The Inviscid Primitive Equations and the Effect of Rotation

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Overview

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- $\mathcal{U} = (\mathcal{V}, w)$, $\mathcal{V} = (u, v)$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Delta_h = \partial_{xx} + \partial_{yy}$.
- $\nu_h \geq 0, \nu_z \geq 0$, dimensionless, horizontal and vertical viscosity.
- $\kappa_h \geq 0, \kappa_z \geq 0$, dimensionless, horizontal and vertical diffusivity.
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- For planetary scales oceanic and atmospheric dynamics, the vertical scale (a few kilometers for ocean, 10–20 kilometers for atmosphere) is much smaller than the horizontal scales (several thousands of kilometers).
- Consider system (1) in a thin domain $\mathcal{D}_\epsilon := \left\{(x, y, z) : 0 \leq z \leq \epsilon, (x, y) \in \mathbb{R}^2\right\}$.

- Following Azérad-Guillén [1] and Li-Titi [42], take $\nu_h = \kappa_h = 1$ and $\nu_z = \kappa_z = \epsilon^2$.
- By rescaling, consider

$$\begin{cases} \mathcal{V}_\epsilon(x,y,z,t) = \mathcal{V}(x,y,\epsilon z,t), & w_\epsilon(x,y,z,t) = \frac{1}{\epsilon} w(x,y,\epsilon z,t), \\ p_\epsilon(x,y,z,t) = p(x,y,\epsilon z,t), & T_\epsilon(x,y,z,t) = \epsilon T(x,y,\epsilon z,t) \end{cases}$$

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- Scaled Boussinesq equations

$$\begin{cases} \partial_t \mathcal{V}_\epsilon + \mathcal{V}_\epsilon \cdot \nabla_h \mathcal{V}_\epsilon + w_\epsilon \partial_z \mathcal{V}_\epsilon - \Delta \mathcal{V}_\epsilon + \Omega \mathcal{V}_\epsilon^\perp + \nabla_h p_\epsilon = 0, \\ \nabla_h \cdot \mathcal{V}_\epsilon + \partial_z w_\epsilon = 0, \\ \epsilon^2 (\partial_t w_\epsilon + \mathcal{V}_\epsilon \cdot \nabla_h w_\epsilon + w_\epsilon \partial_z w_\epsilon - \Delta w_\epsilon) + \partial_z p_\epsilon = T_\epsilon \\ \partial_t T_\epsilon + \mathcal{V}_\epsilon \cdot \nabla_h T_\epsilon + w_\epsilon \partial_z T_\epsilon - \Delta T_\epsilon = 0 \end{cases} \tag{2}$$

in $\mathcal{D} := \left\{(x,y,z) : 0 \leq z \leq 1, (x,y) \in \mathbb{R}^2\right\}$. Here $\mathcal{V}_\epsilon^\perp = (-v_\epsilon, u_\epsilon)$.

- Taking formal limit $\epsilon \to 0^+$, and suppose that $(\mathcal{V}_\epsilon, w_\epsilon, p_\epsilon, T_\epsilon)$ converges to $(\mathcal{V}, w, p, T)$ in a suitable sense, we derive $3D$ **dimensionless** primitive equations (PEs):

$$\begin{cases} \partial_t \mathcal{V} + \mathcal{V} \cdot \nabla_h \mathcal{V} + w\partial_z \mathcal{V} - \Delta \mathcal{V} + \Omega \mathcal{V}^\perp + \nabla_h p = 0, \\ \partial_z p = T, \\ \nabla_h \cdot \mathcal{V} + \partial_z w = 0, \\ \partial_t T + \mathcal{V} \cdot \nabla_h T + w\partial_z T - \Delta T = 0, \end{cases} \tag{3}$$

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- Introduced by Richardson (1922) for weather prediction.
- J.L. Lions, R. Temam, S. Wang (1992) Gave some asymptotic derivation of the PE.
- Rigorous justification of the derivation first by Azérad-Guillén [1] in a weak sense, then by Li-Titi [42] in a strong sense with error estimates in terms of ϵ.

- Global existence of weak solutions in $3D$ by Lions, Temam and Wang [39, 40, 41].
- Uniqueness of weak solution in $2D$ by Bresch, Guillén-González, Masmoudi and Rodríguez-Bellido [10], while remains open for $3D$.
- Local existence of strong solution in $2D$ and $3D$ by Guillén-González, Masmoudi and Rodríguez-Bellido [26], global existence by Bresch, Kazhikhov and Lemoine [11].

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- Global existence of strong solution in $3D$ first by Cao and Titi [18], and later by Kobelkov [31], Kukavica and Ziane [35, 36] for different boundary conditions, Hieber and Kashiwabara [28] for some progress towards relaxing the smoothness on the initial data by using the semigroup method.

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- Global well-posedness of the $3D$ PEs with full viscosity and partial diffusivity by Cao and Titi [19], Cao, Li and Titi [13, 14].
- Global well-posedness of the $3D$ PEs with only horizontal viscosity and partial diffusivity by Cao, Li and Titi [15, 16, 17].

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- Local well-posedness of the inviscid PEs with either real analyticity or some special structures on the initial data by Brenier [8, 9], Grenier [25], Kukavica, Masmoudi, Vicol and Wong [33], Kukavica, Temam, Vicol and Ziane [34], Masmoudi and Wong [43].

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in $\mathcal{D} := \{(x, y, z) : 0 \le z \le 1, (x, y) \in \mathbb{R}^2\}$, subject to

$$w|_{z=0,1} = 0. \tag{5}$$

- For $\Omega = 0$, with initial data

$$u_0(x, y, z) = u_0(x, z), \quad v_0(x, y, z) = v_0(x, z) = 0, \tag{6}$$

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$$\begin{cases} u_t + u\,u_x + wu_z + p_x = 0, \\ p_z = 0, \\ u_x + w_z = 0. \end{cases} \tag{7}$$

Consider in the domain $\mathbb{T} \times [0,1]$, subject to

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- Any shear flow $(U, W, P) = (U(z), 0, 0)$ is a steady solution to system (7).

- Denote by $(u, w, p) = (U + \widetilde{u}, W + \widetilde{w}, P + \widetilde{p}) = (U(z) + \widetilde{u}, \widetilde{w}, \widetilde{p})$.
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- Renardy [44] showed that for certain shear flow $U(z)$ (U is odd and U^{-2} is integrable, for example, $U(z) = \tanh(\frac{z-1/2}{d})$ with d small), the linearized system (10) has solutions of the form

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- Such Kelvin-Helmholtz type instability implies that the inviscid PEs is linearly ill-posed in any Sobolev space,

- Denote by $(u, w, p) = (U + \widetilde{u}, W + \widetilde{w}, P + \widetilde{p}) = (U(z) + \widetilde{u}, \widetilde{w}, \widetilde{p})$.
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$$\begin{cases} \widetilde{u}_t + \widetilde{u}\widetilde{u}_x + U\,\widetilde{u}_x + \widetilde{w}\widetilde{u}_z + \widetilde{w}\,U' + \widetilde{p}_x = 0, \\ \widetilde{p}_z = 0, \\ \widetilde{u}_x + \widetilde{w}_z = 0, \end{cases} \tag{9}$$

and its linearization about the zero steady state solution is

$$\begin{cases} \widetilde{u}_t + U\,\widetilde{u}_x + \widetilde{w}\,U' + \widetilde{p}_x = 0, \\ \widetilde{p}_z = 0, \\ \widetilde{u}_x + \widetilde{w}_z = 0. \end{cases} \tag{10}$$

- Renardy [44] showed that for certain shear flow $U(z)$ (U is odd and U^{-2} is integrable, for example, $U(z) = \tanh(\frac{z-1/2}{d})$ with d small), the linearized system (10) has solutions of the form $\widetilde{u}(x, z, t) = e^{2\pi ikx}e^{\sigma_k t}\widetilde{u}_k(z)$, where $\Re\sigma_k = \lambda k$ for some $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.
- Such Kelvin-Helmholtz type instability implies that the inviscid PEs is linearly ill-posed in any Sobolev space, and in any Gevrey class of order $s > 1$.

- To establish local well-posedness, u_0 should be strongly localized in Fourier, typically for which $|\hat{u}_0(k,z)| \lesssim e^{-\delta|k|}$ with $\delta > 0$.

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- The Prandtl equations have similar structure as PEs. It is shown by Gérard-Varet and Dormy [24] that its linearization around a special background flow has unstable solutions of similar form, but with $\Re\sigma_k \sim \lambda\sqrt{k}$ for $k \gg 1$ arbitrarily large and some positive $\lambda \in \mathbb{R}_+$.

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- Based on the linear ill-posedness, Han-Kwan and Nguyen [27] established nonlinear ill-posedness of system (9) in any Sobolev space.

- For $\Omega \neq 0$, with initial data

$$u_0(x,y,z) = u_0(x,z), \quad v_0(x,y,z) = v_0(x,z), \tag{11}$$

under the ansatz and assumption that the unique smooth solution remains a function of the spatial variables (x,z) only, system (4) reduces to

$$\begin{cases} u_t + u\,u_x + wu_z - \Omega v + p_x = 0, \\ v_t + u\,v_x + wv_z + \Omega u = 0, \\ p_z = 0, \\ u_x + w_z = 0. \end{cases} \tag{12}$$

Consider in the domain $\mathbb{R} \times [0,1]$, subject to

$$w|_{z=0,1} = 0. \tag{13}$$

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- Notice now $v = 0$ is not invariant, unless $u_0 = 0$. This leads to the trivial solution.

- Consider the steady state background flow $(U, V, W, P) = (U(z), -\Omega x, 0, -\frac{1}{2}\Omega^2 x^2)$ to system (12).

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- Consider perturbation $(\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{p})$ around this steady state background flow

$$\begin{cases} \widetilde{u}_t + \widetilde{u}\widetilde{u}_x + U\widetilde{u}_x + \widetilde{w}\widetilde{u}_z + \widetilde{w}U' + \widetilde{p}_x - \Omega\widetilde{v} = 0, \\ \widetilde{v}_t + \widetilde{u}\widetilde{v}_x + U\widetilde{v}_x + \widetilde{w}\widetilde{v}_z = 0, \\ \widetilde{p}_z = 0, \\ \widetilde{u}_x + \widetilde{w}_z = 0, \end{cases} \tag{14}$$

with boundary conditions

$$\begin{aligned} &\widetilde{w}|_{z=0,1} = 0, \\ &\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{p} \text{ are periodic in } x \text{ with period } 1. \end{aligned} \tag{15}$$

- Assume system (14) is well-posed, then by uniqueness, if $\widetilde{v}_0 = 0$, then $\widetilde{v} \equiv 0$. This reduces system (14) back to system (9), where the rotation is not involve.

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- Since system (9) is both linearly and nonlinearly ill-posed in any Sobolev spaces, and linearly ill-posed in any Gevrey class of order $s > 1$, it follows that for arbitrary $\Omega \neq 0$, these ill-posedness results still hold. ([29], Ibrahim-Lin-Titi, 2020).

- $3D$ inviscid PEs:

$$\begin{cases} \partial_t \mathcal{V} + \mathcal{V} \cdot \nabla \mathcal{V} + w \partial_z \mathcal{V} + \Omega \mathcal{V}^{\perp} + \nabla p = 0, \\ \partial_z p = 0, \\ \nabla_h \cdot \mathcal{V} + \partial_z w = 0 \end{cases} \tag{16}$$

in $\mathcal{D} := \left\{(x, y, z) : 0 \leq z \leq 1, (x, y) \in \mathbb{T}^2\right\}$, subject to

$$\begin{cases} \mathcal{V}, w \text{ are periodic in } (x, y, z) \text{ with period } 1, \\ w|_{z=0,1} = 0. \end{cases} \tag{17}$$

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- "Equivalently", we consider system (16) in $\mathbb{T}^3$, subject to

$$\begin{cases} \mathcal{V}, w \text{ are periodic in } (x, y, z) \text{ with period } 1, \\ \mathcal{V} \text{ is even in } z \text{ and } w \text{ is odd in } z. \end{cases} \tag{18}$$

- After finding the solution in $\mathbb{T}^3$, one can restrict the solution in $\mathcal{D} := \left\{(x, y, z) : 0 \leq z \leq 1, (x, y) \in \mathbb{T}^2\right\}$.

- The local well-posedness in the space of analytic functions was established by Kukavica-Temam-Vicol-Ziane [34], giving a time of existence that shrinks to zero as the rotation rate $|\Omega|$ increases toward infinity.

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- This is counter intuitive to the established results about the $3D$ fast rotating Euler, Navier-Stokes and Boussinesq equations by Babin-Mahalov-Nicolaenko [2, 3, 4, 5], where the limit of fast rotation leads to strong dispersion and averaging mechanism that weakens the nonlinear effects and hence allows for establishing the global regularity result in the Navier-Stokes case, and prolongs the life-span of the solution in the Euler case.

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- see also Embid-Majda, [20, 22, 23, 32] and references therein. In Chemin-Desjardines-Gallagher-Grenier [20], they study in the whole space.

Well-posedness strategy

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- Then we will show the effect of the rotation rate Ω on the life-span of the solution in the spirit of the work of Babin et al. for the 3D Euler equations.

- For $f \in L^2(\mathbb{T}^3)$, the Sobolev norm:

$$\|f\|_{H^r} := \Big(\sum_{\mathbf{k}\in\mathbb{Z}^3} (1 + |\mathbf{k}|^{2r}) |\hat{f}_{\mathbf{k}}|^2 \Big)^{1/2}. \tag{19}$$

- Denote by $A = \sqrt{-\Delta}$, subject to periodic boundary conditions.
- For $s \geq 1$, $r \geq 0$, Gevrey class of order s (see [37]) is

$$G^s(\mathbb{T}^3) = \bigcup_{\tau>0} \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)), \tag{20}$$

where

$$\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)) := \{ f \in H^r(\mathbb{T}^3) : \|e^{\tau A^{1/s}} f\|_{H^r} < \infty \}, \tag{21}$$

$$\|e^{\tau A^{1/s}} f\|_{H^r} := \Big(\sum_{\mathbf{k}\in\mathbb{Z}^3} (1 + |\mathbf{k}|^{2r} e^{2\tau |\mathbf{k}|^{1/s}}) |\hat{f}_{\mathbf{k}}|^2 \Big)^{1/2}. \tag{22}$$

- When $s = 1$, $G^1(\mathbb{T}^3)$ corresponds to the space of analytic functions.

- $2D$ Leray projection $\mathbb{P}_h f := f - \nabla_h \Delta_h^{-1} \nabla_h \cdot f$.
  Here, $\phi = \Delta_h^{-1} \psi$ denotes the solution of the elliptic problem:
  $\Delta_h \phi = \psi$ and $\int_{\mathbb{T}^2} \phi dxdy = \int_{\mathbb{T}^2} \psi dxdy = 0$.

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- $\overline{f} := P_0 f$ the **barotropic mode**, $\widetilde{f} := (I - P_0)f$ the **baroclinic mode**.
- Decomposition: $f = P_0 f + (I - P_0)f = \overline{f} + \widetilde{f}$. For $r \geq 0$, $\tau \geq 0$,

$$\begin{aligned} \|f\|_{L^2}^2 &= \|\overline{f}\|_{L^2}^2 + \|\widetilde{f}\|_{L^2}^2, \\ \|e^{\tau A} f\|_{H^r}^2 &= \|e^{\tau A}\overline{f}\|_{H^r}^2 + \|e^{\tau A}\widetilde{f}\|_{H^r}^2. \end{aligned} \tag{23}$$

Local well-posedness of inviscid PEs

- Recall

$$\begin{cases} \partial_t \mathcal{V} + \mathcal{V} \cdot \nabla \mathcal{V} + w \partial_z \mathcal{V} + \Omega \mathcal{V}^{\perp} + \nabla p = 0, \\ \partial_z p = 0, \\ \nabla_h \cdot \mathcal{V} + \partial_z w = 0. \end{cases} \tag{24}$$

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- Integrating the first equation in $\mathbb{T}^3$ and integration by parts, thanks to boundary conditions, we obtain

$$\frac{d}{dt} \int_{\mathbb{T}^3} \mathcal{V} d\mathbf{x} + \Omega \int_{\mathbb{T}^3} \mathcal{V}^{\perp} d\mathbf{x} = 0. \tag{25}$$

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- Assume $\int_{\mathbb{T}^3} \mathcal{V}_0 d\mathbf{x} = 0$, then $\int_{\mathbb{T}^3} \mathcal{V} d\mathbf{x} = 0$ for any time $t \geq 0$.

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- This assumption is just for mathematical simplicity. One can show the same result without this assumption.

- Idea: try to **eliminate pressure terms**, use Leray Projection.
- $\nabla_h \cdot \mathcal{V} \neq 0$, but $\nabla_h \cdot \overline{\mathcal{V}} = -\int_{\mathbb{T}} \partial_z w dz = 0$.

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- $\nabla_h \cdot \mathcal{V} \neq 0$, but $\nabla_h \cdot \overline{\mathcal{V}} = -\int_{\mathbb{T}} \partial_z w dz = 0$.
- Apply $\mathbb{P}_h P_0$ and $I - P_0$ to the first equation in system (24), thanks to the boundary condition, we have the decomposition of the dynamics to the barotropic ($\overline{\mathcal{V}}$) and baroclinic ($\widetilde{\mathcal{V}}$) modes:

$$\begin{cases} \partial_t \overline{\mathcal{V}} + \mathbb{P}_h\Big(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}\Big) + \mathbb{P}_h P_0\Big((\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}\Big) = 0, \\ \partial_t \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} - P_0\Big(\widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} + (\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}\Big) \\ \qquad -\Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}(x,y,s)ds\Big)\partial_z \widetilde{\mathcal{V}} + \Omega \widetilde{\mathcal{V}}^{\perp} = 0 \end{cases} \tag{26}$$

in $\mathbb{T}^3$, subject to

$$\begin{cases} \overline{\mathcal{V}}(x,y), \widetilde{\mathcal{V}}(x,y,z) \text{ are periodic in } \mathbb{T}^3 \text{ and are even in } z; \\ \overline{\mathcal{V}}|_{t=0} = \overline{\mathcal{V}}_0 = P_0 \mathcal{V}_0, \quad \widetilde{\mathcal{V}}_{t=0} = \widetilde{\mathcal{V}}_0 = (I - P_0)\mathcal{V}_0; \\ \nabla_h \cdot \mathcal{V}_0 = 0. \end{cases} \tag{27}$$

Theorem 1 ([30], Ghoul-Ibrahim-Lin-Titi, 2020)

Assume $\overline{\mathcal{V}}_0, \widetilde{\mathcal{V}}_0 \in \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ *with* $r > 5/2$, $\tau_0 > 0$, *and* $\int_{\mathbb{T}^2} \overline{\mathcal{V}}_0 dxdy = 0$, $\nabla \cdot \overline{\mathcal{V}}_0 = 0$. *Let* $\Omega \in \mathbb{R}$ *be arbitrary and fixed. Then there exist non-increasing function* $\tau(t)$

$$\tau(t) = \tau_0 - 2tC_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2_{H^r}), \tag{28}$$

and time

$$\mathcal{T} = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2_{H^r})} > 0, \tag{29}$$

independent of Ω, *such that there exists a unique solution*

$$(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))) \cap L^2(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2}(\mathbb{T}^3)))$$

to system (26)–(27) on $[0, \mathcal{T}]$. *Moreover, the unique solution* $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}})$ *depends continuously on the initial data in the norm* $\|e^{\widetilde{\tau}(t)A} \cdot \|_{H^{r-1/2}}$, *with some* $\widetilde{\tau} \leq \tau$.

Note: thanks to (23), we have similar result for system (24).

Blowup of solutions without rotation

- When $\Omega = 0$, as before, consider

$$\begin{cases} u_t + u\, u_x + w u_z + p_x = 0, \\ p_z = 0, \\ u_x + w_z = 0. \end{cases} \tag{30}$$

in $\mathbb{R} \times [0, 1]$, subject to

$$w|_{z=0,1} = 0. \tag{31}$$

- In both Cao-Ibrahim-Nakanishi-Titi [12] and Wong [45], the authors restrict system (30) on the interval $\{(0, z) : 0 \leq z \leq 1\}$.

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- In [12], the authors considered the periodic and symmetric boundary condition in x variable. In [45], the author considered some additional conditions on the initial data without periodic setting. Denoting by $W(z,t) = w(0,z,t)$, they found that W satisfies

$$W_{tz} - (W_z)^2 + WW_{zz} + 2\int_0^1 (W_z)^2 dz = 0, \quad W(0,t) = W(1,t) = 0. \tag{32}$$

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- Both works [12, 45] first assume there exists a smooth solution to system (30), then established that the smooth solution to (32) blows up in finite time.

Blowup of solutions with rotation

- When $\Omega \neq 0$, we have

$$\begin{cases} u_t + u\, u_x + w u_z - \Omega v + p_x = 0, \\ v_t + u\, v_x + w v_z + \Omega u = 0, \\ p_z = 0, \\ u_x + w_z = 0. \end{cases} \tag{33}$$

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- Consider periodic boundary condition in x variable, and assume u_0, v_0 are odd in x (notice this is invariant). Denoting by $V(z,t) = -v_x(0,z,t)$ and $W(z,t) = w(0,z,t)$, we find that V and W satisfies

$$\begin{cases} V_t - W_z V + WV_z + \Omega W_z = 0, \\ W_{tz} - (W_z)^2 + WW_{zz} + 2\int_0^1 (W_z)^2 dz - \Omega V + \Omega \int_0^1 V(z) dz = 0, \\ W(0,t) = W(1,t) = 0. \end{cases} \tag{35}$$

- When $\Omega \neq 0$, we have

$$\begin{cases} u_t + u\, u_x + w u_z - \Omega v + p_x = 0, \\ v_t + u\, v_x + w v_z + \Omega u = 0, \\ p_z = 0, \\ u_x + w_z = 0. \end{cases} \tag{33}$$

in $\mathbb{R} \times [0,1]$, subject to

$$w\big|_{z=0,1} = 0. \tag{34}$$

- Consider periodic boundary condition in x variable, and assume u_0, v_0 are odd in x (notice this is invariant). Denoting by $V(z,t) = -v_x(0,z,t)$ and $W(z,t) = w(0,z,t)$, we find that V and W satisfies

$$\begin{cases} V_t - W_z V + W V_z + \Omega W_z = 0, \\ W_{tz} - (W_z)^2 + W W_{zz} + 2\int_0^1 (W_z)^2 dz - \Omega V + \Omega \int_0^1 V(z) dz = 0, \\ W(0,t) = W(1,t) = 0. \end{cases} \tag{35}$$

- If $V(z,0) = \Omega$, then $V(z,t) \equiv \Omega$ by uniqueness. Then the equation of W is reduced to (32). Therefore, smooth solutions to system (33) can form singularity in finite time ([29], Ibrahim-Lin-Titi, 2020).

- The initial data in Wong [45] can be analytic, and therefore, the existence of the solution is guaranteed by the local well-posedness result.
- Specifically, we find an example:

$$u_0(x,z) = \lambda(-z^2 + \frac{1}{3}) \sin x, \quad v_0(x,z) = -\Omega \sin x \tag{36}$$

with $\lambda > 0$, and the corresponding upper bound on the blowup time is $\frac{9}{2\lambda}$.

- Here since $\int_0^1 u_0(x,z)dz = 0$ and v_0 is independent of the z variable, the baroclinic mode of the initial data is $(u_0, 0)$, and the barotropic mode of the initial data is $(0, v_0)$.

- When $|\Omega| \gg 1$, and when $\lambda = \frac{1}{|\Omega|}$, this implies a smallness condition on the baroclinic $(u_0, 0) \sim \frac{1}{|\Omega|}$, while the whole initial data satisfies $(u_0, v_0) \sim |\Omega|$. The guaranteed blowup time in this case satisfies $\mathcal{T} \sim |\Omega|$.

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- It remains interesting to know whether for arbitrary Ω there exists a blowup solution with initial data (u_0, v_0) whose barotropic and baroclinic modes are both of order 1. Moreover, to estimate the corresponding blowup time $\mathcal{T}$ as $|\Omega| \to \infty$.

Blowup of solutions with rotation

- When $|\Omega| \gg 1$, and when $\lambda = \frac{1}{|\Omega|}$, this implies a smallness condition on the baroclinic $(u_0, 0) \sim \frac{1}{|\Omega|}$, while the whole initial data satisfies $(u_0, v_0) \sim |\Omega|$. The guaranteed blowup time in this case satisfies $\mathcal{T} \sim |\Omega|$.
- It remains interesting to know whether for arbitrary Ω there exists a blowup solution with initial data (u_0, v_0) whose barotropic and baroclinic modes are both of order 1. Moreover, to estimate the corresponding blowup time $\mathcal{T}$ as $|\Omega| \to \infty$.
- Observe that if the blowup time $\mathcal{T} \sim 1$ as $|\Omega| \to \infty$, this would imply that fast rotation does not prolong the life-span of the solution to the $3D$ inviscid PEs unless, as it has been noted above, a smallness condition on the size of the baroclinic mode is met.

- Observe that when the baroclinic mode $\widetilde{\mathcal{V}} = 0$, the $3D$ inviscid PEs will reduce to $2D$ Euler equations which governs the dynamics of the barotropic mode.

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- Idea: the smaller the analytic norm of $\widetilde{\mathcal{V}}_0$ the longer time of existence of the $3D$ inviscid PEs.
- This result does not take advantage of the rotation.

Long time existence without the effect of rotation

Theorem 2 ([30], Ghoul-Ibrahim-Lin-Titi, 2020)

Assume $\overline{\mathcal{V}}_0 \in \mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$, $\widetilde{\mathcal{V}}_0 \in \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$, *with* $r > 5/2$, $\tau_0 > 0$, *and* $\int_{\mathbb{T}^2} \overline{\mathcal{V}}_0 dxdy = 0$, $\nabla \cdot \overline{\mathcal{V}}_0 = 0$. *Let* $\Omega \in \mathbb{R}$ *be arbitrary and fixed. Suppose* $\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^{r+1}} \leq M$ *for some* $M \geq 0$, *and* $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r} \leq \epsilon$ *for some* $\epsilon \geq 0$. *Then there are constants* $C_M > 1$ *and* $C_r > 1$, *such that if* $\mathcal{T} = \mathcal{T}(\tau_0, \epsilon, M, r)$ *satisfies*

$$\int_0^{\mathcal{T}} e^{K(s)} ds = \frac{\tau_0}{2\epsilon}, \tag{37}$$

where $K(t) = C_M^{\exp(C_r t)}$, *then the unique solution obtained in Theorem 1 exists for* $t \in [0, \mathcal{T}]$. *In particular, from (37),* $\mathcal{T} \gtrsim \ln(\ln(\ln(\frac{1}{\epsilon}))) \to \infty$ *as* $\epsilon \to 0^+$.

- As a corollary of Theorem 2, the solution to $3D$ inviscid PEs converges to the solution to $2D$ Euler equations in the space of analytic functions provided that $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r} \to 0$.

The effect of rotation on the life-span of the solution

- Goal: show fast rotation prolongs the life-span of the solution to the $3D$ inviscid PEs.

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- For the operator $\mathcal{J}\widetilde{\mathcal{V}} := \widetilde{\mathcal{V}}^{\perp}$, $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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- For the operator $\mathcal{J}\widetilde{\mathcal{V}} := \widetilde{\mathcal{V}}^{\perp}$, $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It has eigenvalues $\pm i$, with corresponding eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}$.

  We define

$$P_{+}\mathcal{V} := \left\langle (I - P_0)\mathcal{V}, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}} \right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(\widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}^{\perp}), \tag{38}$$

and

$$P_{-}\mathcal{V} := \left\langle (I - P_0)\mathcal{V}, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}} \right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(\widetilde{\mathcal{V}} - i\widetilde{\mathcal{V}}^{\perp}). \tag{39}$$

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and

$$P_-\mathcal{V} := \left\langle (I - P_0)\mathcal{V}, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}} \right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(\widetilde{\mathcal{V}} - i\widetilde{\mathcal{V}}^\perp). \tag{39}$$

- Similar ideas and projections for $3D$ rotating Euler equations can be found in [22, 32].

- Decomposition of $\mathcal{V}$:

$$\mathcal{V} = P_0\mathcal{V} + P_+\mathcal{V} + P_-\mathcal{V} \tag{40}$$

satisfies

$$P_\pm P_\pm \mathcal{V} = P_\pm \mathcal{V}, \qquad P_0 P_0 \mathcal{V} = P_0 \mathcal{V}, \qquad P_\pm P_\mp \mathcal{V} = P_0 P_\pm \mathcal{V} = P_\pm P_0 \mathcal{V} = 0. \tag{41}$$

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- Applying $P_\pm$ to the baroclinic mode in (26),

$$\begin{aligned}\partial_t P_\pm \mathcal{V} + P_\pm \Big(\widetilde{\mathcal{V}} \cdot \nabla \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla \widetilde{\mathcal{V}} - P_0(\widetilde{\mathcal{V}} \cdot \nabla \widetilde{\mathcal{V}} + (\nabla \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}) \\ -(\int_0^z \nabla \cdot \widetilde{\mathcal{V}}(\mathbf{x}', s) ds)\partial_z \widetilde{\mathcal{V}}\Big) \mp i\Omega P_\pm \mathcal{V} = 0.\end{aligned} \tag{42}$$

- Denote by $\mathcal{V}_+ = e^{-i\Omega t} P_+ \mathcal{V}$, $\mathcal{V}_- = e^{i\Omega t} P_- \mathcal{V}$.

$$
\begin{aligned}
\partial_t \mathcal{V}_+ = & - e^{i\Omega t}\Big(\mathcal{V}_+ \cdot \nabla \mathcal{V}_+ - P_0(\mathcal{V}_+ \cdot \nabla \mathcal{V}_+ + (\nabla \cdot \mathcal{V}_+)\mathcal{V}_+) - (\int_0^z \nabla \cdot \mathcal{V}_+(\mathbf{x}', s)ds)\partial_z \mathcal{V}_+\Big) \\
& - \Big(\overline{\mathcal{V}} \cdot \nabla \mathcal{V}_+ + \frac{1}{2}(\mathcal{V}_+ \cdot \nabla)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp)\Big) - e^{-2i\Omega t}\frac{1}{2}(\mathcal{V}_- \cdot \nabla)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp) \\
& - e^{-i\Omega t}\Big(\mathcal{V}_- \cdot \nabla \mathcal{V}_+ - P_0(\mathcal{V}_- \cdot \nabla \mathcal{V}_+ + (\nabla \cdot \mathcal{V}_-)\mathcal{V}_+) - (\int_0^z \nabla \cdot \mathcal{V}_-(\mathbf{x}', s)ds)\partial_z \mathcal{V}_+\Big).
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&- \Big(\overline{\mathcal{V}} \cdot \nabla \mathcal{V}_+ + \frac{1}{2}(\mathcal{V}_+ \cdot \nabla)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp)\Big) - e^{-2i\Omega t}\frac{1}{2}(\mathcal{V}_- \cdot \nabla)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp) \\
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- By taking complex conjugate, one can find the evolution for $\mathcal{V}_-$.

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$$\begin{aligned}\partial_t \mathcal{V}_+ =& - e^{i\Omega t}\Big(\mathcal{V}_+ \cdot \nabla \mathcal{V}_+ - P_0(\mathcal{V}_+ \cdot \nabla \mathcal{V}_+ + (\nabla \cdot \mathcal{V}_+)\mathcal{V}_+) - (\int_0^z \nabla \cdot \mathcal{V}_+(\mathbf{x}', s) ds)\partial_z \mathcal{V}_+\Big) \\ & - \Big(\overline{\mathcal{V}} \cdot \nabla \mathcal{V}_+ + \frac{1}{2}(\mathcal{V}_+ \cdot \nabla)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^{\perp})\Big) - e^{-2i\Omega t}\frac{1}{2}(\mathcal{V}_- \cdot \nabla)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^{\perp}) \\ & - e^{-i\Omega t}\Big(\mathcal{V}_- \cdot \nabla \mathcal{V}_+ - P_0(\mathcal{V}_- \cdot \nabla \mathcal{V}_+ + (\nabla \cdot \mathcal{V}_-)\mathcal{V}_+) - (\int_0^z \nabla \cdot \mathcal{V}_-(\mathbf{x}', s) ds)\partial_z \mathcal{V}_+\Big). \end{aligned} \tag{43}$$

- By taking complex conjugate, one can find the evolution for $\mathcal{V}_-$.
- For the barotropic mode,

$$\begin{aligned}\partial_t \overline{\mathcal{V}} + \mathbb{P}_h(\overline{\mathcal{V}} \cdot \nabla \overline{\mathcal{V}}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(\mathcal{V}_+ \cdot \nabla \mathcal{V}_+ + (\nabla \cdot \mathcal{V}_+)\mathcal{V}_+\Big) \\ + e^{-2i\Omega t}\mathbb{P}_h P_0\Big(\mathcal{V}_- \cdot \nabla \mathcal{V}_- + (\nabla \cdot \mathcal{V}_-)\mathcal{V}_-\Big) = 0. \end{aligned} \tag{44}$$

The effect of rotation on the life-span of the solution

- Denote by the formal limits of $\mathcal{V}_+, \mathcal{V}_-, \overline{\mathcal{V}}$ to be $V_+, V_-, \overline{V}$.

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$$\partial_t V_\pm = -(\overline{V} \cdot \nabla) V_\pm - \frac{1}{2}(V_\pm \cdot \nabla)(\overline{V} \pm i\overline{V}^\perp), \tag{46}$$

which is a linear transport equation with an additional stretching term.

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- Take the difference between the original system and the limit system (45)–(46), and establish some energy estimates.

- Denote by the formal limits of $\mathcal{V}_+, \mathcal{V}_-, \overline{\mathcal{V}}$ to be $V_+, V_-, \overline{V}$. Taking limit $\Omega \to \infty$ formally, we obtain the limit equation of $\overline{\mathcal{V}}$ is

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- System (45)–(46) is globally well-posed in Sobolev spaces and in the space of analytic functions.
- Take the difference between the original system and the limit system (45)–(46), and establish some energy estimates.
- There are some nonlinear terms requiring the smallness in Sobolev norms of the baroclinic mode of the initial data. We call such initial data *"well-prepared"*.

Theorem 3 ([30], Ghoul-Ibrahim-Lin-Titi, 2020)

Assume $\overline{\mathcal{V}}_0 \in \mathcal{D}(e^{\tau_0 A} : H^{r+3}(\mathbb{T}^3))$, $\widetilde{\mathcal{V}}_0 \in \mathcal{D}(e^{\tau_0 A} : H^{r+2}(\mathbb{T}^3))$, *with* $r > 5/2$, $\tau_0 > 0$, *and* $\int_{\mathbb{T}^2} \overline{\mathcal{V}}_0 dxdy = 0$, $\nabla \cdot \overline{\mathcal{V}}_0 = 0$. *Let* $M \geq 0$ *and* $\delta > 0$, *then there exist constants* $C_{\tau_0} > 1$, $C_{M,\tau_0} > 1$, $C_r > 1$ *that define the function* $\widetilde{K}(t) := e^{C_{M,\tau_0}^{\exp(C_r t)}}$. *Suppose that* $\Omega_0 \in \mathbb{R}$ *is given satisfying* $|\Omega_0| \geq C_{\tau_0} e^{\widetilde{K}(1)}$, *and that* $\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^{r+3}} + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^{r+2}} \leq M$, *with* $\|\widetilde{\mathcal{V}}_0\|_{H^{3+\delta}} \lesssim \frac{1}{|\Omega_0|}$. *Then there exists a time* $\mathcal{T} = \mathcal{T}(\tau_0, |\Omega_0|, M, r) \geq 1$ *satisfying*

$$C_{\tau_0} e^{\widetilde{K}(\mathcal{T})} = |\Omega_0|, \tag{47}$$

such that for all $|\Omega| \geq |\Omega_0|$ *the unique solution obtained in Theorem 1 exists on the interval* $[0, \mathcal{T}]$. *In particular, from (47),* $\mathcal{T} \gtrsim \ln(\ln(\ln(\ln |\Omega_0|))) \to \infty$, *as* $|\Omega_0| \to \infty$.

- To emphasize the difference between smallness in analytic norm and in Sobolev norm, consider
$$\widetilde{\mathcal{V}}_0 = c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad k_3 \neq 0, \tag{48}$$
with $|\mathbf{k}| = \lceil \tau_0^{-1} \ln |\Omega| \rceil$ and $|c_{\mathbf{k}}| = (\ln |\Omega|)^{-r-2} |\Omega|^{-1}$. Then $\|\widetilde{\mathcal{V}}_0\|_{H^{3+\delta}} \leq \|\widetilde{\mathcal{V}}_0\|_{H^{r+2}} \sim |\Omega|^{-1}$, $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^{r+2}} \sim 1$.
- This implies that the initial data is oscillatory in space.
- One can construct a sequence of initial data
$$\{(\widetilde{\mathcal{V}}_0)_\Omega\} = c_{\mathbf{k}(\Omega)} e^{i\mathbf{k}(\Omega)\cdot\mathbf{x}}, \tag{49}$$
where $|\mathbf{k}(\Omega)| = \lceil \tau_0^{-1} \ln |\Omega| \rceil$ and $|c_{\mathbf{k}(\Omega)}| = (\ln |\Omega|)^{-r-2} |\Omega|^{-1}$. Then as $|\Omega| \to \infty$, we have nontrivial result such that with initial condition $\|e^{\tau_0 A}(\widetilde{\mathcal{V}}_0)_\Omega\|_{H^{r+2}} \sim 1$, the solution exists in analytic space on $[0, \mathcal{T}]$ satisfying $\mathcal{T} \to \infty$, as $|\Omega| \to \infty$.

Thank you!

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