An aspect of mass transport of particles towards a target

Boualem Djehiche
KTH Royal Institute of Technology, Stockholm

(Based on joint work with B. Bouchard and I. Kharroubi)

Outline

Stochastic target problem

Consider a system described by a stochastic process $X^{t,x,\nu}$ controlled by ν and starting at x at time t.

Stochastic target problem: Look for the values x such that the system reaches a set K at a terminal time T by choosing an appropriate control ν:

Characterize the reachability sets

$$V(t) = \left\{ x \in \mathbb{R}^d : \quad X_T^{t,x,\nu} \in K \ a.s. \ \text{ for some admissible control } \nu \right\}$$

for $t \in [0, T]$.

Motivating examples

• Optimal reservoir management problem (Upstream Sector in Petroleum Industry): Find the minimal amount x of liquid (e.g. water) to be injected (fracking) in a well at time t, to retrieve a desired amount $X_T^{t,x,\nu}$ of (shale) crude oil or gas, at time T, for some control ν (e.g. pipe dimension, pressure etc..)

Motivating examples

• Optimal reservoir management problem (Upstream Sector in Petroleum Industry): Find the minimal amount x of liquid (e.g. water) to be injected (fracking) in a well at time t, to retrieve a desired amount $X_T^{t,x,\nu}$ of (shale) crude oil or gas, at time T, for some control ν (e.g. pipe dimension, pressure etc..)

Other areas of application include

• Super-replication problem (Finance):
Find the minimal initial endowment such that there exists an investment strategy allowing the terminal wealth to be greater than a given payoff.

• Evacuation strategies in Crowd dynamics

• If we assume the flow property

$$X_t^{t,x} = x, \quad X_s^{t,x,\nu} = X_s^{r, X_r^{t,x,\nu}, \nu}, \quad \text{for any } r \in [t,s], \; x \in \mathbb{R}^d,$$

the Geometric Dynamic Programming Principle (DPP) yields an HJB for the set

$$V(t) = \left\{ x \in \mathbb{R}^d : \; X_T^{t,x,\nu} \in K \; a.s. \; \text{ for some admissible control } \nu \right\} :$$

- If $X_s^{t,x,\nu}$ is Brownian diffusion, $v(t,\cdot) = 1 - \mathbb{1}_{V(t)}(\cdot)$ is shown to solve an HJB equation (Soner and Touzi (2002), Bouchard et al. (2009)).

• In general, the minimal amount of water needed to extract shale oil/gas is too high to be afforded.

Extension of the stochastic target problem

Possible solution: relax the a.s. constraint to obtain a lower price (Föllmer and Leuckert (1999), Bouchard *et al.* (2009)):

- Solve the injection problem under terminal profit & loss constraint:

$$V_\ell(t) = \Big\{ x \in \mathbb{R}^d : \quad \mathbb{E}[\ell(X_T^{t,x,\nu})] \geqslant 0 \text{ for some control } \nu \Big\}.$$

Extension of the stochastic target problem

Possible solution: relax the a.s. constraint to obtain a lower price (Föllmer and Leuckert (1999), Bouchard *et al.* (2009)):

- Solve the injection problem under terminal profit & loss constraint:

$$V_\ell(t) = \Big\{x \in \mathbb{R}^d : \quad \mathbb{E}[\ell(X_T^{t,x,\nu})] \geqslant 0 \text{ for some control } \nu\Big\}.$$

- Example: Take $\ell(x) = 1\!\!1_K(x) - p$ with $p \in [0,1]$ to obtain

$$V_\ell(t) = \Big\{x \in \mathbb{R}^d : \quad \mathbb{P}(X_T^{t,x,\nu} \in K) \geqslant p \text{ for some control } \nu\Big\}.$$

Extension of the stochastic target problem

Possible solution: relax the a.s. constraint to obtain a lower price (Föllmer and Leuckert (1999), Bouchard *et al.* (2009)):

- Solve the injection problem under terminal profit & loss constraint:

$$V_\ell(t) = \left\{ x \in \mathbb{R}^d : \quad \mathbb{E}[\ell(X_T^{t,x,\nu})] \geqslant 0 \text{ for some control } \nu \right\}.$$

- Example: Take $\ell(x) = 1\!\!1_K(x) - p$ with $p \in [0,1]$ to obtain

$$V_\ell(t) = \left\{ x \in \mathbb{R}^d : \quad \mathbb{P}(X_T^{t,x,\nu} \in K) \geqslant p \text{ for some control } \nu \right\}.$$

  - Main idea: use the martingale representation theorem (a type of Riesz representation theorem) to express the expectation constraint as an a.s. constraint of an extended process.
    If Y is a function of the Brownian motion up to time T, then there exists a unique process 'control' α such that

$$Y = \mathbb{E}[Y] + \int_0^T \alpha_s dB_s.$$

Our case-study

Consider the stochastic target problem for controlled diffusion of mean-field type:

$$X_s^{t,\chi,\nu} = \chi + \int_t^s b(X_u^{t,\chi,\nu}, \mathbb{P}_{X_u^{t,\chi,\nu}}, \nu_u)du + \int_t^s \sigma(X_u^{t,\chi,\nu}, \mathbb{P}_{X_u^{t,\chi,\nu}}, \nu_u)dB_u,$$

where

- $\mathbb{P}_{X_u^{t,\chi,\nu}} = X_u^{t,\chi,\nu} \# \mathbb{P}$ denotes the probability law of $X_u^{t,\chi,\nu}$ under $\mathbb{P}$,
- b, σ deterministic functions of (x, y, z).
- B is a standard Brownian motion,
- χ square-integrable and $\mathcal{F}_t$-adapted.

Still with the time consistent constraint $\mathbb{E}[\ell(X_T^{t,x,\nu})] \geqslant 0, \; x \in \mathbb{R}^d$.

The particle picture: mean-field limit

For $i = 1, 2, \ldots,$, the B^i's are independent Brownian motion.
Consider

$$X_s^{t,i,n} = \chi^{i,n} + \int_t^s b(X_u^{t,i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_u^{t,j,n}}, \nu_u)du + \int_t^s \sigma(X_u^{t,\chi,\nu}, \frac{1}{n}\sum_{j=1}^n \delta_{X_u^{t,j,n}}, \nu_u)dB_u^i,$$

If $\chi^{i,n} \simeq \chi^i$ with χ^i independent with the same probability law as χ, then

$$X^{t,i,n} \simeq X^{t,i,\nu}, \qquad \frac{1}{n}\sum_{j=1}^n \delta_{X^{t,j,n}} \simeq \mathbb{P}_{X^{t,\chi,\nu}}, \quad \text{as } n \to \infty,$$

where the $X^{t,i,\nu}$ are independent with the same probability law as $X^{t,\chi,\nu}$.

Extended problem: conditional law

- Problem: While $X^{t,\chi,\nu}$, χ square-integrable r.v., defines a flow, $X^{t,x,\nu}$, $x \in \mathbb{R}^d$ does not have the above flow property!

Extended problem: conditional law

- Problem: While $X^{t,\chi,\nu}$, χ square-integrable r.v., defines a flow, $X^{t,x,\nu}$, $x \in \mathbb{R}^d$ does not have the above flow property!
- In general $X^{t,\chi,\nu} \neq X^{t,x,\nu}|_{x=\chi}$.

Extended problem: conditional law

• Problem: While $X^{t,\chi,\nu}$, χ square-integrable r.v., defines a flow, $X^{t,x,\nu}$, $x \in \mathbb{R}^d$ does not have the above flow property!

• In general $X^{t,\chi,\nu} \neq X^{t,x,\nu}|_{x=\chi}$.

• Condition on the Brownian motion B and apply the martingale representation theorem to obtain

$$\mathbb{E}[\ell(X_T^{t,\chi,\nu})] = \int \ell(x) d\mathbb{P}^B_{X_T^{t,\chi,\nu}}(x) - \int_t^T \alpha_s dB_s$$

for some control α,

where $\mathbb{P}^B_{X_T^{t,\chi,\nu}}(x)$ is the conditional probability law of $X_T^{t,\chi,\nu}$ given B i.e.

$$\mathbb{P}_{X_T^{t,\chi,\nu}}(x) = \int \mathbb{P}^y_{X_T^{t,\chi,\nu}}(x) \mathbb{P}_B(dy).$$

The constraint $\mathbb{E}[\ell(X_T^{t,\chi,\nu})] \geqslant 0$ can be rewritten as

$$L(\mathbb{P}^B_{\widetilde{X}_T^{t,\widetilde{\chi},\widetilde{\nu}}}) \geqslant 0 \qquad \text{or} \qquad \mathbb{P}^B_{\widetilde{X}_T^{t,\widetilde{\chi},\widetilde{\nu}}} \in L^{-1}([0,+\infty)) \qquad \text{a.s.}$$

with $\widetilde{\nu} = (\nu, \alpha)$, $\widetilde{\chi} = (\chi, 0)$,

$$\widetilde{X}^{t,\widetilde{\chi},\widetilde{\nu}} = (X^{t,\chi,\nu}, \int_t^{\cdot} \alpha_s dB_s),$$

$$L(\mu) = \int (\ell(x) - y) m(dx, dy),$$

$$m(dx, dy) := \mathbb{P}^B_{\widetilde{X}_T^{t,\widetilde{\chi},\widetilde{\nu}}}(dx, dy),$$

suggesting a stochastic target problem which involves $\mathbb{P}^B_{X_T^{t,\chi,\nu}}$.

Quenched mean-field SDE

Desintegrating $\mathbb{P}_{X_T^{t,\chi,\nu}}$ w.r.t. B, the dynamics of $X^{t,\chi,\nu}$ can be written as

$$X_s^{t,\chi,\nu} = \chi + \int_t^s b(X_u^{t,\chi,\nu}, \mathbb{P}^B_{X_u^{t,\chi,\nu}}, \nu_u)du + \int_t^s \sigma(X_u^{t,\chi,\nu}, \mathbb{P}^B_{X_u^{t,\chi,\nu}}, \nu_u)dB_u.$$

Such general formulation is related to the probabilistic analysis of large scale particle systems.

In those systems, one is interested in the behavior of particles conditional on the environment ('quenched' behavior/property) (see e.g. Le Doussal and Machta (1989)).

Interpretation of the target problem

By considering a probability law μ as initial condition, instead of χ, our target problem can be interpreted as a transport problem:

What is the collection of initial distributions μ of a system of particles, such that the terminal conditional law $\mathbb{P}^B_{X_T^{t,\chi,\nu}}$, given the environment (modeled by B) satisfies the constraint?

The reachability set reads

$$\mathcal{V}(t) = \Big\{ \mu : \text{ there exists } (\chi, \nu) \text{ s.t. } \mathbb{P}^B_{\chi} = \mu \text{ and } \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G \ a.s. \Big\}.$$

Probabilistic setting

$T > 0$ fixed time horizon.

$\Omega^\circ = \{\omega^\circ \in \mathcal{C}([0,T], \mathbb{R}^d) : \; \omega_0^\circ = 0\}$

$\mathbb{F}^\circ = (\mathcal{F}_t^\circ)_{t \le T}$ filtration generated by the canonical process $B(\omega^\circ) := \omega^\circ$, $\omega^\circ \in \Omega^\circ$.

$\mathbb{P}^\circ$ Wiener measure on $(\Omega^\circ, \mathcal{F}_T^\circ)$.

$\bar{\mathbb{F}}^\circ = (\bar{\mathcal{F}}_t^\circ)_{t \le T}$ the $\mathbb{P}^\circ$-completion of $\mathbb{F}^\circ$.

$\Omega' := [0,1]^d$ endowed with σ-algebra $\mathcal{F}' := \mathcal{B}([0,1]^d)$ and the Lebegues measure $\mathbb{P}'$.

Probability space

We then define the product filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ by

- $\Omega := \Omega^\circ \times \Omega'$,
- $\mathbb{P} = \mathbb{P}^\circ \otimes \mathbb{P}'$,
- $\mathcal{F} = \mathcal{F}_T$ where $\mathbb{F} = (\mathcal{F}_t)_{t \le T}$ is the completion of $(\mathcal{F}_t^\circ \otimes \mathcal{F}')_{t \le T}$.

We canonically extend the random variable ξ defined on Ω' and the process B on Ω by setting $\xi(\omega) = \xi(\omega')$ and $B(\omega) = B(\omega^\circ)$ for any $\omega = (\omega^\circ, \omega') \in \Omega$.

Advantages of this set up

Yields the following key ingredients to show the Geometric DPP and derive the HJB equation using Lions lifting argument:

Advantages of this set up

Yields the following key ingredients to show the Geometric DPP and derive the HJB equation using Lions lifting argument:

• If ν is $\mathbb{F}$-progressively measurable, then

$$\nu_s(\omega^\circ, \omega') = \mathrm{u}(s, B_{\cdot\wedge s}(\omega^\circ), \xi(\omega')), \quad s \in [0, T],$$

with u a Borel function.

Advantages of this set up

Yields the following key ingredients to show the Geometric DPP and derive the HJB equation using Lions lifting argument:

• If ν is $\mathbb{F}$-progressively measurable, then

$$\nu_s(\omega^\circ, \omega^{\text{'}}) = \mathrm{u}(s, B_{\cdot\wedge s}(\omega^\circ), \xi(\omega^{\text{'}})), \quad s \in [0, T],$$

with u a Borel function.

• Jankov-von Neumann's measurable selection theorem: there exists a measurable map ϑ such that (G closed set)

$$\mathbb{P}^B_{X_T^{\theta,\chi',\vartheta(\chi')}} \in G \quad \mathbb{P}^\circ - a.s. \;\text{ for } \mathfrak{P} - a.e. \;\chi'$$

where $\mathfrak{P}$ is the probability measure induced by $\omega^\circ \mapsto X_\theta^{t,\chi,\nu}(\omega^\circ, .)$.

Wasserstein space

We define

$$\mathcal{P}_2 := \left\{ \mu \text{ probability measure on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ s.t. } \int_{\mathbb{R}^d} |x|^2 \mu(dx) < +\infty \right\}.$$

This space is endowed with the 2-Wasserstein distance defined by

$$\mathcal{W}_2(\mu, \mu') := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dy, dy) \ : \ \text{s.t. } \ \pi(\cdot \times \mathbb{R}^d) = \mu \text{ and } \pi(\mathbb{R}^d \times \cdot) = \mu' \right\}^{\frac{1}{2}},$$

for $\mu, \mu' \in \mathcal{P}_2$. For later use, we also define the collection $\mathcal{P}_2^{\bar{\mathbb{F}}^\circ}$ of $\bar{\mathbb{F}}^\circ$-adapted continuous $\mathcal{P}_2$-valued processes.

Controlled quenched diffusion

Let U be a closed subset of $\mathbb{R}^q$ for some $q \geqslant 1$ and $\mathcal{U}$ the set of U-valued $\mathbb{F}$-progressive processes.

Given

- $\theta \in \bar{\mathcal{T}}^\circ$ (the set of $[0, T]$-valued $\bar{\mathbb{F}}^\circ$-stopping times),
- $\chi \in L^2(\Omega, \mathcal{F}_\theta, \mathbb{P}; \mathbb{R}^d)$,
- $\nu \in \mathcal{U}$,

we let $X^{\theta,\chi,\nu}$ denote the solution of

$$X = \mathbb{E}[\chi|\mathcal{F}_{\theta\wedge\cdot}] + \int_\theta^{\theta\vee\cdot} b\big(X_s, \mathbb{P}^B_{X_s}, \nu_s\big)\,ds + \int_\theta^{\theta\vee\cdot} a\big(X_s, \mathbb{P}^B_{X_s}, \nu_s\big)\,dB_s,$$

Existence, uniqueness and stability

We suppose that b, a are continuous, bounded and there exists a constant L such that

$$|b(x,\mu,\cdot)-b(x',\mu',\cdot)|+|a(x,\mu,\cdot)-a(x',\mu',\cdot)|\leqslant L\Big(|x-x'|+\mathcal{W}_2(\mu,\mu')\Big)$$

for all $t\in[0,T]$, $x,x'\in\mathbb{R}^d$ and $\mu,\mu'\in\mathcal{P}_2$.

Existence, uniqueness and stability

We suppose that b, a are continuous, bounded and there exists a constant L such that

$$|b(x,\mu,\cdot) - b(x',\mu',\cdot)| + |a(x,\mu,\cdot) - a(x',\mu',\cdot)| \leqslant L\Big(|x - x'| + \mathcal{W}_2(\mu,\mu')\Big)$$

for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$ and $\mu, \mu' \in \mathcal{P}_2$.

Proposition

For all $\theta \in \bar{\mathcal{T}}^{\circ}$, $\nu \in \mathcal{U}$ and $\chi \in L^2(\mathcal{F}_\theta)$, the SDE admits a unique strong solution $X^{\theta,\chi,\nu}$, and it satisfies

$$\mathbb{E}\Big[\sup_{[0,T]} |X^{\theta,\chi,\nu}|^2\Big] < +\infty,$$

$$\mathbb{P}^B_{X_T^{t,\chi,\nu}} = \mathbb{P}^B_{X_T^{\theta, X_\theta^{t,\chi,\nu},\nu}} \qquad \textit{(Flow property).}$$

Moreover, if $(t_n, \chi_n) \to (t, \chi)$ and $(\nu^n)_n \subset \mathcal{U}$ converges to ν $dt \otimes d\mathbb{P}$-a.e., then

$$\lim_{n\to\infty} \mathbb{E}[\,\mathcal{W}_2(\mathbb{P}^B_{X_T^{t_n,\chi_n,\nu^n}}, \mathbb{P}^B_{X_T^{t,\chi,\nu}})^2\,] = 0.$$

First formulation

Look for the set of initial measures for the conditional law $\mathbb{P}^B_\chi$ such that the terminal conditional law of $X_T^{t,\chi,\nu}$ given B belongs to a fixed closed subset G of $\mathcal{P}_2$:

$$\mathcal{V}(t) = \Big\{\mu \in \mathcal{P}_2 : \text{ there exists } (\chi,\nu) \in L^2(\mathcal{F}_t) \times \mathcal{U} \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\}.$$

This formulation is not convenient for setting a Geometric DPP:

• In $\mathcal{V}(t)$ only the probability distribution μ should matter and not a particular representation χ.

Strong formulation

The following strong formulation allows to take any representing random variable χ for μ.

Proposition

A measure $\mu \in \mathcal{P}_2$ belongs to $\mathcal{V}(t)$ if and only if for all $\chi \in L^2(\mathcal{F}_t)$ such that $\mathbb{P}^B_\chi = \mu$ there exists $\nu \in \mathcal{U}$ for which $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$:

$$\mathcal{V}(t) = \left\{ \mu \in \mathcal{P}_2 : \ \forall \chi \in L^2(\mathcal{F}_t) \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \exists \nu \in \mathcal{U} \ \text{ for which } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G \right\}.$$

This defines a mass transport problem towards a given target along the path of a mean-field diffusion.

Dynamic programming principle

Theorem

Fix $t \in [0, T]$ and $\theta \in \bar{\mathcal{T}}^\circ$ with values in $[t, T]$. Then,

$$\mathcal{V}(t) = \Big\{\mu \in \mathcal{P}_2 : \ \exists(\chi, \nu) \in L^2(\mathcal{F}_t) \times \mathcal{U} \ \textit{s.t.} \ \ \mathbb{P}^B_\chi = \mu \ \ \textit{and} \ \ \mathbb{P}^B_{X_\theta^{t,\chi,\nu}} \in \mathcal{V}(\theta)\Big\}.$$

Note that this DPP holds only for stopping times in $\bar{\mathcal{T}}^\circ$ *i.e.* stopping time w.r.t. the Brownian filtration.

The value function

Let $v : [0, T] \times \mathcal{P}_2 \to \mathbb{R}$ be the indicator function of the complement of the reachability set $\mathcal{V}$:

$$v(t, \mu) \quad = \quad 1 - \mathbb{I}_{\mathcal{V}(t)}(\mu) \ , \quad (t, \mu) \in [0, T] \times \mathcal{P}_2.$$

Aim: provide a characterization of v as a (discontinuous viscosity) solution of a fully non-linear second order parabolic partial differential equation.

Lifting on $\mathcal{P}_2$

Aim: define derivatives for functions defined on $\mathcal{P}_2$.

- Issue: $\mathcal{P}_2$ is not a vector space.

Possible approach: Lions Lifting

For a function $w: \mathcal{P}_2 \to \mathbb{R}$, we define its lift as $W: L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d) \to \mathbb{R}$ such that

$$W(X) = w(\mathbb{P}_X)\,, \qquad \text{for all } X \in L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)\,.$$

Allows to consider functions defined on the Hilbert space $L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)$.

Derivatives on $\mathcal{P}_2$

We then say that w is Fréchet differentiable (resp. $\mathcal{C}^1$) on $\mathcal{P}_2$ if its lift W is (resp. continuously) Fréchet differentiable on $L_2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)$.

Then $DW(X) \in L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)$ admits the representation

$$DW(X) = \partial_\mu w(\mathbb{P}_X)(X)$$

with $\partial_\mu w(\mathbb{P}_X): \ \mathbb{R}^d \to \mathbb{R}^d$ measurable map, called the derivative of w at $\mathbb{P}_X$.

We have $\partial_\mu w(\mu) \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$ for $\mu \in \mathcal{P}_2$.

We denote by $\partial_x \partial_\mu w(\mu)(x)$ the gradient of $x \in \mathbb{R}^d \mapsto \partial_\mu w(\mu)(x)$.

We have the following identification

$$\mathbb{E}\left[D^2 W(X)(YZ)YZ^\top\right] = \mathbb{E}\left[Tr\left(\partial_x \partial_\mu w(\mu)(X)YY^\top\right)\right] \tag{1}$$

for any $Y \in L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^{d\times d})$, $Z \sim N(0, I_d)$ and $Z \perp\!\!\!\perp (X, Y)$

Chain rule

Proposition

Let $w \in \mathcal{C}_b^{1,2}([0,T] \times \mathcal{P}_2)$. Given $(t, \chi, \nu) \in [0,T] \times L^2(\mathcal{F}_t) \times \mathcal{U}$, set $X = X^{t,\chi,\nu}$. Then,

$$\begin{aligned} w(s, \mathbb{P}^B_{X_s}) &= w(t, \mathbb{P}^B_{\chi}) \\ &+ \int_t^s \mathbb{E}_B \left[\partial_t w(r, \mathbb{P}^B_{X_r}) + \partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) b_r \right] dr \\ &+ \frac{1}{2} \int_t^s \mathbb{E}_B \left[Tr\left(\partial_x \partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) a_r a_r^\top \right) \right] dr \\ &+ \frac{1}{2} \int_t^s \mathbb{E}_B \left[\widetilde{\mathbb{E}}_B \left[Tr\left(\partial^2_\mu w(r, \mathbb{P}^B_{X_r})(X_r, \widetilde{X}_r) a_r \widetilde{a}_r^\top \right) \right] \right] dr \\ &+ \int_t^s \mathbb{E}_B \left[\partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) a_r(X_r, \mathbb{P}^B_{X_r}, \nu_r) \right] dB_r \end{aligned}$$

for all $s \in [t,T]$, where $(\widetilde{X}, \widetilde{a})$ is a copy of (X, a) on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

Chain rule on L^2

Given $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, denote $W(t, X)$ the r.v. $\omega^0 \in \Omega^0 \mapsto W(t, X(\omega^0, \cdot))$ a r.v. in $L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)$.

Corollary

Let $W : [0, T] \times L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d) \to \mathbb{R}$ *be* $\mathcal{C}_b^{1,2}$*. Set* $X = X^{t,\chi,\nu}$*. Then,*

$$\begin{aligned} W(s, \widetilde{X}_s) = W(t, \widetilde{\chi}) &+ \int_t^s \widetilde{\mathbb{E}}_B \left[\partial_t W(r, \widetilde{X}_r) + DW(r, \widetilde{X}_r) b_r(\widetilde{X}_r, \widetilde{\mathbb{P}}^B_{X_r}, \widetilde{\nu}_r) \right] dr \\ &+ \frac{1}{2} \int_t^s \widetilde{\mathbb{E}}_B \left[D^2 W(r, \widetilde{X}_r)(X_r) a_r a_r^\top (\widetilde{X}_r, \widetilde{\mathbb{P}}^B_{\widetilde{X}_r}, \widetilde{\nu}_r) \right] dr \\ &+ \int_t^s \widetilde{\mathbb{E}}_B \left[DW(r, \widetilde{X}_r) a_r(\widetilde{X}_r, \widetilde{\mathbb{P}}^B_{\widetilde{X}_r}, \widetilde{\nu}_r)) \right] dB_r, \end{aligned}$$

for all $s \in [0, T]$.

A 'quenched' PDE

We show that $V:[0,T]\times L^2(\Omega',\mathcal{F}',\mathbb{P}';\mathbb{R}^d)\to\mathbb{R}$ (lift of v) is a viscosity solution on $[0,T)\times L^2(\Omega',\mathcal{F}',\mathbb{P}';\mathbb{R}^d)$ of the quenched PDE

$$-\partial_t W(t,\xi)+\mathcal{H}\big(t,\xi,DW(t,\xi),D^2W(t,\xi)\big)=0,$$

where $\quad \mathcal{H}=\lim_{\varepsilon\to 0+}\mathcal{H}_\varepsilon,$

$$\mathcal{L}^u(\xi,P,Q):=\mathbb{E}_B\Big[b^\top(\xi,\mathbb{P}_\xi,u)P+\tfrac{1}{2}Q\big(a(\xi,\mathbb{P}_\xi,u)Z\big)a(\xi,\mathbb{P}_\xi,u)Z\Big],$$

$$\mathcal{H}_\varepsilon(t,\xi,P,Q):=\sup_{u\in\mathcal{N}_\varepsilon(t,\xi,P)}\Big\{-\mathcal{L}^u(\xi,P,Q)\Big\},$$

$$\mathcal{N}_\varepsilon(t,\xi,P):=\Big\{u\in L^0(\Omega,\mathcal{F},\mathbb{P};U)\ :\ |\mathbb{E}_B[a(\chi,\mathbb{P}_\xi,u)P]|\le\varepsilon\Big\},$$

$$P\in L^2(\Omega,\mathcal{F},\mathbb{P};U),\quad Q\,\text{self-adjoint operator on } L^2(\Omega,\mathcal{F},\mathbb{P};U),$$

where $\mathbb{E}_B[\,\cdot\,]$ means conditioning w.r.t. $(B_r,\ r\le T)$.

Continuity assumption

We need the following assumption. It ensures the existence of a regular feedback control 'close' to the kernel $\mathcal{N}_0$.

Continuity Assumption: Let $\mathcal{O}$ be an open subset of $[0,T]\times[L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)]^2$ such that $\mathcal{N}_0\neq\emptyset$ on $\mathcal{O}$. Then, for every $\varepsilon>0$, $(t_0,\chi_0,P_0)\in\mathcal{O}$ and $u_0\in\mathcal{N}_0(t_0,\chi_0,P_0)$, there exists an open neighborhood $\mathcal{O}'$ of (t_0,χ_0,P_0) and a measurable map $\hat{u}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\Omega'\to U$ such that:

(i) $\mathbb{E}_B[|\hat{u}_{t_0}(\chi_0,P_0,\xi)-u_0|]\leqslant\varepsilon$,

(ii) there exists $C>0$ for which

$$\mathbb{E}[|\hat{u}_t(\chi,P,\xi)-\hat{u}_t(\chi',P',\xi)|^2]\leq C\mathbb{E}[|\chi-\chi'|^2+\mathcal{W}_2^2(\mathbb{P}_P,\mathbb{P}_{P'})]$$

for all $(t,\chi,P),(t,\chi',P')\in\mathcal{O}'$,

(iii) $\hat{u}_t(\chi,P,\xi)\in\mathcal{N}_0(t,\chi,P)$ $\mathbb{P}^\circ-a.e.$, for all $(t,\chi,P)\in\mathcal{O}'$,

Viscosity property

We also suppose that there exists a constant C and a function $m:\ \mathbb{R}_+ \to \mathbb{R}$ such that $m(t) \underset{t\to 0}{\longrightarrow} 0$ and

$$|b(x,\mu,u) - b_{t'}(x,\mu,u')| + |a(x,\mu,u) - a(x,\mu,u')| \leqslant m(t-t') + C|u-u'|.$$

for all $t, t' \in [0,T]$, $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2$ and $u, u' \in U$.

Theorem

The function V is a viscosity supersolution of the HJB equation.

If in addition the **Continuity Assumption** *holds, then V is also a viscosity subsolution of the HJB equation.*

Parabolic boundary conditions

Define the function g by

$$g(\xi) = 1 - \mathbb{I}_G(\mathbb{P}_\xi)\ , \quad \xi \in L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)$$

and g_* and g^* its lower and upper semi-continuous envelopes.

Theorem

Under **(H1)**, *the function V satisfies*

$$V^*(T,.) \quad = g^* \quad \text{and} \quad V_*(T,.) \quad = g_*$$

on $L^2(\Omega', \mathcal{F}', \mathbb{P}'; \mathbb{R}^d)$.

Conclusion and perspectives

- Introduced a seemingly new stochastic target problem with potential financial and engineering applications.
- Obtained a dynamic programming principle
- Obtained a random PDE and derived some of its properties

Extensions and open problems

- Uniqueness (or a comparison result) for the PDE
- Target problem for $\mathbb{P}_{X_T}$ (unconditional law)
- Numerics for the quenched PDE.
- Processes quenched by other environments such as jump processes, long memory processes (fractional BM etc..)

References

D. P. Bertsekas and S. E. Shreve. *Stochastic Optimal Control. The Discrete-Time Case*. Academic Press, New York, 1978.

M. Birkner and Rongfeng Sun. *Annealed vs quenched critical points for a random walk pinning model*. Annales de l'Institut Henri Poincarè, Probabilitès et Statistiques, 46(2):414-441, 2010.

B. Bouchard, R. Elie, and N. Touzi. *Stochastic target problems with controlled loss*. SIAM Journal on Cont. and Opt., 48(5):3123-3150, 2009.

P. Cardaliaguet. *Notes on Mean Field Games (from P.-L. Lions' lectures at Collége de France)*. https://www.ceremade.dauphine.fr/ cardalia/MFG20130420.pdf, 2012.

References

H. Föllmer and P. Leukert. *Quantile hedging*. Finance and Stochastics, 3(3):251-273, 1999.

G. Giacomin. *Random polymer models*. Imperial College Press, 2007.

P. Le Doussal and J. Machta. *Annealed versus quenched diffusion coefficient in random media*. Physics Review B, 40(13):9427-9430, 1989.

H. M. Soner and N. Touzi. *Stochastic target problems, dynamic programming, and viscosity solutions*. SIAM Journal on Cont. and Opt., 41(2):404-424, 2002.

Thank You!