Riemann's non-differentiable function and the binormal flow

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Summary

(1) The analytical problem:

"Continuous functions nowhere differentiable"

(2) The geometric problem:

"Evolution of vortex filaments (smoke rings)"

$$\chi_t = cb$$

How does a corner evolve? And several corners?

(3) The PDE problem:

- 1d–Cubic NLS: $u_0 = \sum_{j\in\mathbb{Z}} a_j \delta(x-j)$
- Schrödinger map

(4) Is there a connection with turbulence?

Conjecture: Yes (Non–circular jets)

Multifractal formalism (Frisch–Parisi)

Intermittency

Riemann's function

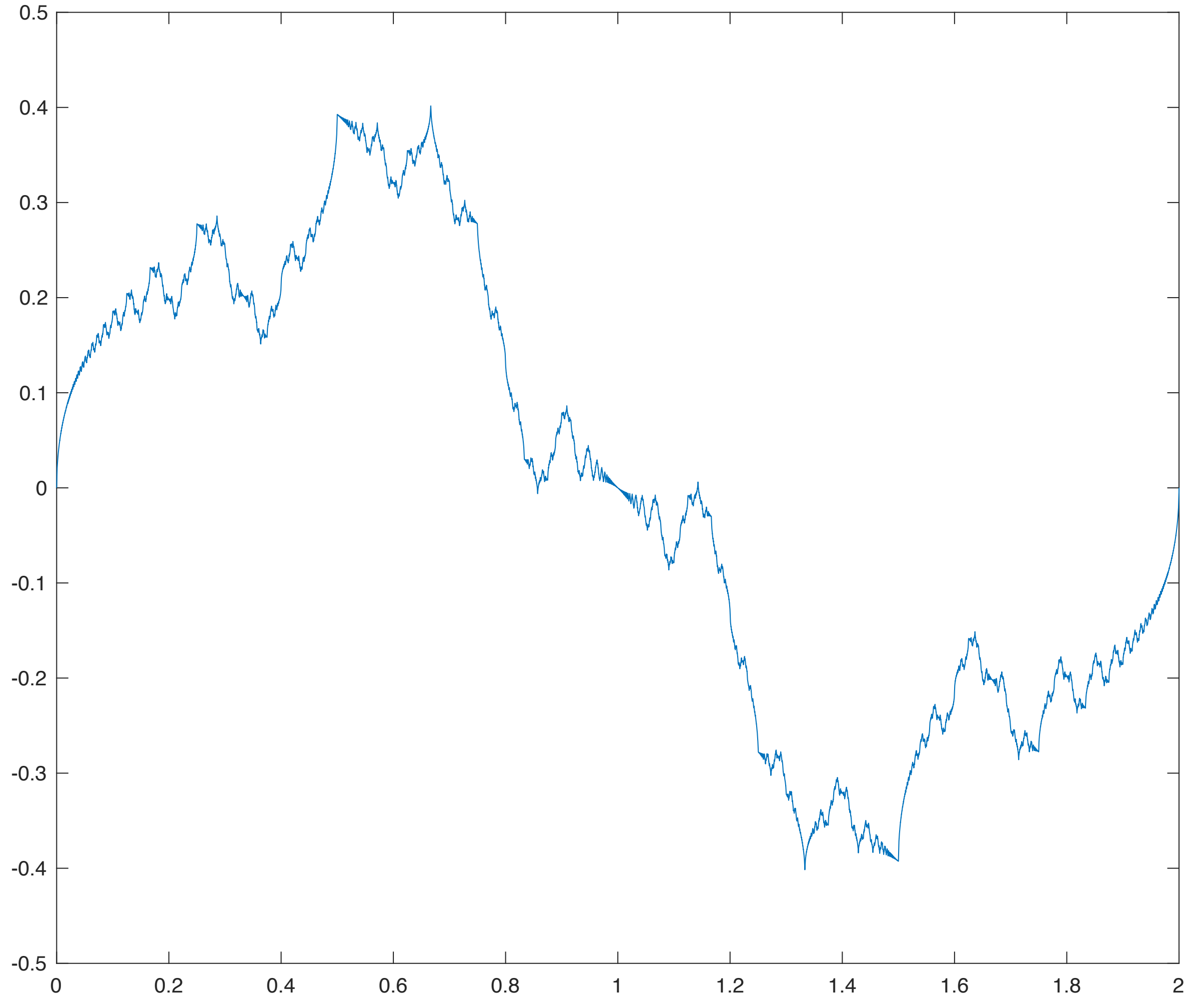
$$\varphi_R(t) = \sum_{j=1}^{\infty} \frac{\sin(tj^2)}{j^2} \qquad (\sim 1860)$$

- Hardy 1915 (H–Littlewood circle method)
- Gerver 1960 (Riemann was wrong)

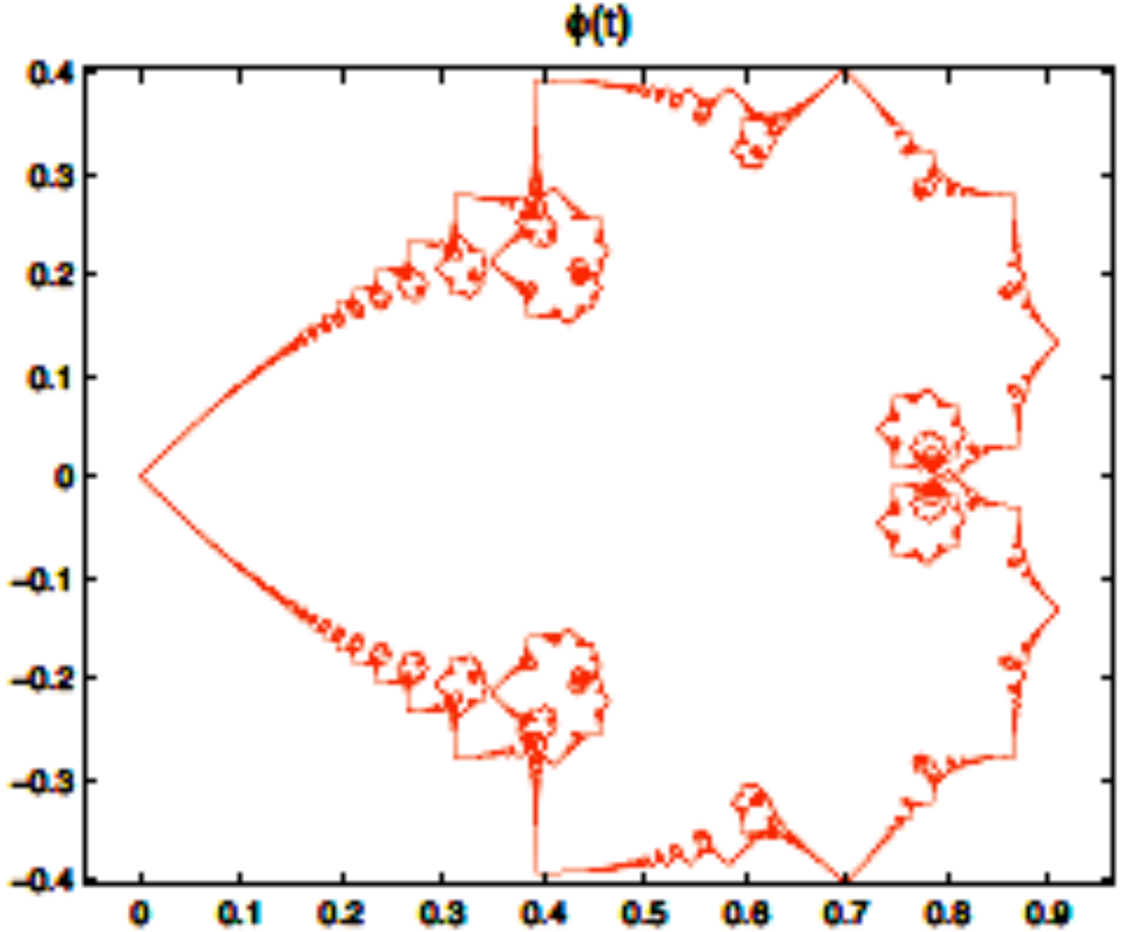
At $t_{p,q} = \pi p/q$ $\quad p, q$ odd, the derivative exists and is $-1/2$

$$\varphi_D(t) = \sum_{j=1}^{\infty} \frac{e^{itj^2}}{ij^2} \qquad \text{Duistermaat 1991}$$

Fractal behavior of the graph.


0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

$$\phi(t) = \sum_{k \neq 0} \frac{e^{\pi i k^2 t}}{i\pi k^2}, \quad t \in [0, 2]$$



- Jaffard
- Multifractal (Frisch–Parisi conjecture)

Multifractal formalism

Spectrum of singularities (of a function f):

$$d_f(\beta) = H\text{-dim } E_\beta$$

$$E_\beta = \{t_0 \in [0,2] \; : \; f \text{ is } \beta\text{-Hölder at } t_0\}$$

i. e. $$\sup\{\alpha \; : \; f \in \mathcal{C}^\alpha(t_0)\}$$

$$\left| f(t) - P(t - t_0) \right| < C|t - t_0|^\alpha$$

Example: Weierstrass functions

$$\mathcal{W}_{a,b}(t) = \sum_{n \neq 0} a^n \cos(b^n t) \qquad a < 1 < ab$$

- Nowhere differentiable
- $\alpha = -\lg a / \lg(b)$ (monofractal)

- Antonia, Hopfinger, Gagne, and Anselment experiment 1984
- Frisch–Parisi: multifractal model
- Multifractal formalism (Frisch–Parisi conjecture)

$$d_f(\beta) = \inf_p \big(\beta p - \eta_f(p) + 1\big)$$
$$\eta_f(p) = \sup\big\{s \,:\, f \in B_p^{s/p,\infty}\big\}$$

- Jaffard: (1996)

$$d_{\varphi_R}(\beta) = 4\beta - 2 \qquad \tfrac{1}{2} \le \beta \le \tfrac{3}{4}$$

Multifractal formalism is true for φ_R

- $\displaystyle\sum_k \frac{\sin t(ck+d)^2}{(ck+d)^2} \qquad c, d \in \mathbb{Z}$ Oskolkov 2013, Chamizo–Ubis 2013

(2) The binormal curvature flow

(BF) • $\chi_t = \chi_s \wedge \chi_{ss} = cb \quad c : \text{curvature} \quad b : \text{binormal}$

(SM) • $\chi_s = T$ Schrödinger map $\qquad T_t = T \wedge T_{ss}$

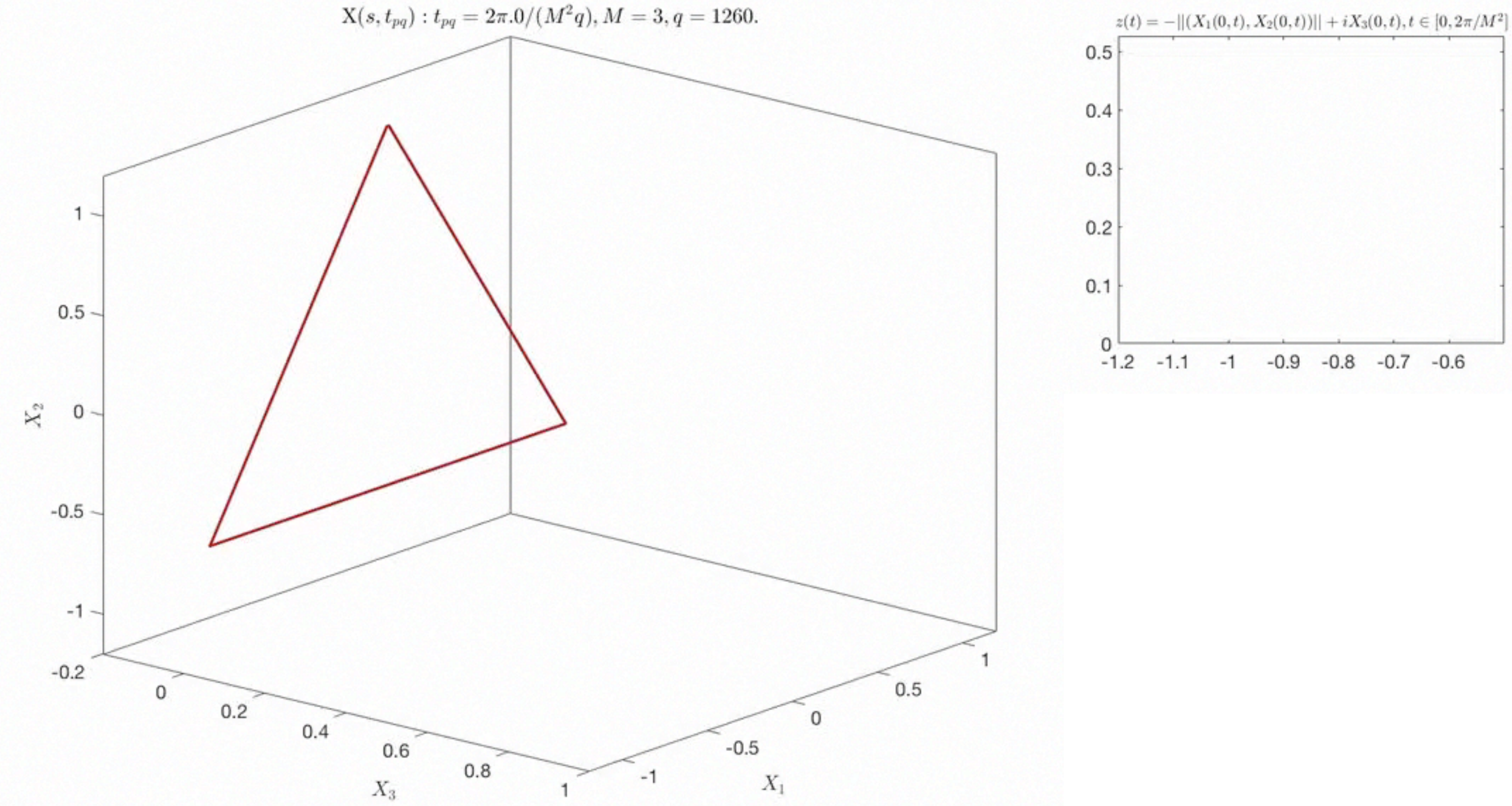
$$\begin{aligned} T_s &= \quad cn \\ n_s &= -cT + \tau b \\ b_s &= -\tau n \end{aligned}$$

Examples:

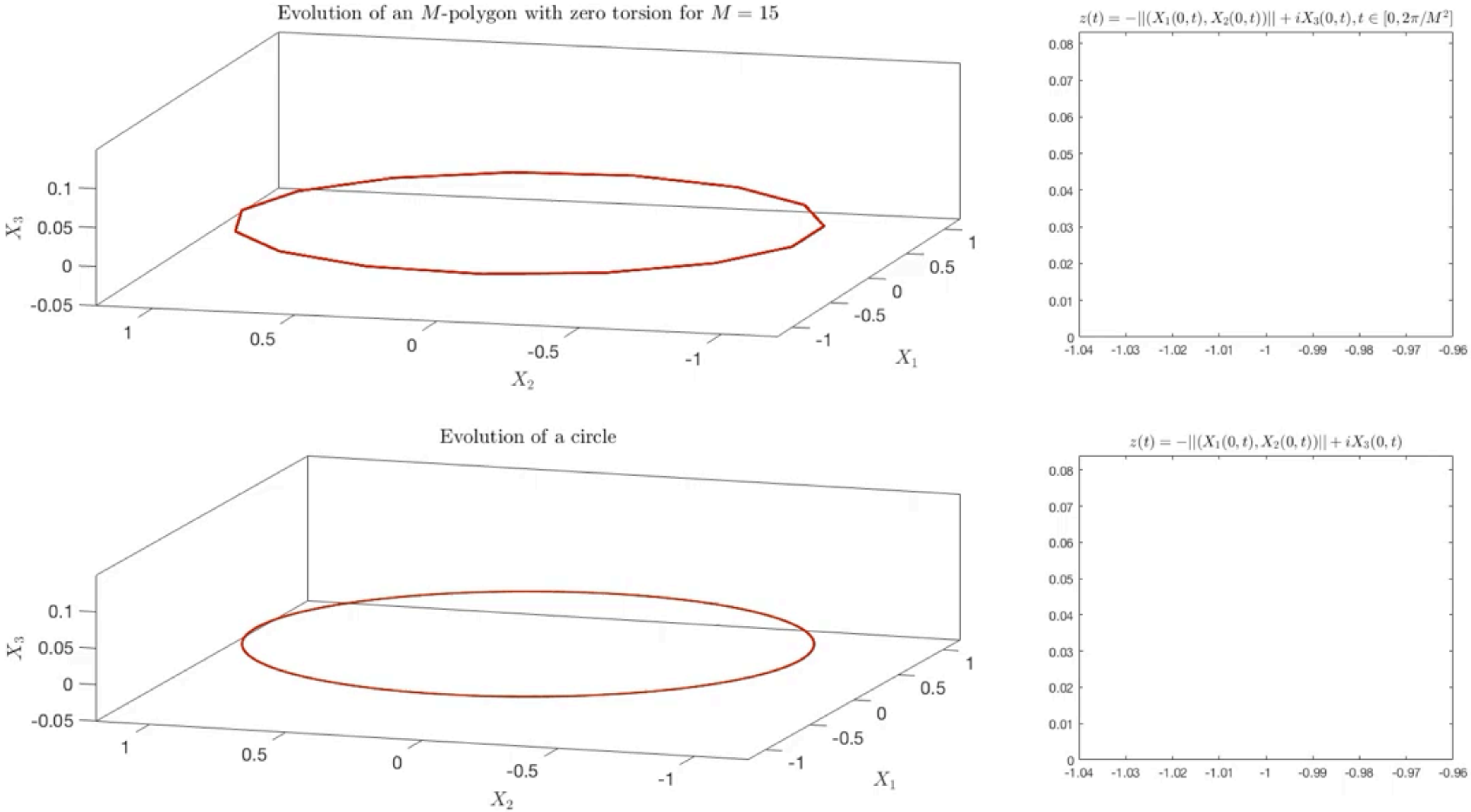
(i) Straight lines.

(ii) Circles.

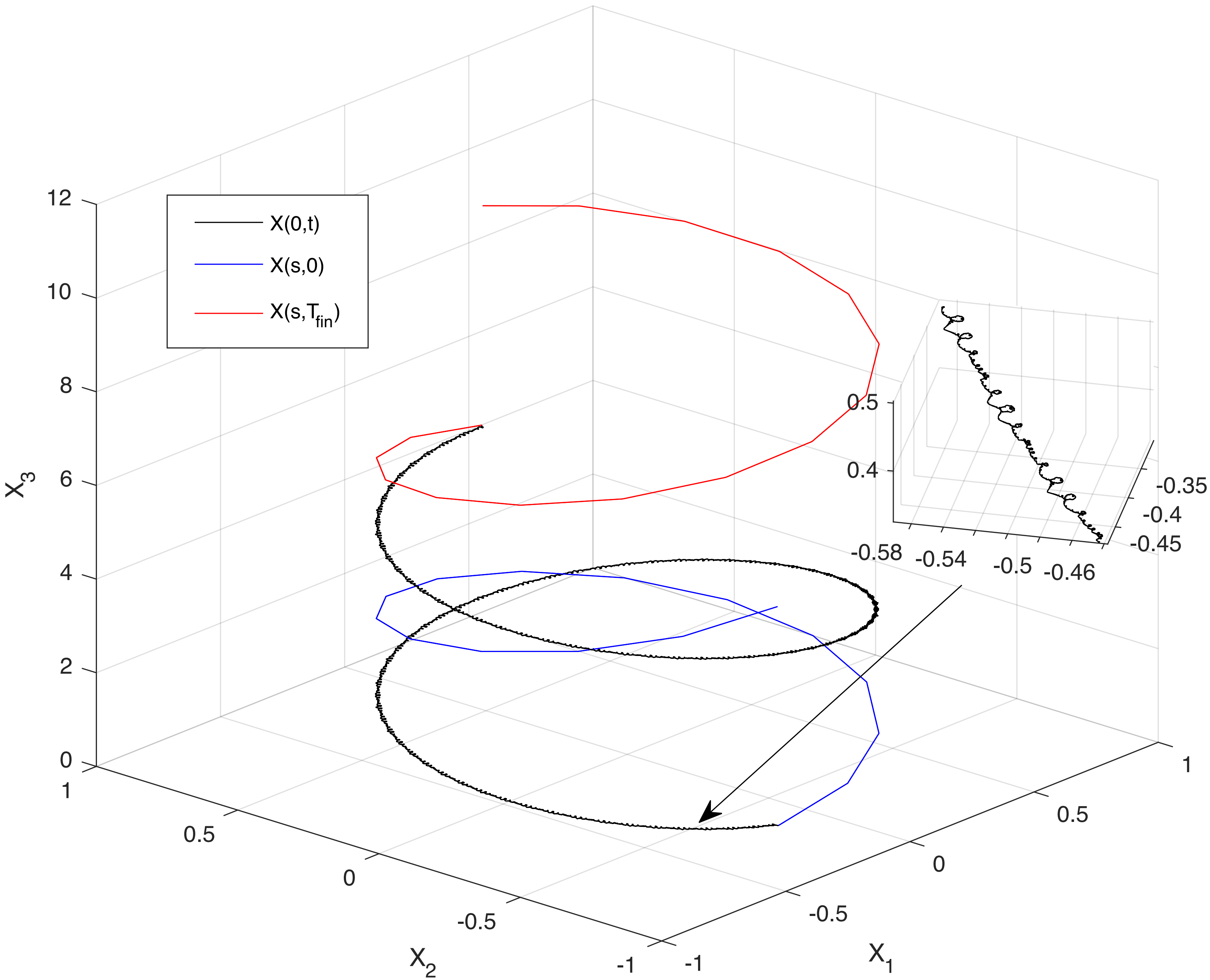
(iii) Helices.



$X(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2 q), M = 3, q = 1260.$
X_2
X_3
X_1
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t), t \in [0, 2\pi/M^2]$



Evolution of an M-polygon with zero torsion for $M = 15$
X_3
X_2
X_1
$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t), t \in [0, 2\pi/M^2]$
Evolution of a circle
X_3
X_2
X_1
$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t)$



X(0,t)
X(s,0)
X(s,T_fin)
X_3
X_2
X_1
12
10
8
6
4
2
0
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0.5
0.4
-0.58
-0.54
-0.5
-0.46
-0.35
-0.4
-0.45

- Buttke 1988
- Peskin, Mcqueen 1994
- Jerrard, Smets 2011
- De La Hoz, Vega 2013, 2018
- Kumar, De La Hoz, Vega 2020

The PDEs

(BF) • $\chi_t = \chi_s \wedge \chi_{ss} = cb$ $\quad c$: curvature $\quad b$: binormal

(SM) • $\chi_s = T$ Schrödinger map $\qquad T_t = T \wedge T_{ss}$

(NLS) • ψ Hasimoto wave function 1d **NLS** (cubic focusing)

$\chi(0, s)$: skew polygonal line

$T(0, s)$: sequence of points T_j such that $\lim_{j \to \pm\infty} T_j = A^{\pm}$

$$\psi(0, s) = \sum a_j \delta(s - j) \qquad \sum_j |j|^{1^+} |a_j|^2 < +\infty$$

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_s = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}$$

$$\psi = \alpha + i\beta$$

Schrödinger equation

Hasimoto transformation:

$$\psi(s,t) = c(s,t)e^{i\int_0^s \tau(s',t)ds'}$$

$c = c(s,t)$ curvature

$\tau = \tau(s,t)$ torsion

$$\partial_t \psi(s,t) = i\left(\partial_s^2 \psi \pm \frac{1}{2}(|\psi|^2 + A(t))\psi\right) \qquad A(t) \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} |\psi(s,t)|^2 ds = \int_{-\infty}^{\infty} |\psi(s,0)|^2 ds = \int_{-\infty}^{\infty} c^2(s,0) ds$$

In our case

$$\psi(s,t) = \frac{a}{\sqrt{t}} e^{i\frac{s^2}{4t}} \qquad , \qquad \int_{-\infty}^{\infty} |\psi|^2 ds = +\infty.$$

Selfsimilar solutions: The ODE

$$\chi_t = \chi_s \wedge \chi_{ss} = cb$$

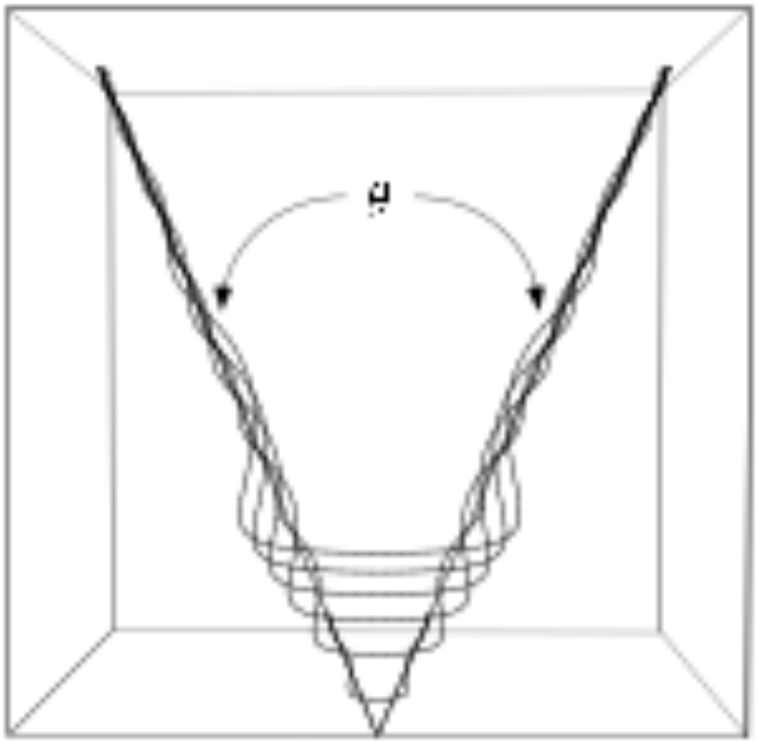
- $\chi(s,t) = \sqrt{t}\,G\big(s/\sqrt{t}\big)$
- $\dfrac{1}{2}G - \dfrac{s}{2}G' = G' \wedge G''$
- The curvature is constant $c(s) = a_0$
- $G(0) = 2a_0\, b(0)$ (trajectory of the corner)
- Given the angle of the corner θ_0, what is a_0?

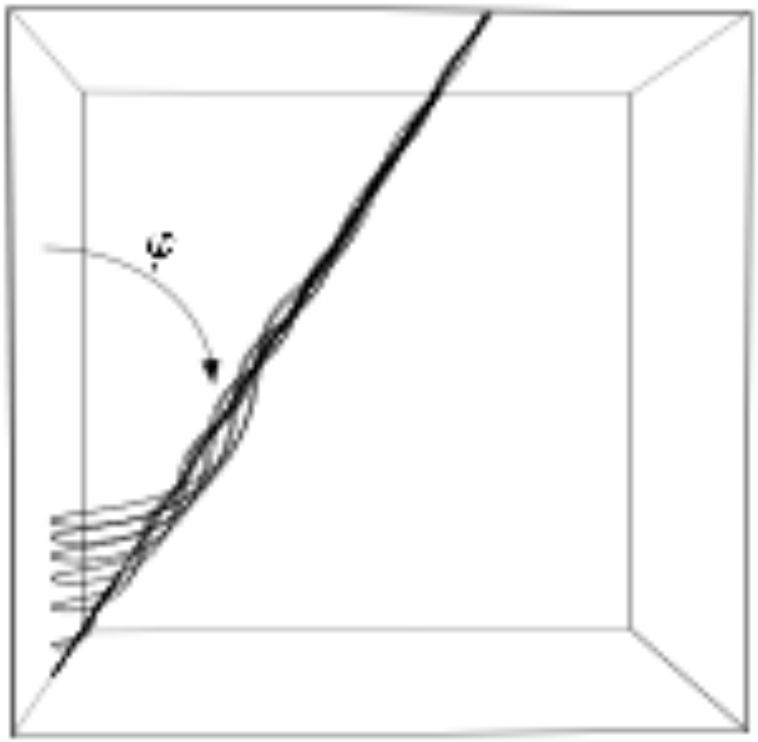
$$\sin\frac{\theta}{2} = e^{-\pi\frac{a_0^2}{2}} \qquad \text{(Gutiérrez, Rivas, V. 2003)}$$

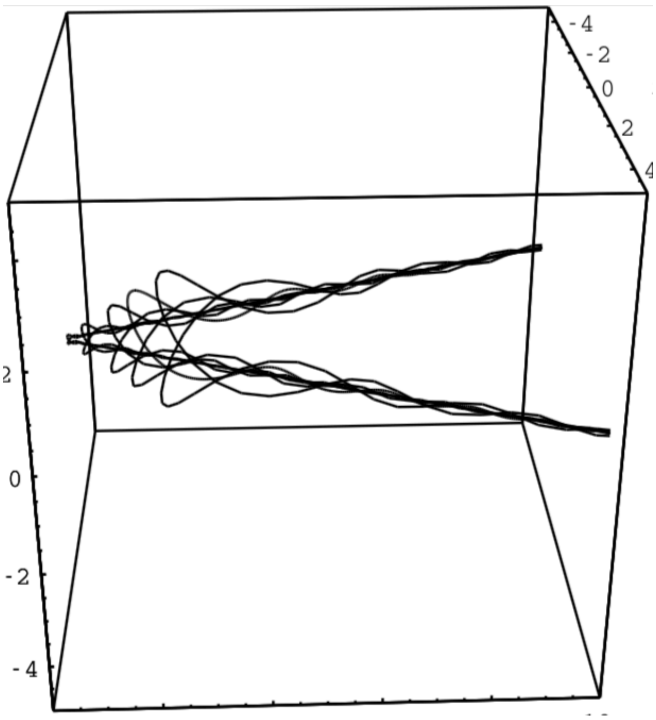
- New approach with De la Hoz, 2018

  If $Z(\eta) = \eta\widehat{G}(\sqrt{\eta})$ then

$$Z'' + \left(1 - \frac{a_0^2}{\eta}\right) Z = 0 \qquad \eta > 0.$$






-4
-2
0
2
4
2
0
-2
-4

1–D cubic NLS with several Dirac masses

We consider distributions

$$f = \sum_{k\in\mathbb{Z}} \alpha_k \delta_k, \qquad \delta_k(\cdot) = \delta(\cdot - k)$$

Their Fourier transform on $\mathbb{R}$ writes

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-ik\xi},$$

and in particular $\hat{f}$ is $2\pi-$periodic. Imposing

$$\|\{\alpha_k\}\|_{l^{2,s}} := \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |\alpha_k|^2 < \infty$$

translates into $\hat{f} \in H^s(0,2\pi)$.
We define

$$H^s_{pF} := \{f \in \mathcal{S}'(\mathbb{R}),\ \hat{f}(\xi+2\pi) = \hat{f}(\xi), \hat{f} \in H^s(0,2\pi)\}.$$

Result for 1–D cubic NLS with several Dirac data

Theorem (Banica-V. 18).– Let $s > \frac{1}{2}$, $0 < \gamma < 1$, $\{\alpha_k\} \in l^{2,s}$ and $M = \sum_{k\in\mathbb{Z}} |\alpha_k|^2$.
We consider the 1-D cubic NLS renormalized equation:

$$i\partial_t u + \Delta u \pm \frac{1}{2}(|u|^2 - \frac{M}{2\pi t})u = 0.$$

There exists $T > 0$ and a unique solution on $(0, T)$ of the form

$$u(t,x) = \sum_{k\in\mathbb{Z}} e^{\mp i \frac{|\alpha_k|^2}{4\pi} \log t}(\alpha_k + R_k(t))e^{it\Delta}\delta_k(x),$$

with

$$\sup_{0<t<T} t^{-\gamma}\|\{R_k(t)\}\|_{l^{2,s}} + t\,\|\{\partial_t R_k(t)\}\|_{l^{2,s}} < C.$$

Moreover, if $s \geq 1$ then the solution can be extended to $(0, \infty)$.

Evolution of polygonal lines by the binormal flow

Theorem (Banica-V. 18).– Let $\chi_0(x)$ be a polygonal line parametrized by arc length with corners located at $x = k \in \mathbb{Z}$, of angles θ_k s.t. $\{a_k\}$ defined by $\sin(\frac{\theta_k}{2}) = e^{-\pi \frac{a_k^2}{2}}$ belongs to $l^{2,3}$. Then there exists $\chi(t)$ smooth solution of the binormal flow on $\mathbb{R}^*$,

solution in the weak sense on $\mathbb{R}$ with

$$|\chi(t,x) - \chi_0(x)| \leq C\sqrt{t}, \quad \forall x \in \mathbb{R}, |t| \leq 1.$$

Refined analysis for some families of polygonal lines

Let $n \in \mathbb{N}^*$, $\nu \in]0,1]$, $\Theta > 0$.

From now on we focus on particular classes of initial data: polygonal lines $\chi_n(0)$ with finite but many corners located at $j \in \mathbb{Z}$ with $|j| \leq n^\nu$, of same torsion ω_0 and angles θ_n such that

$$\lim_{n\to\infty} n(\pi - \theta_n) = \Theta,$$

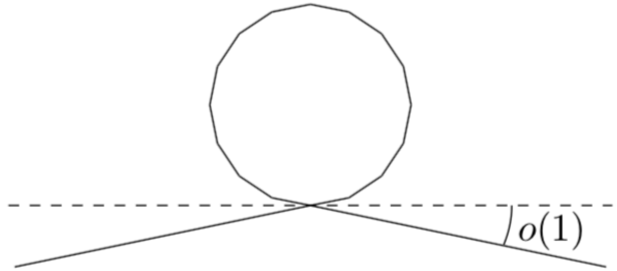
and we suppose without loss of generality

$$\chi_n(0,0) = (0,0,0), \quad \partial_x \chi_n(0,0^\pm) = (\sin\frac{\theta_n}{2}, \pm\cos\frac{\theta_n}{2}, 0).$$

- In the NLS fixed point argument we use l^1-spaces, take advantage of the sizes of nonresonant phases, get better estimates and a time of existence $T > 0$ independent of n.

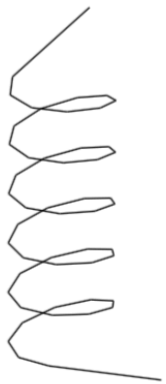
The initial data are the polygonal lines:

$\chi_n(0)$ planar approximation of a line



$\chi_n(0)$ planar approximation of a (multi-)loop

$\chi_n(0)$ non-planar approximation of a line



$\chi_n(0)$ approximation of multi-turns of helices

Theorem (a Frish-Parisi multifractal behaviour)

For the previous solutions with torsion $\omega_0 \in \pi\mathbb{Q}$ we have the following description of the trajectory of the corner $\chi_n(t,0)$, uniformly on $(0,T)$:

$$n\,\chi_n(t,0) - (0, \Re(\tilde{\mathfrak{R}}(t)), \Im(\tilde{\mathfrak{R}}(t))) \overset{n\to\infty}{\longrightarrow} 0.$$

The function $\tilde{\mathfrak{R}}$ is multifractal, and its spectrum of singularities $d_{\tilde{\mathfrak{R}}}$ satisfies the multifractal formalism of Frisch-Parisi:

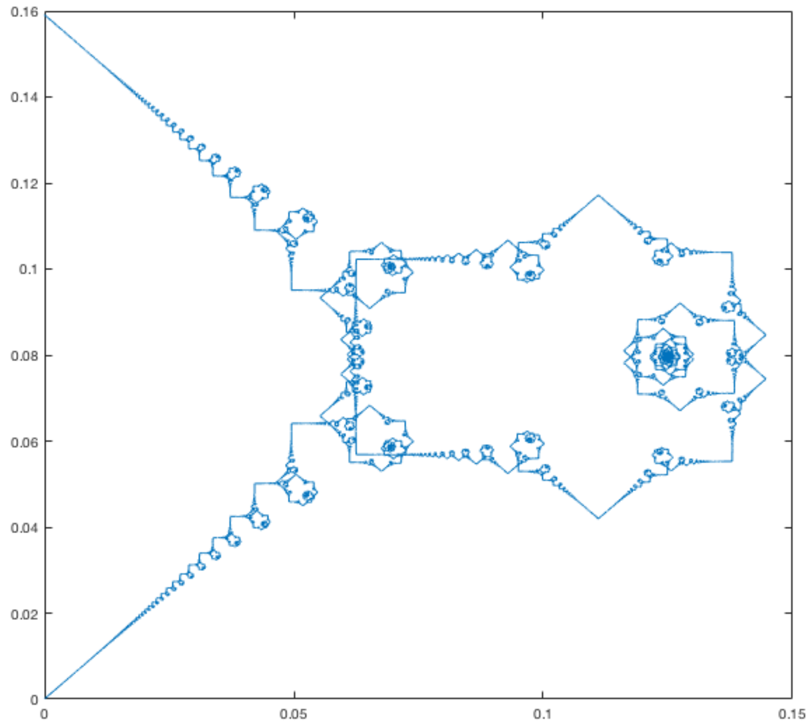
$$d_{\tilde{\mathfrak{R}}}(\beta) := \dim_{\mathcal{H}}\{t, \tilde{\mathfrak{R}} \in \mathcal{C}^{\beta}(t)\} = \inf_{p}(\beta p - \eta_{\tilde{\mathfrak{R}}}(p) + 1),$$

$$\eta_{\tilde{\mathfrak{R}}}(p) := \sup\{s,\, \tilde{\mathfrak{R}} \in B_p^{\frac{s}{p},\infty}\},$$

a model for predicting the structure function exponents in turbulent flows.

In the torsion-free case $\tilde{\mathfrak{R}}(t) = -\Theta\frac{\mathfrak{R}(4\pi^2 t)}{4\pi^2}$, where $\mathfrak{R}(t) = \sum\limits_{j\in\mathbb{Z}} \frac{e^{itj^2}-1}{ij^2}$ is a complex version of Riemann's non-differentiable function.

- Graph on $[0, 2\pi]$ of Riemann's function $\mathfrak{R}(t) = \sum_{j \in \mathbb{Z}} \frac{e^{itj^2} - 1}{ij^2}$:



- $\mathfrak{R}$ satisfies the multifractal formalism of Frisch-Parisi (Jaffard 96) is intermittent (Boritchev-Eceizabarrena-Da Rocha 19), its graph has no tangents (at the end Riemann was right!!) and has Haussdorf dimension $\leq \frac{4}{3}$ (Eceizabarrena 19)

- The theorem gives a non-obvious non-linear geometric interpretation for Riemman's function.

THANK YOU FOR YOUR ATTENTION