On the stability of equilibria for infinitely many particles

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From many body quantum dynamics to Hartree equation

The evolution of $\Psi_N(x_1, ..., x_N)$ the wave function for a system of N spinless fermions :

$$i\partial_t \Psi_N(t) = H_n \Psi_N(t), \quad H_N := \underbrace{-\Delta}_{\text{Dispersion}} + \underbrace{\frac{1}{N}\sum_{i<j} w(x_i, x_j)}_{\text{Pairwise interaction, e.g. Coulomb}} .$$

Problem : high dimensionality as $N \to \infty$.

Hartree-Fock : derivation of an ***effective equation***.

Assuming the Slater determinant form (Pauli exclusion principle) :

$$\Psi_N = u_1 \wedge u_2 \wedge ... \wedge u_N,$$

in the **mean field limit** $N \to \infty$:

$$i\partial_t u_j \approx -\Delta u_j + \frac{1}{N}\left((w * (\sum_{j=1}^{N} |u_k|^2) \right) u_j, \quad j = 1, ..., N.$$

[Bardos-Erdos-Golse-Mauser-Yau '02, Bardos-Golse-Gottlieb-Mauser '02, Elgart-Erdos-Schlein-Yau '03, Frohlich-Knowles '11, Benedikter-Porta-Schlein '14, Benedikter-Jaksic-Porta-Saffirio-Schlein '16]

Hartree equation for single functions, and for density matrices

For bosons, slater determinant is replaced by permanent :

$$\Psi_N = u \otimes u \otimes \dots \otimes u,$$

in the mean field limit $N \to \infty$ Hartree equation for a ***single function*** :

$$i\partial_t u = -\Delta u + (w * |u|^2)u.$$

Here :

$$i\partial_t u_j \approx -\Delta u_j + \frac{1}{N}\left((w * (\sum_{j=1}^{N} |u_k|^2) \right) u_j, \quad j = 1, \dots, N.$$

Set $\gamma = \frac{1}{N}\sum_{j=1}^{N} |u_j\rangle\langle u_j|$, that is, $\gamma(v) = \frac{1}{N}\sum_{j=1}^{N} u_j \langle u_j, v_j \rangle$.

Then Hartree equation for ***density matrices*** (operators) :

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma]$$ ~~+ exchange term~~

$\rho_\gamma = \gamma(t, x, x)$ the diagonal of the kernel of γ (charge density).

A Hartree equation for random fields

de Suzzoni '15 proposed another formulation :

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0,x,\omega) = X_0(x,\omega) \end{cases}$$

where :

- $(\Omega, d\omega)$ is a probability space and $\mathbb{E}(|X|^2) = \int_\Omega |X|^2 d\omega$.
- $X : I \times \mathbb{R}^d \times \Omega \to \mathbb{C}$ is a time dependent random field.
- w is a real valued pair interaction potential, and $*$ denotes convolution.

Here :

w is a finite Borel measure on $\mathbb{R}^3$.

If $w = \pm\delta$, nonlinear Schrödinger equation for random field :

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X \pm \mathbb{E}(|X|^2)X, \\ X(t=0,x,\omega) = X_0(x,\omega) \end{cases}$$

Link with the Hartree equation for density matrices

Derivation. Starting with :

$$i\partial_t u_j \approx -\Delta u_j + \frac{1}{N}\left((w * (\sum_{j=1}^{N} |u_k|^2) \right) u_j, \quad j = 1, ..., N.$$

Let $(g_j)_{1\leq j\leq N}$ an orthonormal family in $L^2(\Omega)$ (e.g. independent normalised complex Gaussians).

Set $X = \frac{1}{\sqrt{N}} \sum_1^N u_j(x) g_j(\omega)$ then X solves :

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0, x, \omega) = X_0(x, \omega) \end{cases}$$

In general, the covariance operator $\gamma_X(t) := \mathbb{E}(|X(t)\rangle, \langle X(t)|)$ (i.e. with kernel $\tilde{\gamma}(x,y) = \mathbb{E}\left(X(t,x)\bar{X}(t,y)\right)$ solves :

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma], \quad \rho_\gamma(x) = \tilde{\gamma}(x,x)$$

- It is not an equation with random forcing. It is not like other equations with random data since the evolution depends on the whole information on Ω.
- The condition $\|X\|_{L^2(\Omega, L^2(\mathbb{R}^d))} = \infty$ models an infinite number of particles.

A Hartree equation for random fields

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0,x,\omega) = X_0(x,\omega) \end{cases}$$

For $w = \delta_{x=0}$, if $X(t,x)$ is a solution, then $\lambda X(\lambda^2 t, \lambda x)$ is a solution : $(randomHartree)$ is $H^{d/2-1}$ critical.

Proposition [de Suzzoni '16]

For $d \geq 2$, the equation $(randomHartree)$ is locally well-posed in $L^2(\Omega, H^{d/2-1})$. Moreover, if $d = 3$ and $X_0 \in L^2(\Omega, H^1(\mathbb{R}^3))$, then the solution is global and there exists $X_{\pm\infty} \in L^2(\Omega, H^1(\mathbb{R}^3))$ such that :

$$\|X(t) - e^{it\Delta} X_{\pm\infty}\|_{L^2(\Omega, H^1(\mathbb{R}^3))} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

Steady states

The random field :

$$Y_f(t,x,\omega) = \int_{\xi\in\mathbb{R}^d} f(\xi) e^{i\xi.x - it(m+|\xi|^2)} dW(\xi), \quad m := \int_{\mathbb{R}^d} |f|^2 d\xi$$

- is a steady state (solution + time translation invariant law) for (*randomHartree*).
- is a centred stationary Gaussian field (space translation invariant covariance) with covariance operator the Fourier multiplier :

$$\gamma_f := \mathbb{E}(|\cancel{X}(t)\rangle, \langle \cancel{X}(t)|) = \mathcal{F}^{-1}(|f(\xi)|^2 \mathcal{F})$$

- is non-localised : $\mathbb{E}(|Y_f(x)|^2) = m$ (Fermionic gas).

Above dW is the Wiener integral ("sum of infinitesimal complex Gaussians") :

$$\mathbb{E}\left(\overline{\int_{\mathbb{R}^d} f(\xi')dW(\xi')} \int_{\mathbb{R}^d} g(\xi)dW(\xi)\right) = \int_{\mathbb{R}^d} \bar{f}(\xi)g(\xi)d\xi.$$

For example, f the Fermi-Dirac equilibrium momentum distribution :

$$|f(\xi)|^2 = \frac{1}{e^{\frac{|\xi|^2-\mu}{T}}+1}$$

Main result

Notations :

- Lebesgue spaces :

$$\|u\|_{L^p_x L^q_\omega} = \left(\int_{\mathbb{R}^3} (\mathbb{E}(|u|^q))^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}}$$

- Sobolev spaces :

$$\|u\|_{L^2_\omega H^s_x} = \left(\mathbb{E} \left(\|u\|^2_{H^s(\mathbb{R}^3)} \right) \right)^{\frac{1}{2}}$$

- Schatten spaces :

$$\|\gamma\|_{\mathfrak{S}^p} = |\mathrm{Trace}(\sqrt{\gamma^*\gamma}^p)|^{\frac{1}{p}}$$

- Sobolev-Schatten spaces :

$$\|\gamma\|_{\mathfrak{S}^{s,p}} = \|\langle\nabla\rangle^s \gamma \langle\nabla\rangle^s\|_{\mathfrak{S}^p}$$

Main result

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0,x,\omega) = X_0(x,\omega) \end{cases} \qquad (1)$$

Initial datum :

$$X_0 = Y_f(t=0) + Z_0, \qquad Y_f(t,x,\omega) = \int_{\xi\in\mathbb{R}^d} f(\xi) e^{i\xi.x - it(m+|\xi|^2)} dW(\xi) \text{ steady state} \qquad (2)$$

Theorem [C.-de Suzzoni '20] Part 1

Assume $d=3$, f radial, decreasing, with additional smoothness properties, w is a finite even Borel measure on $\mathbb{R}^3$ with $\|\hat{w}_-\|_{L^\infty} + \hat{w}(0)_+ \leq C(f)$.
There exists $\epsilon > 0$ such that for any $Z_0 \in L^2_\omega H^{1/2}_x \cap L^{3/2}_x L^2_\omega$ with

$$\|Z_0\|_{L^2_\omega H^{1/2}_x} + \|Z_0\|_{L^{3/2}_x L^2_\omega} \leq \epsilon,$$

the solution to (3) with initial datum (4) is global and scatters to Y_f :

$$X(t) = Y_f(t) + W_V(Y)(t) + e^{i(\Delta - m)t} Z_\pm + o_{L^2_\omega H^{1/2}_x}(1) \quad \text{as } t \to \pm\infty.$$

$$W_V(Y) = S(t)\left(\tilde{Z}_\pm + o_{L^3_x L^2_\omega}(1)\right) \quad \text{as } t \to \pm\infty.$$

...

Main result

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0,x,\omega) = X_0(x,\omega) \end{cases} \tag{3}$$

Initial datum :

$$X_0 = Y_f(t=0) + Z_0, \qquad Y_f(t,x,\omega) = \int_{\xi\in\mathbb{R}^d} f(\xi) e^{i\xi.x - it(m+|\xi|^2)} dW(\xi) \text{ steady state} \tag{4}$$

Associated operators (density matrix) :

$$\gamma = \mathbb{E}(|X\rangle\langle X|), \qquad \gamma_f = \mathbb{E}(|Y_f\rangle\langle Y_f|) = m_{|f|^2(\xi)}$$

Theorem [C.-de Suzzoni '20] Part 2

...

there exists $\gamma_\pm \in \mathfrak{S}^{1/2,4+}$ and the convergence :

$$\gamma = \gamma_f + e^{i\Delta t}\gamma_\pm e^{-i\Delta t} + o_{\mathfrak{S}^{\frac{1}{2},4+}}(1) \qquad \text{as } t \to \pm\infty.$$

Some previous results in the context of density matrices

Hartree for density matrices :

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma], \quad \gamma_f := \mathcal{F}^{-1}(|f(\xi)|^2 \mathcal{F})$$

Under various different hypotheses on the momentum distribution function f and interaction potential w :

- [Lewin-Sabin '15] Global well posedness near γ_f in Schatten spaces for $d = 2$.
- [Chen-Hong-Pavlovic '17] Global well posedness near γ_f in Schatten spaces for $d \geq 3$, w allowed to be a Dirac interaction.
- [Lewin-Sabin '14] Scattering near γ_f in Schatten spaces for $d = 2$.
- [Chen-Hong-Pavlovic '18] Scattering near γ_f in Schatten spaces for $d \geq 3$.
- [Pusateri-Sigal '19] Scattering near 0 in Schatten spaces for $d \leq 3$, (new approach : vector fields and Sobolev-Klainerman inequalities).
- The present work extends these results to the Hartree equation for random fields for $d = 3$. We can reach optimal regularity for the perturbation and the limits of the density matrices, and a Dirac interaction potential. Previous works for $d = 4$ by [C.-de Suzzoni '18].
- [Lewin-Sabin '20] prove the convergence of Hartree to Vlasov in semi-classical regime, and obtain from their global well-posedness result for Hartree near steady state, a global well-posedness result for Vlasov near steady state.

Proof : Set-up

Write the perturbation : $X = Y_f + \tilde{Z}$:

$$i\partial_t \tilde{Z} = (m-\Delta)\tilde{Z} + \Big(w * \underbrace{(2Re\mathbb{E}(\bar{Y}\tilde{Z})) + \mathbb{E}(|\tilde{Z}|^2)}_{:=V(t,x)}\Big)(Y+\tilde{Z})$$

Change unknown to couple $(\tilde{Z}, V)$:

$$\begin{cases} i\partial_t \tilde{Z} = (m-\Delta)\tilde{Z} + (w*V)(Y+\tilde{Z}), & \tilde{Z}(t=0) = Z_0, \\ V = \mathbb{E}(|\tilde{Z}|^2) + 2\mathrm{Re}\ \mathbb{E}(\bar{Y}\tilde{Z}). \end{cases}$$

Worst term in first equation :

$$i\partial_t \bar{Z} = (m-\Delta)\bar{Z} + (w*V)Y, \qquad \bar{Z}(t=0) = 0,$$

Notations : $S(t)Z_0$ is the solution to $i\partial_t \bar{Z} = (m-\Delta)\bar{Z}, \qquad \bar{Z}(t=0) = Z_0.$

$W_{V'}(f)$ denotes solution to $i\partial_t \bar{Z} = (m-\Delta)\bar{Z} + V'f, \qquad \bar{Z}(t=0) = 0.$

Change unknown to : $X = Y_f + W_V(Y) + Z$:

$$\begin{cases} Z = S(t)Z_0 + W_V^2(Y) + W_V(Z), \\ V = 2\mathrm{Re}\ \mathbb{E}(\bar{Y}S(t)Z_0) + 2\mathrm{Re}\ \mathbb{E}(\bar{Y}W_V(Y)) + \mathbb{E}(|Z|^2) + 2\mathrm{Re}\ \mathbb{E}(\overline{W_V(Y)}Z + \bar{Y}W_V(Z)) \\ \quad +\mathbb{E}(|W_V(Y)|^2) + 2\mathrm{Re}\ \mathbb{E}(\bar{Y}W_V^2(Y)). \end{cases}$$

Previous results for the Gross-Pitaevskii equation

Gross Pitaevskii equation :

$$i\partial_t u = -\Delta u + (|u|^2 - 1)u \qquad \lim_{|x|\to\infty} |u(x)| = 1.$$

Equilibrium $v(x) = e^{i\theta}$, $\theta \in \mathbb{R}$.

- [Bethuel-Saut '99, Gerard '06] Global well-posedness in H^1, GWP in energy space, $d = 2, 3$.
- [Gustafson-Nakanishi-Tsai '06] Scattering around v in dimension $d \geq 4$.
- [Gustafson-Nakanishi-Tsai '07 and '09] Scattering around v in dimension $d = 3$ under spatial decay assumption.
- [Guo-Hani-Nakanishi '18] Scattering around v in dimension $d = 3$ under angular regularity assumption.

Proof : Linear problem

$$\begin{cases} i\partial_t \tilde{Z} = (m-\Delta)\tilde{Z} + (w * V)Y, & \tilde{Z}(t=0) = Z_0, \\ V = 2\mathrm{Re}\ \mathbb{E}(\bar{Y}\tilde{Z}). \end{cases} \tag{5}$$

Proposition

For $Z_0 \in L^2_\omega H^{1/2}_x \cap L^{3/2}_x L^2_\omega = \Theta_0$ the solution to (6) is :

$$\tilde{Z} = S(t)(Z_0) + W_{w*V}(Y)$$

with :

$$\|S(t)Z_0\|_{\underbrace{L^2_\omega \mathcal{C}(\mathbb{R}, H^{1/2}_x)}_{\text{Continuity in } H^s} \cap \underbrace{L^2_\omega L^5_{t,x} \cap L^2_\omega L^{10/3}_t W^{1/2,10/3}_x}_{\text{Strichartz + Sobolev}}} + \|V\|_{\underbrace{L^2_t H^{1/2}_x}_{*} \cap \underbrace{L^{5/2}_{t,x}}_{\text{S+S}}} \lesssim \|Z_0\|_{\Theta_0}$$

$$\|W_{w*V}(Y)\|_{\underbrace{L^\infty_t L^3_x L^2_\omega}_{*}} + \|\langle\nabla\rangle^{1/2} W_{w*V}(Y)\|_{\underbrace{L^5_{t,x} L^2_\omega}_{\text{S+S}}} \lesssim \|Z_0\|_{\Theta_0}.$$

$*$ = Improved specific decay for problem.

Proof : Linear problem

Perturbation created by potential

$W_{w*V}(Y)$

Perturbation of solution $\tilde{Z}$

Perturbation of potential V

$2\mathrm{Re}\mathbb{E}(\bar{Y}\tilde{Z})$

Potential created by perturbation

Proof : Linear problem

Local perturbation of momenta distribution function :

$$Z_g(\omega,x)=\int_{\mathbb{R}^3} dW(\xi)e^{i\xi.x}g(x,\xi), \qquad \|Z_g\|_{\dot H^s_{x,\omega}}=\|g\|_{L^2_\xi \dot H^s_x}.$$

Proposition

For any $s\in\mathbb{R}$:

[Perturbation created by potential] $\quad \|W_U(Y)\|_{L^\infty_t \dot H_x^{s+\cancel{\frac12}}}\lesssim \|U\|_{L^2_t \dot H_x^{s-\frac12}}$

[Potential created by perturbation] $\quad \|\mathbb{E}\left(\bar Y(t)S(t)Z_g\right)\|_{L^2_t \dot H_x^{s+\frac12}}\lesssim \|Z_g\|_{\dot H^s_{x,\omega}}$

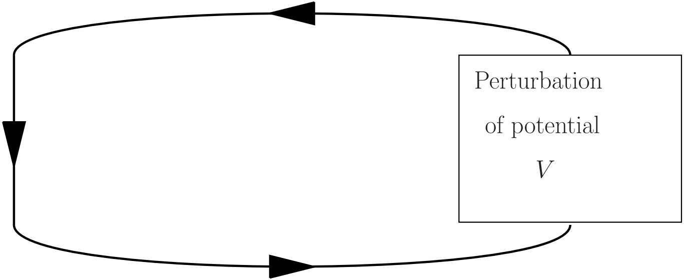
Proof : Spreading effect by interaction with Y.

$$\left[\Delta, e^{i\xi.x}\right]=e^{i\xi.x}\Big(\underbrace{|\xi|^2}_{\text{Phase renormalisation}} - \underbrace{2\xi.\nabla}_{\text{Transport in direction } \xi}\Big)$$

$$\mathbb{E}(\bar Y S(t)Z_g)=\mathcal{F}^{-1}\Big(\int_{\mathbb{R}^3} f(\xi)\underbrace{e^{-it(|\eta|^2+2\eta.\xi)}}_{\text{Non stationary phase in } \eta}\hat g(\xi,\eta)d\xi\Big).$$

$+\quad TT^*$ and Christ-Kiselev

Proof : Linear problem



$$2\mathrm{Re}\mathbb{E}(\bar{Y} W_{w*V}(\bar{Y})) =: L(V)$$

Linear response of steady state Y

to a potential V

Proof : Linear problem

[Lindhard '54], [Lewin-Sabin] '14

Proposition [Linear response of steady state to a potential]

L_2 is a Fourier multiplier in time and space.

Under the hypotheses of Theorem, L_2 is continuously invertible on $L^2_{t,x}$

Remark : **no loss** of 1/2 derivative at low frequencies as previously !

Formal computation : ***Low frequency linear cancellation***. Given $V(t,x) \approx V(t)$ low frequency,

$$W_V(Y)(t) \approx -iY(t)\int_0^t V(\tau)d\tau,$$

$$\mathfrak{RE}(\bar{Y}W_V(Y)) = \mathfrak{RE}\left(\bar{Y}\left(-iY\int_0^t V(\tau)d\tau\right)\right) = -\mathfrak{RE}\left(i|Y|^2\int_0^t V(\tau)d\tau\right) = 0,$$

The contribution to low frequency perturbation is *out of phase* with Y !

Proof : Linear problem

$$\begin{cases} i\partial_t \tilde{Z} = (m-\Delta)\tilde{Z} + (w * V)Y, & \tilde{Z}(t=0) = Z_0, \\ V = 2\mathrm{Re}\ \mathbb{E}(\bar{Y}\tilde{Z}). \end{cases} \quad (6)$$

Proposition

For $Z_0 \in L^2_\omega H^{1/2}_x \cap L^{3/2}_x L^2_\omega = \Theta_0$ the solution to (6) is :

$$\tilde{Z} = S(t)(Z_0) + W_{w*V}(Y)$$

with :

$$\|S(t)Z_0\|_{\underbrace{L^2_\omega \mathcal{C}(\mathbb{R}, H^{1/2}_x)}_{\text{Continuity in } H^s} \cap \underbrace{L^2_\omega L^5_{t,x} \cap L^2_\omega L^{10/3}_t W^{1/2,10/3}_x}_{\text{Strichartz + Sobolev}}} + \|V\|_{\underbrace{L^2_t H^{1/2}_x}_{*} \cap \underbrace{L^{5/2}_{t,x}}_{\text{S+S}}} \lesssim \|Z_0\|_{\Theta_0}$$

$$\|W_{w*V}(Y)\|_{\underbrace{L^\infty_t L^3_x L^2_\omega}_{*}} + \|\langle\nabla\rangle^{1/2} W_{w*V}(Y)\|_{\underbrace{L^5_{t,x} L^2_\omega}_{\text{S+S}}} \lesssim \|Z_0\|_{\Theta_0}.$$

$*$ = Improved specific decay for problem.

Proof : Nonlinear problem

Recall the set-up :
Solution of the form : $X = Y_f + \underbrace{W_V(Y) + Z}_{=\tilde{Z}}$, $V = 2Re\mathbb{E}(\bar{Y}\tilde{Z}) + \mathbb{E}(|\tilde{Z}|^2)$:

$$\begin{cases} Z = S(t)Z_0 + W_V^2(Y) + \underbrace{W_V(Z)}_{\text{linear } L}, \\ V = 2\mathrm{Re}\ \mathbb{E}(\bar{Y}S(t)Z_0) + \underbrace{2\mathrm{Re}\ \mathbb{E}(\bar{Y}W_V(Y))}_{\text{linear } L} + \mathbb{E}(|Z|^2) + 2\mathrm{Re}\ \mathbb{E}(\overline{W_V(Y)}Z + \bar{Y}W_V(Z)) \\ \quad + \mathbb{E}(|W_V(Y)|^2) + 2\mathrm{Re}\ \mathbb{E}(\bar{Y}W_V^2(Y)). \end{cases}$$

Fixed point formulation :

$$\begin{pmatrix} Z \\ V \end{pmatrix} = (Id - L)^{-1}\left[C_0 + Q\begin{pmatrix} Z \\ V' \end{pmatrix}\right].$$

$$C_0 = \begin{pmatrix} S(t)Z_0 \\ 2\mathrm{Re}\ \mathbb{E}(\bar{Y}S(t)Z_0) \end{pmatrix}, \quad Q\begin{pmatrix} Z \\ V' \end{pmatrix} = \begin{pmatrix} W_{V'}(Z) + W_{V'}^2(Y) \\ \mathbb{E}(|Z|^2) + Q_1(Z, V') + Q_2(V') \end{pmatrix},$$

Proof : Nonlinear problem

Worst quadratic part :

$$Q_2(V') = \mathbb{E}(|W_{V'}(Y)|^2) + 2\mathrm{Re}\ \mathbb{E}(\bar{Y} W_{V'}^2(Y)).$$

Proposition [Adaptation of Lewin-Sabin '14]

There holds :

$$\|Q_2(V,U)\|_{L^2_{t,x} \cap L^\infty_t L^2_x} \lesssim C(h,w) \|V\|_{L^2_t H^{1/2}_x} \|U\|_{L^2_t H^{1/2}_x}.$$

Quadratic cancellation at low frequencies : for $V(t,x) \approx V(t)$ and $U(t,x) \approx U(t)$:

$$\begin{aligned} & \overline{W_U(Y)} W_V(Y) + \bar{Y} W_V \circ W_U(Y) + \bar{Y} W_U \circ W_V(Y) \\ = \quad & |\bar{Y}|^2(t) \int_0^t U \cdot \int_0^t V - |Y|^2(t) \int_0^t ds V(s) \int_0^s U(s')ds' - |Y|^2(t) \int_0^t ds U(s) \int_0^s V(s')d \\ = \quad & 0 \end{aligned}$$

$\Rightarrow$ linearised potential generated by the second iterate $W_V \circ W_V$ cancels with the quadratic potential created by W_V. Specific to nonlinearities $f(\mathbb{E}|X|^2)$.

Proof : Nonlinear problem

Fixed point :

$$\begin{pmatrix} Z \\ V \end{pmatrix} = (Id - L)^{-1} \left[C_0 + Q \begin{pmatrix} Z \\ V' \end{pmatrix} \right].$$

$$C_0 = \begin{pmatrix} S(t) Z_0 \\ 2\mathrm{Re}\, \mathbb{E}(\bar{Y} S(t) Z_0) \end{pmatrix}, \qquad Q \begin{pmatrix} Z \\ V' \end{pmatrix} = \begin{pmatrix} W_{V'}(Z) + W^2_{V'}(Y) \\ \mathbb{E}(|Z|^2) + Q_1(Z, V') + Q_2(V') \end{pmatrix},$$

solved for :

$$Z \in L^2_\omega \mathcal{C}(\mathbb{R}, H^{1/2}_x) \cap L^2_\omega L^5_{t,x} \cap L^2_\omega L^{10/3}_t W^{1/2,10/3}_x,$$

$$V \in L^2_t H^{1/2}_x \cap L^{5/2}_{t,x}.$$

Low frequencies issues handled by previous analysis.

High frequencies issues handled by Strichartz (usual) + Fractional Leibniz $\Rightarrow$ reach critical $H^{1/2}_x$ regularity.

Scattering is obtained easily from global bounds (Ginibre-Velo).

Proof of scattering for associated density matrix

Theorem [C.-de Suzzoni '20] Part 2

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there exists $\gamma_\pm \in \mathfrak{S}^{1/2,4+}$ and the convergence :

$$\gamma = \mathbb{E}(|X\rangle\langle X|) = \gamma_f + e^{i\Delta t}\gamma_\pm e^{-i\Delta t} + o_{\mathfrak{S}^{\frac{1}{2},4+}}(1) \quad \text{as } t \to \pm\infty.$$

Recall that we have obtained :

$$X(t) = Y_f(t) + W_V(Y)(t) + e^{i(\Delta - m)t} Z_\pm + o_{L^2_\omega H_x^{1/2}}(1) \quad \text{as } t \to \pm\infty.$$

Let

$$\gamma_\pm = \mathbb{E}\Big(|\underbrace{W_{V,\pm}Y_0}_{\text{worst linear}} + Z_\pm\rangle\langle Y_0| + |Y_0\rangle\langle \underbrace{W_{V,\pm}Y_0}_{\text{worst linear}} + Z_\pm| + \underbrace{|W_{V,\pm}Y_0 + Z_\pm\rangle\langle W_{V,\pm}Y_0 + Z_\pm|}_{\text{quadratic}}$$

First **worst** linear term :

$$\mathbb{E}(|W_{V,\pm}Y_0\rangle\langle Y_0|) = -iW_{V,\pm}\circ \underbrace{\gamma_{f^2}}_{\text{Fourier multiplier by } |f(\xi)|^2}$$

where

$$W_{V,\pm}u = \int_0^{\pm\infty} S(-s)(V(s)S(s)(u))ds$$

Proof of scattering for associated density matrix

Worst term for scattering for density matrices :

$$W_{V,\pm}u = \int_0^{\pm\infty} S(-s)(V(s)S(s)(u))ds$$

Theorem [Frank Sabin '17]

For any :

$$\frac{1+d}{2} < q' \leq \infty \quad \text{and} \quad \frac{2}{p'} + \frac{d}{q'} = 2$$

there holds :

$$\|W_{V,\pm}\|_{\mathfrak{S}^{2q'}(L^2(\mathbb{R}^d))} \lesssim \|V\|_{L_t^{p'}L^{q'}(\mathbb{R}^d)}.$$

Previous work by [Frank-Lewin-Lieb-Seiringer] '14. Dual version of the following *Strichartz inequality for a system of orthonormal functions* $(u_j)_{j\in\mathbb{N}}$:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \left| \sum_j n_j (e^{it\Delta}u_j)(x)|^2 \right|^q \right)^{\frac{p}{q}} \leq C_{d,q}^p \left(\sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{p(q+1)}{2q}}.$$

Our regularity $\gamma_\pm \in \mathfrak{S}^{1/2,4+}$ seems optimal.

Conclusion

- Studied an equivalent formulation of the Hartree equation for density matrices

$$(randomHartree) \quad \begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0,x,\omega) = X_0(x,\omega) \end{cases}$$

  around non-localised steady states

$$Y_f(t,x,\omega) = \int_{\xi \in \mathbb{R}^d} f(\xi) e^{i\xi.x - it(m+|\xi|^2)} dW(\xi)$$

  and proved scattering for localised perturbation, borrowing ideas from [Lewin-Sabin] '14, [Chen-Hong-Pavlovic] '17.

- Reached a regularity that includes the Dirac potential $w = \delta_0$, and other optimal regularities for initial perturbation, asymptotic of associated density matrix (novelties).

- Relies on the study of the particle density (Lindhard functional) V.
  Linear dynamics do not inflate at low frequencies because of cancellation.
  Quadratic terms also have low frequencies cancellation.
  In addition to the usual decay (Strichartz) of linear Schrödinger, an additional decay due to interaction with Y_f.
  Used Strichartz estimates for orthonormal systems from
  [Frank-Lewin-Lieb-Seiringer] '14 and [Frank-Sabin] '17.

Thank you for your attention ! !
And I thank the organisers ! !