Long time dispersive estimates for perturbations of a kink solution of one dimensional cubic wave equations

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(Joint work with Nader Masmoudi)

1. Motivations

Goal: Stability of stationary solution H of a nonlinear wave equation (in dimension 1).

Example: • $H \equiv 0$ stationary solution of $(\partial_t^2 - \partial_x^2 + 1)u = N(u) = O(u^2),\ u \to 0$. If Cauchy data are of size $\epsilon \ll 1$ in a space of smooth decaying functions, then $\|u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}}),\ t \to +\infty$ (Lindblad-Soffer, D., Stingo).

• The *kink* $H(x) = \tanh(x/\sqrt{2})$ is a stationary solution of the ϕ^4 model

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3.$$

Write $\phi(t,x) = H(x) + \varphi(t\sqrt{2}, x\sqrt{2})$. Then the perturbation φ solves

$$\left(D_t^2 - (D_x^2 + 1 + 2V(x))\right)\varphi = \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3$$

where

$D_t = \frac{1}{i}\partial_t, V(x) = -\frac{3}{4}\cosh^{-2}\left(\frac{x}{2}\right) \in \mathcal{S}(\mathbb{R}), \kappa(x) = \frac{3}{2}\tanh\left(\frac{x}{2}\right)$.

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• Kowalczyk, Martel and Muñoz: Asymptotic stability locally in space.

Operator $-\partial_x^2 + 2V(x)$ has $[0, +\infty[$ as absolutely continuous spectrum and two eigenvalues -1 and $-\frac{1}{4}$.

Restriction: From now on we consider only odd perturbations φ. Then if Y is a normalized eigenfunction associated to eigenvalue $-\frac{1}{4}$, we may decompose

$$(\varphi(t,x), \partial_t \varphi(t,x)) = \underbrace{(u_1(t,x), u_2(t,x))}_{\text{Proj. on a.c. spectrum}} + (z_1(t), z_2(t))Y(x).$$

Theorem (KMM)

If $(\varphi, \partial_t \varphi)|_{t=0}$ is small in $H^1 \times L^2$ and odd, then

$$\int_{-\infty}^{+\infty} \left[|z_1(t)|^4 + |z_2(t)|^4 \right] dt < +\infty$$

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}} \left[(\partial_x u_1)^2 + u_1^2 + u_2^2 \right](t,x) e^{-c_0|x|}\, dx dt < +\infty.$$

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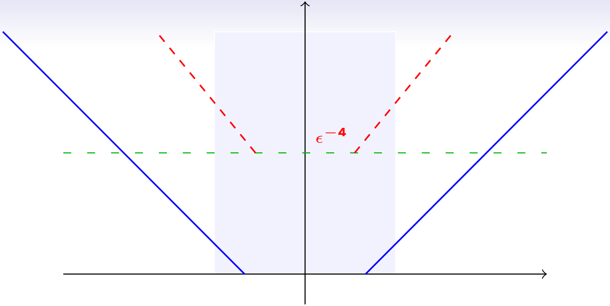

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2. Statement of main theorem

Recall that the (odd) perturbation φ solves

$$(E) \qquad \big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3.$$

We decompose $\varphi = P_{\rm ac}\varphi + a(t)Y$, with $P_{\rm ac}$ = spectral projector on a.c. spectrum and $a(t) = \langle \varphi, Y\rangle$. One gets

$$(S) \qquad \begin{aligned} &\Big(D_t^2 - \frac{3}{4}\Big)a(t) = \langle \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, Y\rangle \\ &\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)(P_{\rm ac}\varphi) = P_{\rm ac}\big(\kappa(x)\varphi^2 + \frac{1}{2}\varphi^3\big). \end{aligned}$$

We want to solve (E) or (S) with initial data at $t = 1$, that are smooth enough, decaying and small i.e. that will be of the form $\varphi|_{t=1} = \epsilon\varphi_0, \partial_t\varphi|_{t=1} = \epsilon\varphi_1$ with

$$\|\varphi_0\|_{H^{s+1}} + \|\varphi_1\|_{H^s} + \|x\varphi_0\|_{H^1} + \|x\varphi_1\|_{L^2} \leq 1,$$

with $\epsilon \ll 1$ and s large enough.

Theorem

There is $\rho_0 \in \mathbb{N}$ *and for any* $\rho \geq \rho_0$, *any* $c > 0$, *any* $\theta' \in]0, \frac{1}{2}[$, *any large enough* N, s, *there are* $\epsilon_0 > 0, C > 0$ *such that the solution to* (S) *with odd initial data of size* $\epsilon < \epsilon_0$ *is defined for* $t \in [1, \epsilon^{-4+c}[$ *and one has the estimates*

$$a(t) = \langle Y, \varphi \rangle = e^{it\frac{\sqrt{3}}{2}} g_+(t) - e^{-it\frac{\sqrt{3}}{2}} g_-(t)$$

$$|g_\pm(t)| \leq \frac{C\epsilon}{\sqrt{1 + t\epsilon^2}}, \ |\partial_t g_\pm(t)| \leq \frac{C\epsilon}{\sqrt{1 + t\epsilon^2}} t^{-\frac{1}{2}}$$

$$\|P_{\mathrm{ac}}\varphi(t, \cdot)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{1}{2}}(\epsilon^2\sqrt{t})^{\theta'}$$

$$\|\langle x \rangle^{-2N} D_t^j P_{\mathrm{ac}}\varphi(t, \cdot)\|_{W^{\rho,\infty}} \leq Ct^{-\frac{3}{4}}(\epsilon^2\sqrt{t})^{\theta'}, \ j = 0, 1.$$

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Remarks: • For an equation $\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = NL(x,\varphi)$ and a potential *without bound states*, Germain and Pusateri prove a $O(\epsilon t^{-\frac{1}{2}})$ L^∞ bound.

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Remarks: • If $t \gg \epsilon^{-4}$, possible logarithmic loss $\|P_{\mathrm{ac}} \varphi(t, \cdot)\|_{L^\infty} = O\Big(\frac{\log t}{\sqrt{t}}\Big)$. See Lindblad, Lührmann and Soffer for $\big(D_t^2 - (D_x^2 + 1)\big)\varphi = a(x)\varphi^2 + b(x)\varphi^3$ Lindblad, Lührmann, Soffer and Schlag for $\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = P_{\mathrm{ac}}(a(\cdot)u^2)$.

• Our bounds imply the integral bounds of KMM (up to time ϵ^{-4}).

3. The case of NLKG

Recall the bootstrap argument for small perturbations of the zero solution of NLKG i.e. small solutions to $(D_t - p(D_x))u = |u|^2 u$, with $p(\xi) = \sqrt{1+\xi^2}$. If $L_+ = x + tp'(D_x)$, one gets the optimal dispersive estimates proving the set of inequalities

$$(E_A) \qquad \begin{aligned} \|u(t,\cdot)\|_{H^s} &\leq A\epsilon t^{\delta} \\ \|L_+ u(t,\cdot)\|_{L^2} &\leq A\epsilon t^{\delta} \\ \|u(t,\cdot)\|_{L^\infty} &\leq \frac{A\epsilon}{\sqrt{t}}. \end{aligned}$$

Bootstrap: Assume (E_A) with $0 < \delta \ll 1, s \gg 1, \epsilon \ll 1, A \gg 1$ on $[1, T]$. Using the equation, show that then $(E_{A/2})$ holds on $[1, T]$.

Idea of proof to show the second inequality in $(E_{A/2})$: Use that $[L_+, D_t - p(D_x)] = 0$ to write

$$(D_t - p(D_x))(L_+ u) = L_+(|u|^2 u) = O_{L^2}(\|u\|_{L^\infty}^2 \|L_+ u\|_{L^2}).$$

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4. Some ideas of proof

The equation satisfied by $P_{\mathrm{ac}}\varphi$ is

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)(P_{\mathrm{ac}}\varphi) = P_{\mathrm{ac}}(\text{Non linearity}).$$

Idea: Reduce that equation to $(D_t - p(D_x))u = |u|^2 u$.

1st Step: *Conjugation through wave operators.*
Define $A = -\frac{1}{2}\partial_x^2 + V(x)$, $A_0 = -\frac{1}{2}\partial_x^2$,
$W_+ = \mathrm{s} - \lim_{t\to+\infty} e^{itA}e^{-itA_0}$. Then one knows that
$W_+^*[D_x^2 + 2V(x)]P_{\mathrm{ac}} = D_x^2 W_+^* P_{\mathrm{ac}}$. Setting $w = W_+^* P_{\mathrm{ac}}\varphi$, this implies

$$\big(D_t^2 - (D_x^2 + 1)\big)w = b(x, D_x)^*\left[\kappa(x)[a(t)Y + b(x, D_x)w]^2\right] + \frac{1}{2}b(x, D_x)^*\left[(a(t)Y + b(x, D_x)w)^3\right]$$

where $b(x,\xi)$ is a pseudo-differential symbol of order zero with $\frac{\partial b}{\partial x}(x,\xi) = O(\langle x\rangle^{-\infty}), x \to \pm\infty$.
Setting $u_\pm = (D_t \pm p(D_x))w$, one may rewrite this second order equation as a first order system.

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Step 2: *Elimination of quadratic terms by normal form.*
Equation on the unknown u_+:

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+ = a(t)^2 \underbrace{Y_2}_{\in\mathcal{S}(\mathbb{R})} &+ a(t)^3 \underbrace{Y_3}_{\in\mathcal{S}(\mathbb{R})}\\ &+ a(t) \underbrace{Q_1'(x,u_+,\bar{u}_+)}_{\text{linear in }(u_+,\bar{u}_+)\text{ with coeff. in }\mathcal{S}(\mathbb{R})}\\ &+ \underbrace{Q_2(x,u_+,\bar{u}_+)}_{\text{quadratic in }(u_+,\bar{u}_+)\text{ with coeff. with derivative in }\mathcal{S}(\mathbb{R})}\\ &+ \underbrace{Q_3(x,u_+,\bar{u}_+).}_{\text{cubic in }(u_+,\bar{u}_+)\text{ with coeff. with derivative in }\mathcal{S}(\mathbb{R})}\end{aligned}$$

The worst term above is Q_2. One eliminates it through a "time normal form" à la Shatah, finding a new quadr. form $\tilde{Q}_2$ such that

$$\big(D_t - p(D_x)\big)[u_+ - \tilde{Q}_2(x,u_+,\bar{u}_+)]$$
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Step 3: *Elimination of source terms.*
Let u^{app} be the solution of the linear equation

$$\big(D_t - p(D_x)\big)\, u^{\mathrm{app}} = a(t)^2 Y_2 + a(t)^3 Y_3$$
$$u^{\mathrm{app}}|_{t=1} = 0.$$

Then one gets

$$\big(D_t - p(D_x)\big)[u_+ - \tilde{Q}_2(x, u_+, u_-) - u^{\mathrm{app}}]$$
$$= a(t) Q_1'(x, u_+, \bar{u}_+) + Q_2'(x, u_+, \bar{u}_+) + Q_3(x, u_+, \bar{u}_+).$$

Proposition

One may decompose $u^{\mathrm{app}} = u'^{\mathrm{app}}_+ + u''^{\mathrm{app}}_+ + u'^{\mathrm{app}}_- + u''^{\mathrm{app}}_-$ *with for* $t \le \epsilon^{-4}$

$$\|L_+ u'^{\mathrm{app}}_\pm(t,\cdot)\|_{L^2} \le C t^{\frac{1}{4}} (\epsilon^2 \sqrt{t})$$
$$\|L_+ u''^{\mathrm{app}}_\pm(t,\cdot)\|_{L^\infty} \le C \log t \log(1 + \epsilon^2 t)$$
$$|u'^{\mathrm{app}}_\pm(t,x)| \le C \frac{\epsilon^2 \sqrt{t}}{\sqrt{t}} \Big(1 + t^{-\frac{1}{2}} \Big| x \pm t\sqrt{\frac{2}{3}} \Big|\Big)^{-1}$$

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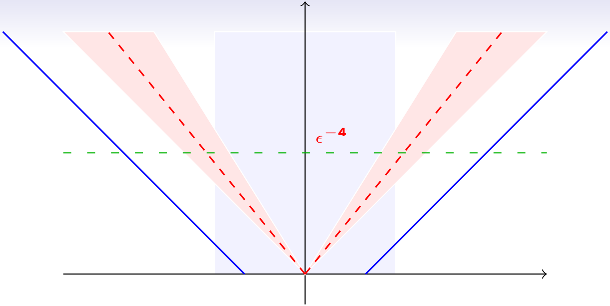

Proposition

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Step 3: *Elimination of source terms.*

Let u^{app} be the solution of the linear equation

$$\big(D_t - p(D_x)\big)u^{\mathrm{app}} = a(t)^2 Y_2 + a(t)^3 Y_3$$
$$u^{\mathrm{app}}|_{t=1} = 0.$$

Then one gets

$$\big(D_t - p(D_x)\big)\big[\underbrace{u_+ - \tilde{Q}_2(x, u_+, u_-) - u^{\mathrm{app}}}_{\tilde{u}_+}\big]$$
$$= a(t)Q_1'(x, u_+, \bar{u}_+) + Q_2'(x, u_+, \bar{u}_+) + Q_3(x, u_+, \bar{u}_+).$$

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Step 4: *Elimination of linear term.*

Set $\tilde{u} = \begin{bmatrix} \tilde{u}_+ \\ \tilde{u}_- \end{bmatrix}$ with $\tilde{u}_- = -\overline{\tilde{u}}_+$ and define in the same way $\tilde{u}^{\mathrm{app}}$.

Then the preceding equation may be rewritten

$$(*) \qquad \left(D_t - \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix} - \mathcal{V}(t) \right) \tilde{u} = \underbrace{\mathcal{M}_3(\tilde{u}, \tilde{u}^{\mathrm{app}})}_{\text{cubic}} + \underbrace{\mathcal{M}_2'(\tilde{u}, \tilde{u}^{\mathrm{app}})}_{\text{quadratic}}$$

where $\mathcal{V}(t)$ is a matrix of linear operators (whose entries are of the form $b(t, x, D_x)$, with $b(t, x, \xi)$ symbol of order zero, with coefficients rapidly decaying in x). Set $L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}$ with $L_\pm = x \pm tp'(D_x)$. Then L does not commute to the $\mathcal{V}(t)$ term in $(*)$. One constructs a "wave operator" $C(t)$ such that

$$C(t)\left(D_t - \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix} - \mathcal{V}(t) \right) = \left(D_t - \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix} \right) C(t).$$

Step 5: *Further step of normal forms and bootstrap.*
The preceding reductions lead to

$$\left(D_t - \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix}\right) C(t)\tilde{u} = C(t)\mathcal{M}_3 + C(t)\mathcal{M}_2'.$$

One performs "space-time normal forms" in order to reduce the cubic term essentially to $\begin{bmatrix} |\tilde{u}_+|^2 \tilde{u}_+ \\ |\tilde{u}_-|^2 \tilde{u}_- \end{bmatrix}$ and to eliminate, up to remainders, the quadratic one. One is then in position to perform a bootstrap argument relying on estimates of the form

$$\|\tilde{u}_+(t,\cdot)\|_{H^s} = O(\epsilon t^\delta)$$

$$\|L_+\tilde{u}_+(t,\cdot)\|_{L^2} = O((\epsilon^2\sqrt{t})^\theta t^{\frac{1}{4}})$$

$$\|\tilde{u}_+(t,\cdot)\|_{L^\infty} = O\Big(\frac{(\epsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\Big).$$

The second estimate is obtained commuting L_+ to the equation.
The last one follows deducing from the PDE an ODE.

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Step 6: *Study of the non dispersive component.*

Recall that we had to study a system made of the coupling between a PDE and an ODE given by:

$$\left(D_t^2 - \frac{3}{4}\right)a(t) = \langle \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, Y\rangle$$

where $\varphi = P_{\mathrm{ac}}\varphi + a(t)Y$. One needs to obtain bounds in $C\epsilon(1+t\epsilon^2)^{-\frac{1}{2}}$ for a. After normal forms one is reduced to

$$D_t g(t) = \left(\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\right)|g(t)|^2 g(t) + \text{ remainders}$$

where α is real and Y_2 is an explicit function in $\mathcal{S}(\mathbb{R})$. To get a solution defined for any $t \geq 1$, one needs a condition on Y_2, namely that $\hat{Y}_2(\sqrt{2})^2 < 0$. This is the Fermi Golden Rule. It is evident that $\hat{Y}_2(\sqrt{2})$ is purely imaginary, so that the condition to check reduces to $\hat{Y}_2(\sqrt{2}) \neq 0$. This condition had been checked numerically by Kowalczyk, Martel and Muñoz. Actually, it reduces to the computation of the integral of an explicit function, that may be done by residues.