lambda)$ have negative real part if and only if all the leading minors A_i, $1 \leq i \leq 4$ of the matrix

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

are strictly positive. In our case, we have

$$\begin{aligned} A_1 &= 1 + \tau|\xi|^2 \\ A_2 &= |\xi|^2\left(|\xi|^2\tau + \beta - \tau + 1\right) \\ A_3 &= |\xi|^4(\beta - \tau)\left(\tau|\xi|^4 + (\beta + 1)|\xi|^2 + 1\right) \\ A_4 &= |\xi|^4 A_3. \end{aligned}$$

Hence, the condition $0 < \tau < \beta$ is a necessary and sufficient condition to have $A_3 > 0$.

We consider the nonlinear Jordan–Moore–Gibson–Thompson equation:

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - \beta \Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right), \tag{58a}$$

where $x \in \mathbb{R}^N$ (Cauchy problem), and $t > 0$. We consider the initial conditions:

$$u(t=0) = u_0, \qquad u_t(t=0) = u_1 \qquad u_{tt}(t=0) = u_2. \tag{58b}$$

We rewrite the right-hand side of equation (58a) in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right) = \frac{1}{c^2}\frac{B}{A} u_t u_{tt} + 2\nabla u \nabla u_t,$$

and introduce the new variables

$$v = u_t \qquad \text{and} \qquad w = u_{tt},$$

Without loss of generality, we put $c = 1$. Then equation (58a) can be rewritten as the following first order system

$$\begin{cases} u_t = v, \\ v_t = w, \\ \tau w_t = \Delta u + \beta \Delta v - w + \dfrac{B}{A} vw + 2\nabla u \nabla v, \end{cases} \tag{59}$$

with the initial data

$$u(t=0) = u_0, \qquad v(t=0) = v_0, \qquad w(t=0) = w_0. \tag{60}$$

We introduce the energy norm, $\mathcal{E}_k(t)$, and the corresponding dissipation norm, $\mathcal{D}_k(t)$, as follows:

$$\begin{aligned}\mathcal{E}_k^2(t) \quad = \quad & \sup_{0\leq\sigma\leq t}\Big(\left\|\nabla^k(v+\tau w)(\sigma)\right\|_{H^1}^2+\left\|\Delta\nabla^k v(\sigma)\right\|_{L^2}^2+\left\|\nabla^{k+1}v(\sigma)\right\|_{L^2}^2 \\ & +\left\|\Delta\nabla^k(u+\tau v)(\sigma)\right\|_{L^2}^2+\left\|\nabla^{k+1}(u+\tau v)(\sigma)\right\|_{L^2}^2+\|\nabla^k w(\sigma)\|_{L^2}^2\Big), \end{aligned} \tag{61}$$

and

$$\begin{aligned}\mathcal{D}_k^2(t) \quad = \quad & \int_0^t\Big(\left\|\nabla^{k+1}v(\sigma)\right\|_{L^2}^2+\left\|\Delta\nabla^k v(\sigma)\right\|_{L^2}^2+\|\nabla^k w(\sigma)\|_{L^2}^2 \\ & +\left\|\Delta\nabla^k\left(u+\tau v\right)(\sigma)\right\|_{L^2}^2+\left\|\nabla^{k+1}(v+\tau w)(\sigma)\right\|_{L^2}^2\Big)d\sigma. \end{aligned} \tag{62}$$

For some positive integer $s\geq 1$ that will be fixed later on, we define

$$\mathrm{E}_s^2(t)=\sum_{k=0}^{s}\mathcal{E}_k^2(t) \qquad \text{and} \qquad \mathrm{D}_s^2(t)=\sum_{k=0}^{s}\mathcal{D}_k^2(t). \tag{63}$$

Theorem (Local existence)

Assume that $0 < \tau < \beta$ *and let* $s > \frac{5}{2} = \frac{N}{2} + 1$. *Let* $\mathbf{U}_0 = (u_0, v_0, w_0)^T$ *be such that*

$$\begin{aligned} \mathrm{E}_s^2(0) &= \|(v_0 + \tau w_0)\|_{H^{s+1}}^2 + \|\Delta v_0\|_{H^s}^2 + \|\nabla v_0\|_{H^s}^2 \\ &\quad + \|\Delta(u_0 + \tau v_0)\|_{H^s}^2 + \|\nabla(u_0 + \tau v_0)\|_{H^s}^2 + \|w_0\|_{H^s}^2 \leq \tilde{\delta}_0 \end{aligned} \tag{64}$$

for some $\tilde{\delta}_0 > 0$. *Then, there exists a small time* $T = T(\mathrm{E}_s(0)) > 0$ *such that problem* (58) *has a unique solution* u *on* $[0, T) \times \mathbb{R}^3$ *satisfying*

$$\mathrm{E}_s^2(T) + \mathrm{D}_s^2(T) \leq C_{\tilde{\delta}_0},$$

where $\mathrm{E}_s^2(T)$ *and* $\mathrm{D}_s^2(T)$ *are given in* (63), *determining the regularity of* u, *and* $C_{\tilde{\delta}_0}$ *is a positive constant depending on* $\tilde{\delta}_0$.

We write

$$\mathbf{U}(t) = \Phi(\mathbf{U})(t) = e^{t\mathcal{A}}\mathbf{U}_0 + \int_0^t e^{(t-r)\mathcal{A}}\mathcal{F}(\mathbf{U}, \nabla\mathbf{U})(r)dr. \tag{65}$$

and show that using the contraction mapping principle that there exists a unique solution of (65).

Theorem (Global existence)

Assume that $0 < \tau < \beta$ and let $s > \frac{5}{2}$. Assume that $u_0, v_0, w_0 \in H^s(\mathbb{R}^3)$. Then there exists a small positive constant α, such that if

$$\begin{aligned} \mathrm{E}_s^2(0) &= \|(v_0 + \tau w_0)\|_{H^{s+1}}^2 + \|\Delta v_0\|_{H^s}^2 + \|\nabla v_0\|_{H^s}^2 \\ &\quad + \|\Delta(u_0 + \tau v_0)\|_{H^s}^2 + \|\nabla(u_0 + \tau v_0)\|_{H^s}^2 + \|w_0\|_{H^s}^2 \le \alpha, \end{aligned}$$

then the local solution u exists globally in time.

Theorem (Decay estimates)

Assume that $0 < \tau < \beta$ *and* $s > \frac{5}{2}$. *Let* u *be the global solution of* (58). *Let* $v_0 = u_t(t = 0)$, $v_1 = u_{tt}(t = 0)$ *and* $v_2 = u_{ttt}(t = 0)$ *satisfying* $v_0, v_1, v_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ *and* $(v_1, v_2) \in L^{1,1}(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N} v_i(x)dx = 0$, $i = 1, 2$. *Assume that* $\|\mathbf{V}_0\|_{H^s \cap L^1}$ *is small enough. Then, the following decay estimates hold:* ($\mathbf{V} := (v + \tau w, \nabla(u + \tau v), \nabla v)$)

$$\|\nabla^j \mathbf{V}(t)\|_{L^2} \leq C(1 + t)^{-N/4 - j/2},$$

for all $0 \leq j \leq s$.

We define

$$Y_s(t) := \mathrm{E}_s^2(t) + \mathrm{D}_s^2(t).$$

The main goal is to prove by a continuity argument that for s large enough, $Y_s(t)$ is uniformly bounded for all time if the initial energy $\mathrm{E}_s^2(0) = Y_s(0)$ is sufficiently small. Due to the presence of the term $-\beta\Delta_t u$ in (58a) and the special nonlinearity, the global existence is proved without using the decay of the linearized problem.

Proposition

Let $s > \frac{5}{2}$, then the following estimate holds for t in an interval $[0, T]$ of local existence:

$$Y_s(t) \leq Y_s(0) + C{Y_s}^{3/2}(t), \tag{66}$$

where C is a positive constant that does not depend on t, T.

Proposition

Let $s > \frac{5}{2}$. Then, the following estimate holds:

$$\mathrm{E}_s^2(t) + \mathrm{D}_s^2(t) \leq C\mathrm{E}_s^2(0) + C\mathrm{E}_s(t)\mathrm{D}_s^2(t). \tag{67}$$

Lemma

The energy functional associated to system (59) *is*

$$E_1(t) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|v+\tau w|^2 + \tau(\beta-\tau)|\nabla v|^2 + |\nabla(u+\tau v)|^2\right) dx \tag{68}$$

and satisfies, for all $t \geq 0$, *the identity*

$$\frac{d}{dt}E_1(t) + (\beta-\tau)\|\nabla v\|_{L^2}^2 = R_1, \tag{69}$$

where

$$R_1 := \int_{\mathbb{R}^N} \left(\frac{B}{A}vw + 2\nabla u \nabla v\right)(v+\tau w)\, dx.$$

Next, we define the energy of second order

$$E_2(t) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla(v+\tau w)|^2 + \tau(\beta-\tau)|\Delta v|^2 + |\Delta(u+\tau v)|^2\right) dx. \tag{70}$$

The following lemma is proved analogously.

Lemma

The energy functional $E_2(t)$ satisfies, for all $t \geq 0$, the identity

$$\frac{d}{dt}E_2(t) + (\beta-\tau)\,\|\Delta v\|_{L^2}^2 = R_2, \tag{71}$$

where

$$R_2 := -\int_{\mathbb{R}^N} \left(\frac{B}{A}vw + 2\nabla u \nabla v\right)\Delta(v+\tau w)dx.$$

Now, we define the functional $F_1(t)$ as

$$F_1(t) := \int_{\mathbb{R}^N} \nabla(u + \tau v)\nabla(v + \tau w)dx. \tag{72}$$

Then, we have

Lemma

For any $\epsilon_0 > 0$, we have

$$\begin{aligned} & \frac{d}{dt}F_1(t) + (1 - \epsilon_0)\int_{\mathbb{R}^N} |\Delta\,(u + \tau v)\,|^2 dx \\ \leq \; & \int_{\mathbb{R}^N} |\nabla(v + \tau w)|^2 dx + C(\epsilon_0)\int_{\mathbb{R}^N} |\Delta v|^2 dx + |\tilde{R}_2| \end{aligned} \tag{73}$$

with

$$\tilde{R}_2 = -\int_{\mathbb{R}^N} \left(\frac{B}{A}vw + 2\nabla u\nabla v\right)\Delta\,(u + \tau v)\,dx.$$

Next, we define the functional $F_2(t)$ as

$$F_2(t) := -\tau \int_{\mathbb{R}^N} \nabla v \nabla (v + \tau w) dx. \tag{74}$$

Lemma

For any $\epsilon_1, \epsilon_2 > 0$, we have

$$\begin{aligned} & \frac{d}{dt} F_2(t) + (1 - \epsilon_1) \int_{\mathbb{R}^N} |\nabla (v + \tau w)|^2 dx \\ \leq \quad & C(\epsilon_1, \epsilon_2) \int_{\mathbb{R}^N} (|\nabla v|^2 + |\Delta v|^2) dx + \epsilon_2 \int_{\mathbb{R}^N} |\Delta (u + \tau v)|^2 dx + R_3, \end{aligned} \tag{75}$$

where

$$R_3 = \tau \left| \int_{\mathbb{R}^N} \left(\frac{B}{A} v w + 2 \nabla u \nabla v \right) \Delta v dx \right|.$$

Crucial energy estimate

Now, multiplying the third equation in (59) by w and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\tau\left|w\right|^2 dx+\int_{\mathbb{R}^3}\left|w\right|^2 dx &= \int_{\mathbb{R}^3}(\Delta u+\beta\Delta v)wdx\\ &\quad+\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)wdx\\ &\leq C(\|\Delta u\|_{L^2}+\|\Delta v\|_{L^2})\|w\|_{L^2}+\tilde{R}_1\\ &\leq C(\|\Delta(u+\tau v)_{L^2}+\|\Delta v\|_{L^2})\|w\|_{L^2}+\tilde{R}_1\end{aligned}$$

with

$$\tilde{R}_1:=\int_{\mathbb{R}^3}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)wdx.$$

Applying Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\tau\left|w\right|^2 dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|w\right|^2 dx\leq C(\|\Delta(u+\tau v)^2_{L^2}+\|\Delta v\|^2_{L^2})+|\tilde{R}_1|. \quad (76)$$

We define the Lyapunov functional $L(t)$ as

$$L(t) := \gamma_0 \left(E_1(t) + E_2(t) + \epsilon_0 \tau \|w\|_{L^2}\right) + F_1(t) + \gamma_1 F_2(t), \tag{77}$$

we have $L(t) \sim E_1(t) + E_2(t)$, for γ_0 large enough.
We have

$$\begin{aligned} & \frac{d}{dt} L(t) + \left(\gamma_0(\beta - \tau) - \gamma_1 C(\epsilon_1, \epsilon_2)\right) \|\nabla v\|_{L^2}^2 \\ & + \left(\gamma_0(\beta - \tau) - C(\epsilon_0) - \gamma_1 C(\epsilon_1, \epsilon_2)\right) \|\Delta v\|_{L^2}^2 \\ & + (1 - \epsilon_0 - \gamma_1 \epsilon_2) \|\Delta (u + \tau v)\|_{L^2}^2 \\ & + \left(\gamma_1 (1 - \epsilon_1) - 1\right) \|\nabla (v + \tau w)\|_{L^2}^2 \\ \leq \quad & \gamma_0 \left(|R_1| + |R_2|\right) + |R_2| + |\tilde{R}_2| + \gamma_1 |R_3|. \end{aligned} \tag{78}$$

Consequently, we deduce that there exists a positive constant γ_2 such that for all $t \geq 0$,

$$\begin{aligned} & \frac{d}{dt} L(t) + \gamma_2 \left(\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta (u + \tau v)\|_{L^2}^2 + \|\nabla (v + \tau w)\|_{L^2}^2 + \|w\|_{L^2} \right) \\ & \leq C \sum_{i=1}^{3} |R_i| + C \tilde{R}_2. \end{aligned} \tag{79}$$

Integrating (79) form 0 to t, we obtain

$$\begin{aligned} \mathcal{E}_0^2(t) + \mathcal{D}_0^2(t) \leq \mathcal{E}_0^2(0) \quad + \quad & C \int_0^t \Big(|R_1(\sigma)| + |\tilde{R}_1(\sigma)| + |R_2(\sigma)| \\ & + |\tilde{R}_2(\sigma)| + |R_3(\sigma)| \Big) d\sigma, \end{aligned} \tag{80}$$

where

Estimate of $\int_0^t R_j$

Basic tools:

- Ladyzhenskaya interpolation inequality in 3D

$$\|f\|_{L^4} \leq c\|f\|_{L^2}^{1/4}\|\nabla f\|_{L^2}^{3/4} \tag{81}$$

- Sobolev embedding in 3D:

$$\|u\|_{L^6} \leq C\|\nabla u\|_{L^2}.$$

- Gagliardo–Nirenberg interpolation inequality:

$$\left\|\nabla^j u\right\|_{L^p} \leq C\left\|\nabla^m u\right\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \qquad 1 \leq p,\ q\,,r \leq \infty \tag{82}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}, \quad 0 \leq j < m \tag{83}$$

for α satisfying $j/m \leq \alpha \leq 1$

In particular

$$\|f\|_{L^3} \leq C\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} \tag{84}$$

Lemma

Let $1 \leq p, q, r \leq \infty$ *and* $1/p = 1/q + 1/r$. *Then, we have*

$$\|\nabla^k(uv)\|_{L^p} \leq C(\|u\|_{L^q}\|\nabla^k v\|_{L^r} + \|v\|_{L^q}\|\nabla^k u\|_{L^r}), \quad k \geq 0, \tag{85}$$

and the commutator estimate

$$\begin{aligned} \|[\nabla^k, f]g\|_{L^p} &= \|\nabla^k(fg) - f\nabla^k g\|_{L^p} \\ &\leq C(\|\nabla f\|_{L^q}\|\nabla^{k-1}g\|_{L^r} + \|g\|_{L^q}\|\nabla^k f\|_{L^r}), \quad k \geq 1, \end{aligned} \tag{86}$$

for some constant $C > 0$.

$$
\begin{aligned}
|R_1| &= \left|\int_{\mathbb{R}^3}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)(v+\tau w)\,dx\right|\\
&\leq C\left|\int_{\mathbb{R}^3} vw(v+\tau w)dx\right| + C\left|\int_{\mathbb{R}^3}\nabla u\nabla v\,(v+\tau w)\,dx\right|\\
&\equiv I_1 + I_2.
\end{aligned}
$$

First, we estimate I_1 as follows:

$$
\begin{aligned}
I_1 &= C\left|\int_{\mathbb{R}^3} vw(v+\tau w)dx\right|\\
&\leq C\|w\|_{L^2}\|v\|_{L^4}^2 + C\|v\|_{L^2}\|w\|_{L^4}^2.
\end{aligned}
$$

Using the Ladyzhenskaya interpolation inequality, we have

$$
\begin{aligned}
\|w\|_{L^2}\|v\|_{L^4}^2 &\leq C\|w\|_{L^2}\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{3/2}\\
&= C\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}\|w\|_{L^2}\\
&\leq C(\|v\|_{L^2}+\|\nabla v\|_{L^2})\|\nabla v\|_{L^2}\|w\|_{L^2} \qquad (87)
\end{aligned}
$$

and

$$
\|v\|_{L^2}\|w\|_{L^4}^2 \leq C\|v\|_{L^2}\|w\|_{L^2}^{1/2}\|\nabla w\|_{L^2}^{3/2}. \qquad (88)
$$

Thus, using (88), we get

$$\begin{aligned} & \int_0^t \|v(\sigma)\|_{L^2} \|w(\sigma)\|_{L^2}^{1/2} \|\nabla w(\sigma)\|_{L^2}^{3/2} d\sigma \\ \leq \quad & C \sup_{0\leq\sigma\leq t} \|v(\sigma)\|_{L^2} \Big(\int_0^t \|w(\sigma)\|_{L^2}^2 d\sigma \Big)^{1/4} \Big(\int_0^t \|\nabla w(\sigma)\|_{L^2}^2 d\sigma \Big)^{3/4} \\ \leq \quad & C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \end{aligned} \tag{89}$$

Then, we deduce that

$$\int_0^t I_1(\sigma) d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t). \tag{90}$$

We can estimate I_2 as follows:

$$\begin{aligned} I_2 = \Big| \int_{\mathbb{R}^3} \nabla u \nabla v \, (v + \tau w) \, dx \Big| \quad & \leq \quad \Big| \int_{\mathbb{R}^3} v \nabla u \nabla v dx \Big| + \Big| \int_{\mathbb{R}^3} \tau \nabla u \nabla v w dx \Big| \\ & = \quad J_1 + J_2. \end{aligned} \tag{91}$$

It is clear that

$$J_2 \leq C \|\nabla u\|_{L^\infty} \|\nabla v\|_{L^2} \|w\|_{L^2}.$$

Then, Hölder's inequality implies

$$\int_0^t J_2(\sigma) d\sigma \leq C \sup_{0\leq\sigma\leq t} \|\nabla u(\sigma)\|_{L^\infty} \mathcal{D}_0^2(t).$$

Now, we need to estimate the term J_1. This is done in the following lemma.

Lemma

We have the estimate

$$\int_0^t J_1(\sigma)d\sigma \leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t).$$

Proof.

First, we have, by Hölder's inequality

$$J_1 \leq C\|v\|_{L^6}\|\nabla u\|_{L^3}\|\nabla v\|_{L^2}. \tag{92}$$

Now, applying the interpolation inequality, which holds for $N = 3$,

$$\|f\|_{L^3} \leq C\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} \tag{93}$$

we obtain

$$\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}. \tag{94}$$

Consequently, using the above estimates, (92) becomes

$$J_1 \quad \leq \quad C\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^2$$

Now, using the fact that

$$\|\nabla^k u\|_{L^2} \leq C(\|\nabla^k(u + \tau v)\|_{L^2} + \|\nabla^k v\|_{L^2}), \qquad k \geq 1,$$

together with (95), we obtain

$$\begin{aligned}\int_0^t J_1(\sigma) d\sigma &\leq \sup_{0 \leq \sigma \leq t} (\|\nabla u(\sigma)\|_{L^2} + \|\nabla^2 u(\sigma)\|_{L^2}) \int_0^t \|\nabla v(\sigma)\|_{L^2}^2 d\sigma \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0^2(t).\end{aligned}$$

This completes the proof.

Applying the operator ∇^k, $k \geq 1$ to (59), we get for $U := \nabla^k u$, $V := \nabla^k v$ and $W := \nabla^k w$

$$\begin{cases} \partial_t U = V, \\ \partial_t V = W, \\ \tau\partial_t W = \Delta U + \beta\Delta V - W + \frac{B}{A}[\nabla^k, v]w + \frac{B}{A}vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u \nabla V, \end{cases} \tag{96}$$

where $[A, B] = AB - BA$.
We define the first energy of order k as

$$\begin{aligned} E_1^{(k)}(t) \quad ; = \quad & \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla^k v + \tau\nabla^k w|^2 + \tau(\beta-\tau)|\nabla^{k+1} v|^2 + |\nabla^{k+1} u + \tau\nabla^{k+1} v|^2\right) dx \\ = \quad & \frac{1}{2}\int_{\mathbb{R}^N} \left(|V + \tau W|^2 + \tau(\beta-\tau)|\nabla V|^2 + |\nabla(U+\tau V)|^2\right) dx. \end{aligned} \tag{97}$$

Lemma

For all $t \geq 0$, *it holds that*

$$\frac{d}{dt}E_1(t)^{(k)} + (\beta - \tau)\,\|\nabla V\|_{L^2}^2 = \int_{\mathbb{R}^N} R_1^{(k)}(t)\,(V + \tau W)\,dx, \tag{98}$$

where

$$R_1^{(k)}(t) = \frac{B}{A}[\nabla^k, v]w + \frac{B}{A}vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u \nabla V. \tag{99}$$

As in the case $k = 0$, we define the second energy of order k as follows:

$$E_2^{(k)}(t) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla(V + \tau W)|^2 + \tau(\beta - \tau)|\Delta V|^2 + |\Delta(U + \tau V)|^2\right) dx. \tag{100}$$

Lemma

The energy functional $E_2(t)$ *satisfies, for all* $t \geq 0$, *the identity*

$$\frac{d}{dt}E_2^{(k)}(t) + (\beta - \tau)\,\|\Delta v\|_{L^2}^2 = -\int_{\mathbb{R}^N} R_1^{(k)}\Delta(V + \tau W)dx. \tag{101}$$

We define now the functional $F_1^{(k)}(t)$ as

$$F_1^{(k)}(t) := \int_{\mathbb{R}^N} \nabla(U+\tau V)\nabla(V+\tau W)dx. \tag{102}$$

Lemma

For any $\epsilon_0' > 0$, we have

$$\begin{aligned} &\frac{d}{dt}F_1^{(k)}(t) + (1-\epsilon_0')\int_{\mathbb{R}^N} |\Delta(U+\tau V)|^2 dx \\ \leq\ & \int_{\mathbb{R}^N} |\nabla(V+\tau W)|^2 dx + C(\epsilon_0')\int_{\mathbb{R}^N} |\Delta V|^2 dx + \int_{\mathbb{R}^N} |R_1^{(k)}||\Delta(U+\tau V)|dx \end{aligned} \tag{103}$$

$$\mathcal{E}_k^2(t) + \mathcal{D}_k^2(t) \leq \mathcal{E}_k^2(0) + \sum_{i=1}^{5}\int_0^t I_i^{(k)}(\sigma)d\sigma. \tag{104}$$

where

$$\begin{aligned} \sum_{j=1}^{5} I_j^{(k)} =& \int_{\mathbb{R}^3} |R_1^{(k)}(t)||(V+\tau W)|dx + C\int_{\mathbb{R}^3} |\nabla R_1^{(k)}||\nabla(V+\tau W)|dx \\ &+ C\int_{\mathbb{R}^3} |R_1^{(k)}||\Delta(U+\tau V)|dx + C\int_{\mathbb{R}^3} |\nabla R_1^{(k)}||\nabla V|dx + \int_{\mathbb{R}^3} |R_1^{(k)}||W|dx. \end{aligned}$$

$$R_1^{(k)}(t) = \frac{B}{A}[\nabla^k, v]w + \frac{B}{A}vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u\nabla V.$$

$$\mathcal{T}_1 = \int_{\mathbb{R}^3} |[\nabla^k, v]w| \, (V + \tau W) \, |dx \leq \left\| [\nabla^k, v]w \right\|_{L^{6/5}} \|(V + \tau W)\|_{L^6}. \tag{105}$$

Applying, the commutator estimate (86), we have

$$\left\| [\nabla^k, v]w \right\|_{L^{6/5}} \leq C(\|\nabla v\|_{L^2} \|\nabla^{k-1} w\|_{L^3} + \|w\|_{L^2} \|\nabla^k v\|_{L^3}) \tag{106}$$

To estimate the term $\|\nabla^{k-1} w\|_{L^3}$, we apply (82) to find

$$\|\nabla^{k-1} w\|_{L^3} \leq C \|\nabla^k w\|_{L^2}^{\frac{2k-1}{2k}} \|w\|_{L^2}^{\frac{1}{2k}}. \tag{107}$$

Similarly, we have

$$\|\nabla^k v\|_{L^3} \leq C \|\nabla^{k+1} v\|_{L^2}^{\frac{2k+1}{2(k+1)}} \|v\|_{L^2}^{\frac{1}{2(k+1)}}. \tag{108}$$

Plugging (107) and (108) into (106), we obtain

$$\begin{aligned} \left\| [\nabla^k, v]w \right\|_{L^{6/5}} \leq \; & C \|\nabla v\|_{L^2} \|\nabla^k w\|_{L^2}^{\frac{2k-1}{2k}} \|w\|_{L^2}^{\frac{1}{2k}} + C \|w\|_{L^2} \|\nabla^{k+1} v\|_{L^2}^{\frac{2k+1}{2(k+1)}} \|v\|_{L^2}^{\frac{1}{2(k+1)}} \\ = \; & C \|\nabla v\|_{L^2}^{\frac{2k-1}{2k}} \|w\|_{L^2}^{\frac{1}{2k}} \|\nabla v\|_{L^2}^{\frac{1}{2k}} \|\nabla^k w\|_{L^2}^{\frac{2k-1}{2k}} \\ & + C \|w\|_{L^2}^{\frac{2k+1}{2(k+1)}} \|v\|_{L^2}^{\frac{1}{2(k+1)}} \|\nabla^{k+1} v\|_{L^2}^{\frac{2k+1}{2(k+1)}} \|w\|_{L^2}^{\frac{1}{2(k+1)}}. \end{aligned} \tag{109}$$

Plugging (109) into (105), making use of Sobolev embedding theorem and applying Young's inequality, we get

$$\begin{aligned}\int_0^t \mathcal{T}_1(\sigma)d\sigma \quad &\leq \quad C \sup_{0\leq\sigma\leq t} (\|\nabla v(\sigma)\|_{L^2} + \|w(\sigma)\|_{L^2}) \\ &\quad \times \int_0^t \|\nabla v(\sigma)\|_{L^2}^{\frac{1}{2k}} \|\nabla^k w(\sigma)\|_{L^2}^{\frac{2k-1}{2k}} \|\nabla(V+\tau W)\|_{L^2} \\ &\quad + C \sup_{0\leq\sigma\leq t} (\|v(\sigma)\|_{L^2} + \|w(\sigma)\|_{L^2}) \\ &\quad \times \int_0^t \|\nabla^{k+1} v(\sigma)\|_{L^2}^{\frac{2k+1}{2(k+1)}} \|w(\sigma)\|_{L^2}^{\frac{1}{2(k+1)}} \|\nabla(V+\tau W)\|_{L^2}. \qquad (110)\end{aligned}$$

Applying Hölder's inequality together with Young's inequality, we obtain

$$\int_0^t \mathcal{T}_1(\sigma)d\sigma \leq C\mathcal{E}_0(t)(\mathcal{D}_0^2(t) + \mathcal{D}_k^2(t)). \qquad (111)$$

Let

$$\Lambda_3(t) = (\|v\|_{W^{1,\infty}} + \|w\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty})(t).$$

Then, we have

$$\begin{aligned} \mathcal{E}_k^2(t) + \mathcal{D}_k^2(t) &\leq \mathcal{E}_k^2(0) + C(\Lambda_3(t) + \mathcal{E}_0(t) + \mathcal{E}_k(t))(\mathcal{D}_0^2(t) + \mathcal{D}_{k-1}^2(t) + \mathcal{D}_k^2(t)) \\ &\leq \mathcal{E}_k^2(0) + C(\Lambda_3(t) + \mathcal{E}_s(t))(\mathcal{D}_0^2(t) + \mathcal{D}_{k-1}^2(t) + \mathcal{D}_k^2(t)). \end{aligned} \tag{112}$$

Summing up (112) over $k = 1, \ldots, s$ we find

$$\mathrm{E}_s^2(t) + \mathrm{D}_s^2(t) \leq \mathrm{E}_s^2(0) + C(\Lambda_3(t) + \mathrm{E}_s(t))\mathrm{D}_s^2(t). \tag{113}$$

Now, we need to estimate $\Lambda_3(t)$ by $\mathrm{E}_s(t)$ by using Sobolev embeddings. Due to the embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ for $s > 5/2$ we have

$$\Lambda_3(t) \leq C\mathrm{E}_s(t). \tag{114}$$

Plugging (114) into (113), we conclude that (67) holds true.

We define (recall that $\mathbf{V} := (v + \tau w, \nabla(u + \tau v), \nabla v)$)

$$\mathcal{M}(t) := \sup_{0 \le \sigma \le t} \sum_{j=0}^{s} (1+\tau)^{N/4+j/2} (\|\nabla^j \mathbf{V}(\sigma)\|_{L^2} + \|\nabla^j v(\sigma)\|_{L^2}).$$

We also define the quantities

$$\begin{aligned} M_0(t) &:= \sup_{0 \le \sigma \le t} (1+\sigma)^{\frac{N}{2}} \left(\|\mathbf{V}(\sigma)\|_{L^\infty} + \|v(\sigma)\|_{L^\infty} \right), \\ M_1(t) &:= \sup_{0 \le \sigma \le t} (1+\sigma)^{\frac{N}{2}+\frac{1}{2}} \|\nabla \mathbf{V}(\sigma)\|_{L^\infty}. \end{aligned}$$

So, our goal is to show that $\mathcal{M}(t)$ is bounded uniformly in t if $\|\mathbf{V}_0\|_{H^s \cap L^1} = \|\mathbf{V}_0\|_{H^s} + \|\mathbf{V}_0\|_{L^1}$ is small enough. We have for $\mathbf{U} = (u, v, w)$, and for $0 \le j \le s$,

$$\begin{aligned} \|\nabla^j \mathbf{U}(t)\|_{\mathcal{H}} &\le \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + \int_0^t \left\| \nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r) \right\|_{\mathcal{H}} dr \\ &= \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + \int_0^{t/2} \left\| \nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r) \right\|_{\mathcal{H}} dr \\ &\quad + \int_{t/2}^t \left\| \nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r) \right\|_{\mathcal{H}} dr \\ &\equiv \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + J_1 + J_2. \end{aligned}$$

$$\|\nabla^j \mathbf{U}(t)\|_{\mathcal{H}} \leq C(1+t)^{-N/4-j/2}\left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)}\right)$$
$$+ C\mathcal{M}^2(t)(1+t)^{-N/4-j/2} + C(1+t)^{-N/4-j/2}(M_0(t)+M_1(t))\mathcal{M}(t)$$

This yields

$$\mathcal{M}(t) \leq C\left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)}\right)$$
$$+ C\mathcal{M}^2(t) + C(M_0(t)+M_1(t))\mathcal{M}(t).$$

Applying for $m > \frac{N}{2}$, the estimate

$$\|\mathbf{V}\|_{L^\infty} \leq C\,\|\nabla^m \mathbf{V}\|_{L^2}^{\frac{N}{2m}}\,\|\mathbf{V}\|_{L^2}^{1-\frac{N}{2m}},$$

and similar estimates can be used for $\|v\|_{L^\infty}$. This yields $M_0(t) \leq C\mathcal{M}(t)$, provided that $s \geq m > \frac{N}{2}$.

Next, to estimate $M_1(t)$, we have for $m > \frac{N+2}{2}$,

$$\|\nabla\mathbf{V}\|_{L^\infty} \leq C\,\|\nabla^m \mathbf{V}\|_{L^2}^{\frac{N+2}{2m}}\,\|\mathbf{V}\|_{L^2}^{1-\frac{N+2}{2m}}.$$

This leads to $M_1(t) \leq C\mathcal{M}(t)$, provided that $s \geq m > \frac{N}{2}+1$. Hence, since $M_0(t)+M_1(t) \leq C\mathcal{M}(t)$, then (115) implies that

$$\mathcal{M}(t) \leq C\left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)}\right) + C\mathcal{M}^2(t).$$

Consequently, this gives the desired result, provided that $\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)}$ is small enough for all $0 \leq j \leq s$.

Other related result

1. V. Nikolic and B. Said-Houari, On the Jordan–Moore–Gibson–Thompson wave equation in hereditary fluids with quadratic gradient nonlinearity, Journal of Mathematical Fluid Mechanics, 2020
2. V. Nikolic and B. Said-Houari, Mathematical analysis of memory effects and thermal relaxation in nonlinear sound waves on unbounded domains Journal of Differential Equations, 2021.