Construction of blow-up solution for Complex Ginzburg-Landau equation in some critical case.

Long Time Behavior and Singularity Formation in PDEs, NUY-AD.

Nejla Nouaili

January 13, 2021

Joint work with G.K.Duong (NYU-Abu Dhabi) and H.Zaag (Paris 13).

The complex Ginzburg Landau (CGL) equation

We consider the following equation

$$\begin{array}{rcl} \partial_t u & = & (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u - \gamma u \\ u(x,0) & = & u_0(x) \text{ for } x \in \mathbb{R}^N, \end{array} \qquad (CGL)$$

where

- $p > 1$, β, δ and γ are reals.
- $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{C}$.
- $u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C})$.

Content

Physical motivation and Mathematical relevance for CGL

- Physical motivation: When $p = 3$,
  The world of the complex Ginzburg-Landau equation, Aranson and Kramer 2002.
  CGL appears:
  - in the context of plane Poiseuille flow, see Stewartson and Stuart (1971) and Hocking, Stuart and Stewartson (1971);
  - in the context of the binary mixture, see Kolodner, Bensimon, and Surko (1988).
- Mathematical relevance: Classical tools break down:
  - Maximum principle;
  - Variational formulation;
  - Energy methods.

History of blow-up in CGL equation

$p = 3$, Formal approach

- Existence of blow-up solutions and blow-up behavior was obtained by Hocking and Stewartson (1972),
- Popp, Stiller, Kuznetsov and Kramer (1998), under some condition on β and δ:
  - Existence of blow-up solutions;
  - Determination of the blow-up Behavior.

Rigorous approach for $p > 1$

- Construction, profile and stability, under some conditions on β and δ,
  - when $\beta = 0$, see Zaag (1998);
  - when $\beta \neq 0$, see Masmoudi and Zaag (2008).
- Case $\beta = \delta$: This is variational. Results by Cazenave, Dickstein and Weissler 2012.
- Plechac and Sverak (2001)
  Using a combination of rigourous results and numerical computations describe a countable family of self-similar singularities.
- Budd , Rottschafer and Williams (2005),
  construct, both asymptotically and numerically, multi-bump, blow-up, self-similar.

Cauchy problem and blow-up

- Cauchy problem welpossedness in $L^\infty(\mathbb{R}^N, \mathbb{C})$, Ginibre and Velo (1996-1997), Cazenave (2003) and Ogawa and Yokota (2004).
- Blow-up solutions If $T < \infty$, then $\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty$.
- Blow-up point The point a is a blow-up point if and only if there exists $(a_n, t_n) \to (a, T)$ as $n \to +\infty$ such that $|u(a_n, t_n)| \to +\infty$.

Content

Case $\beta = \delta = 0$, the heat equation

- The generic profile is given by

$$(T-t)^{\frac{1}{p-1}} u(x,t) \sim f_0(z) \text{ as } t \to T$$

where $f_0(z) = (p - 1 + b_0|z|^2)^{-\frac{1}{p-1}}$,

$$z = \frac{x}{\sqrt{(T-t)|\log(T-t)|}} \text{ and } b_0 = \frac{(p-1)^2}{4p}$$

See Hocking and Stewartson (1972), Galaktionov-Posashkov (1985), Berger-Kohn (1988), Herrero-Velzquez (1993).
The constructive existence proof by Bricmont-Kupiainen (1994), Merle-Zaag. (1997) is based on:
  - The reduction of the problem to a finite-dimensional one.
  - The solution of the finite-dimensional problem thanks to the degree theory.
- Other profiles are possible.

Subcriticale case $\beta \neq 0$ and $\delta \neq 0$

If

$$p - \delta^2 - \beta\delta(p+1) > 0, \qquad (Subcritical)$$

then, Masmoudi and Zaag (2008) proved that

$$(T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} u(z\sqrt{(T-t)|\log(T-t)|}, t) \sim f(z),$$

where $f(z) = \kappa^{-i\delta}(p-1+b|z|^2)^{-\frac{1+i\delta}{p-1}}$, $\kappa = (p-1)^{-\frac{1}{p-1}}$

$$b = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta(p+1))} \text{ and } \mu = -\frac{2b\beta}{(p-1)^2}(1+\delta^2)$$

Critical case $\beta = 0$ and $\delta \neq 0$

Theorem (N. and Zaag,18)

If

$$p = \delta^2,$$

then, there exists a solution $u(x,t)$, s.t.

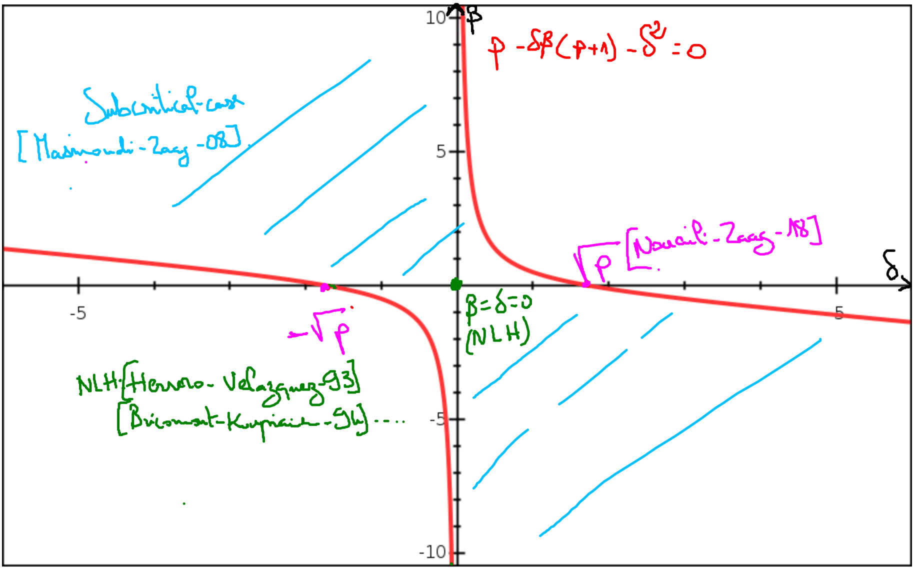
- *Blow-up profile*

$$(T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} u(x,t) \sim f_c(z) \text{ as } t \to T,$$

where

$$f_c(z) = (p-1+b_c|z|^2)^{-\frac{1+i\delta}{p-1}},\ z = \frac{x}{\sqrt{T-t}|\log(T-t)|^{1/4}},$$

$$b_c = \frac{(p-1)^2}{8\sqrt{p(p+1)}} \text{ and } \mu = \frac{8\delta b^2}{(p-1)^4}(1+p).$$


10
5
-5
-10
-5
5
β
δ
p - δβ(p+1) - δ² = 0
Subcritical case
[Masmoudi-Zaag-08]
√p [Nouaili-Zaag-18]
-√p
β=δ=0
(NLH)
NLH [Herrero-Velazquez-93]
[Bricmont-Kupiainen-94]....

Critical case $\beta \neq 0$ and $\delta \neq 0$

Theorem (Duong, N. and Zaag, 20)

If

$$p - \delta^2 - \beta\delta(p+1) = 0,$$

then, there exists $\mu = \mu(\beta, \delta, p)$*, s.t. CGL-eq has a solution* $u(x,t)$*, s.t.*

- *Blow-up profile*

$$(T-t)^{\frac{1+i\delta}{p-1}} e^{-i\nu\sqrt{|\log(T-t)|}} |\log(T-t)|^{-i\mu} u(x,t) \sim f_{cri}(z) \text{ as } t \to T,$$

where

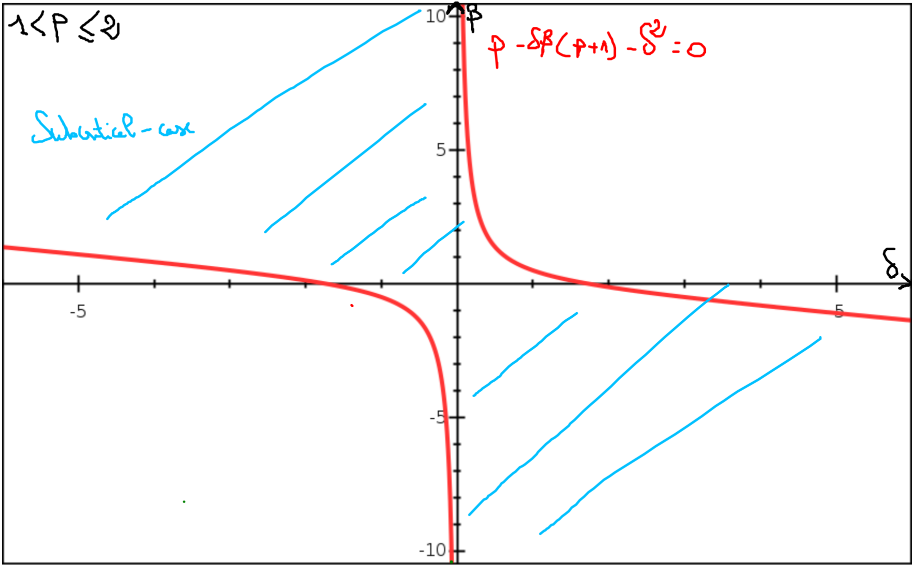
$$f_{cri}(z) = (p - 1 + b_{cri}|z|^2)^{-\frac{1+i\delta}{p-1}},\ z = \frac{x}{\sqrt{T-t}|\log(T-t)|^{1/4}},$$

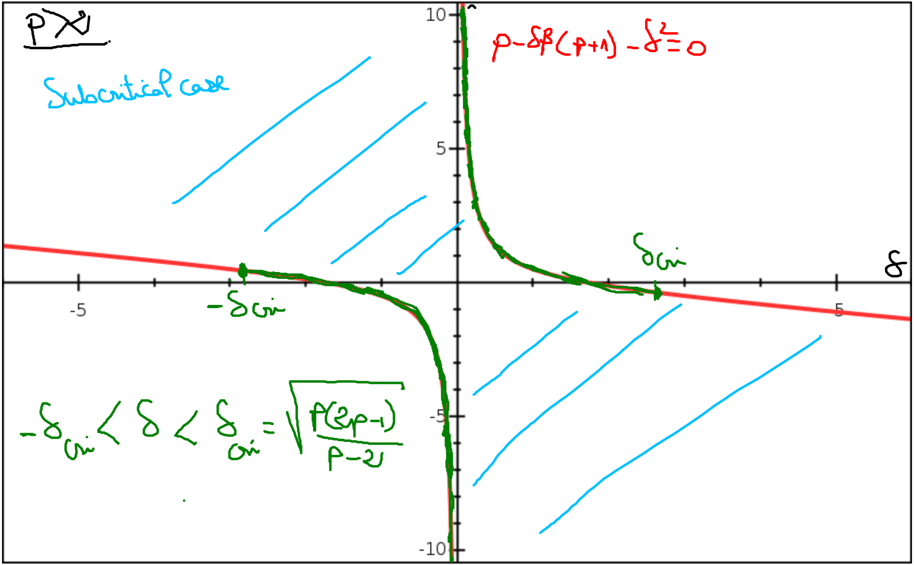
$$\nu = -\frac{4b\beta(1+\delta^2)}{(p-1)^2}.$$

$$b_{cri}^2 = \frac{(p-1)^4(p+1)^2\delta^2}{16(1+\delta^2)(p(2p-1)-(p-2)\delta^2)((p+3)\delta^2+p(3p+1))} > 0,$$

for all $\delta \in (-p_{cri}, p_{cri})$ and

$$p_{cri} = \begin{cases} \sqrt{\frac{p(2p-1)}{p-2}} & \text{if } p > 2 \\ +\infty & \text{if } p \in (1,2] \end{cases}$$


1<p ≤ 2
p − δβ(p+1) − δ² = 0
Subcritical - case
β
δ
10
5
-5
-10
-5
5


P>2
Subcritical case
p-δβ(p+1)-δ²=0
δcri
-δcri
δ
-δcri < δ < δcri = √(p(2p-1)/(p-2))
10
5
-5
-10
-5
5

Comments

The exhibited behavior is new in two respects:

- The scaling law: $\sqrt{(T-t)|\log(T-t)|^{\frac{1}{4}}}$ instead of the laws of subcritical case , $\sqrt{(T-t)|\log(T-t)|}$.
- The profile function: $f_{cri}(z) = (p-1+b_{cri}|z|^2)^{-\frac{1+i\delta}{p-1}}$ is different from the profile of the subcritical case , namely $f(z) = (p-1+b|z|^2)^{-\frac{1+i\delta}{p-1}}$, in the sense that $b_{cri} \neq b$

Idea of the proof

We follow the the constructive existence proof used by Bricomont-Kupiainen (1994), Merle-Zaag (1997) for standard semilinear heat equation and Masmoudi and Zaag (2008) for the CGL equation in the subcritical case.

The method is base on:

- The reduction of the problem to a finite-dimensional one ($N+1$ parameters);
- The solution of the finite-dimensional problem thanks to the degree theory.

Stability of the constructed solution

Thanks to the interpretation of the $(N+1)$ parameters of the finite-dimensional problem in terms of the blow-up time in $(\mathbb{R})$ and the blow-up point (in $\mathbb{R}^N$), the existence proof yields the following:

Theorem (Duong, N. and Zaag: Stability)

The constructed solution is stable with respect to perturbation in initial data.
Consider initial data $\hat{u}_0$ of the solution of (CGL) with blow-up time $\hat{T}$, blow-up point $\hat{a}$ and profile f_{cri} centred at $(\hat{T}, \hat{a})$.

Then, $\exists \mathcal{V}$ neighborhood of $\hat{u}_0$ s.t. $\forall u_0 \in \mathcal{V}$, $u(x,t)$ the solution of (CGL) blows up at time T, at a point a, with the profile f_{cri} centred at(T, a).

Content

A formal approach to find the ansatz ($N = 1$)

• The method of matched asymptotic expansions; Herrero, Galaktionov and Velázquez (1991), Tayachi and Zaag (2015), N. and Zaag (2018).
• Following the standard semilinear heat equation case, we work in similarity:

$$w(y,s) = (T-t)^{\frac{1+i\delta}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\log(T-t).$$

We need to find a solution for the following equation defined for all $s \geq s_0$ and $y \in \mathbb{R}$:

$$\partial_s w = (1+i\beta)\partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1} w,$$

such that

$$0 < \varepsilon_0 \leq \|w(s)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\varepsilon_0}.$$

We suppose that $w(rs^{1/4}, s) = R(r,s)e^{i\varphi(r,s)}$where $r = |y|$.

$$\left\{\begin{array}{rcl} \partial_s R & = & \frac{1}{\sqrt{s}}\left[R'' - R(\varphi')^2 - \beta(2R'\varphi' + R\varphi'')\right] \\ & & +R' r\left(\frac{1}{4s} - \frac{1}{2}\right) - \frac{R}{p-1} + |R|^{p-1}R, \\ \partial_s \varphi & = & \frac{1}{\sqrt{s}}\left[\varphi'' - \beta(\varphi')^2 + \frac{1}{R}\left(2R'\varphi' + \beta R''\right)\right] \\ & & +\varphi' r\left(\frac{1}{4s} - \frac{1}{2}\right) - \frac{\delta}{p-1} + \delta|R|^{p-1}. \end{array}\right.$$

we consider the following ansatz, inspired by the work of Popp and *al*

$$\begin{array}{rcl} R(r,s) & = & R_0(r) + \dfrac{R_1(r)}{\sqrt{s}} + \dfrac{R_2(r)}{s} + ... \\ \varphi(r,s) & = & \Phi(s) + \varphi_0(r) + \dfrac{\varphi_1(r)}{\sqrt{s}} + \dfrac{\varphi_2(r)}{s} + ... \end{array}$$

where $\Phi(s) = \nu\sqrt{s} + \mu \ln s$ and ν, μ unknown.

order 1 of the system gives us

$$\begin{aligned} R_0(r) &= \left(p-1+br^2\right)^{-\frac{1}{p-1}}, \\ \varphi_0(r) &= -\frac{\delta}{p-1}\ln\left(p-1+br^2\right). \end{aligned}$$

order $\frac{1}{s^{1/2}}$, gives us:

$$-\frac{1}{2}R_1'r-\frac{R_1}{p-1}+p|R_0|^{p-1}R_1+R_0''-R_0\varphi_0'^2-\beta\left(2R_0'\varphi_0'+R_0\varphi_0''\right)=0,$$

$$\begin{aligned} R_1(r) &= \frac{r^2}{(p-1+br^2)^{\frac{p}{p-1}}}\times\Bigg[-\frac{2b(\delta\beta-1)}{p-1}r^{-2} \\ &+\frac{8b^2(p-(p+1)\delta\beta-\delta^2)}{(p-1)^3}\left(\ln|r|-\frac{\ln(p-1+br^2)}{2}\right)+\mathcal{C}(\beta,\delta)\Bigg] \end{aligned}$$

The resolution of equation on φ at order $\frac{1}{s^{1/2}}$ gives:

$$\varphi_1(r) = \left[-\nu - \frac{4b\beta(1+\delta^2)}{(p-1)^2}\right] \ln|r| + \frac{2\beta(1+\delta^2)b}{(p-1)^2} \ln(p-1+br^2)$$
$$-\frac{2b}{(p-1)^2}\left((p+3)\delta + \beta(2p+\delta^2(p-3)) + \frac{\mathcal{C}\delta(p-1)^3}{2b^2}\right)(p-1+br^2)^{-1}.$$

By the regularity of φ_1 at 0, the contribution of $\ln|r|$ need to be removed.

$$\nu = -\frac{4b\beta(1+\delta^2)}{(p-1)^2}.$$

The order $1/s$ of the equation on R gives us

$$b_{cri} = \frac{(p-1)^4(p+1)^2\delta^2}{16(1+\delta^2)(p(2p-1)-(p-2)\delta^2)((p+3)\delta^2+p(3p+1))},$$

and

$$\mu = f(p,\beta,\delta) + \frac{2\beta(1+\delta^2)}{p-1}\mathcal{C}.$$

From the above approach, we can formally derive the profile of our solution

$$w(y,s) \sim e^{i(\nu\sqrt{s}+\mu\ln s)}\left(p-1+b_{cri}\frac{|y|^2}{s^{1/2}}\right)^{-\frac{1+i\delta}{p-1}}.$$

Theorem in selfsimilar variables

Theorem (Duong,N. and Zaag)

$$\sup_{|y|<Ms^{\frac{1}{4}}} \left| W(y,s)e^{-i\left(\nu\sqrt{s}+\mu\log s+\theta(s)\right)} - \left\{ f_{cri}\left(\frac{y}{s^{1/4}}\right) + \frac{a(1+i\delta)}{s^{\frac{1}{2}}} + \frac{1}{s}\mathcal{F}(y) \right\} \right|$$
$$\leq C\frac{(1+|y|^5)}{s^{\frac{3}{2}}},$$

and $\theta(s) \to \theta_0$ *as* $s \to \infty$ *(*$t \to T$*), such that*

$$|\theta(s) - \theta_0| \leq \frac{C}{s^{\frac{1}{4}}}$$

with

$$f_{cri}(z) = \left(p - 1 + b_{cri}z^2\right)^{-\frac{1+i\delta}{p-1}},$$

and

$$\mathcal{F}(y) = \mathcal{A}_0(\delta)h_0(y) + \mathcal{A}_2(\delta)h_2(y) + \tilde{\mathcal{A}}_2(\delta)\tilde{h}_2(y).$$

Strategy of the proof

The use of topological methods in the analysis of singularities for blow-up phenomena seems to have been introduced by Bressan (1992). Bricomont and Kupiainen (1994) then Merle and Zaag (1997) for the semilinear heat equation.

The Solutions we construct are obtained through a **topological "shooting method"**:

• The reduction of the problem to a finite-dimensional one;

• The solution of the finite-dimensional problem thanks to the degree theory.

This strategy was adapted for:

- Degenerate neckpinches in mean curvature flow, Angenent and Velázquez 97,
- the Heat equation with subcritical gradient exponent in Ebde and Zaag (2011), with critical power nonlinear gradient termTayachi and Zaag 2015 ;
- The complex heat equation N. and Zaag (2015).
- the CGL equation: in Zaag (1998) and Masmoudi and Zaag (2008);
- the supercritical gKdV and NLS in Côte, Martel and Merle (2011);
- the semilinear wave equation in Côte and Zaag (2013), for the construction of a blow-up solution showing multi-solitons.

The strategy of the proof ($N = 1$)

We recall our aim: To consruct a solution $w(y,s)$ of the equation in similarity variables:

$$\partial_s w = (1+i\beta)\partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1} w,$$

such that

$$w(y,s) \sim e^{i\left(\nu\sqrt{s}+\mu\log s\right)}\varphi(y,s)$$

where

$$\varphi(y,s) = \kappa^{-i\delta}(p-1+b\frac{y^2}{s^{1/2}})^{-\frac{1+i\delta}{p-1}} + (1+i\delta)\frac{a}{s^{1/2}}$$

Idea:

We linearize around φ, introducing $q(y,s)$ and $\theta(s)$

$$w(y,s) = e^{i(\nu\sqrt{s}+\mu\log s+\theta(s))}(\varphi(y,s) + q(y,s))$$

In that case, our aim becomes to find $\theta \in C^1([-\log T, \infty),)$ such that $q(y,s)$ is defined for all $(y,s) \in \times[-\log T, \infty)$ and

$$\|q(s)\|_{L^\infty} \to 0 \text{ as } s \to \infty$$

Modulation

$$\Im\left(\int q\rho_\beta dy\right) - \delta\Re\left(\int q\rho_\beta dy\right) = 0.$$

This choice of $\theta(s)$ kills one neutral mode given by $h_0(y)$.

Decomposition of $q(y,s)$ into inner and outer parts

The variable $z = \frac{y}{s^{1/4}}$ plays a fundamental role. Thus we will consider the dynamics for the outer region $|z| > K$ and the inner region$|z| < 2K$.
Consider a cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{s^{1/4}}\right),$$

where $\chi_0 \in C^\infty([0,\infty),[0,1])$, s.t. $supp(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$, in $[0,1]$.
Then, we introduce

$$q = q_{inner} + q_{outer}$$

Remark: q_{outer} is easily controlled, because the spectrum of $\mathcal{L}_{\beta,\delta} + V_1 + V_2$ is negative in the outer region.

$q(y,s)$ satisfies for all $s \geq s_0$ and $y \in \mathbb{R}$,

$$\partial_s q = \mathcal{L}_{\beta,\delta} q - i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q + V_1 q + V_2 \bar{q} + B(q,y,s) + R^*(\theta', y, s),$$

where

$$\begin{array}{rcl}
\mathcal{L}_{\beta,\delta} q & = & (1+i\beta)\partial_y^2 q - \frac{1}{2} y \cdot \partial_y q + (1+i\delta)\Re q, \\
V_1(y,s) & = & (1+i\delta)\frac{p+1}{2}\left(|\varphi|^{p-1} - \frac{1}{p-1}\right), \\
V_2(y,s) & = & (1+i\delta)\frac{p-1}{2}\left(|\varphi|^{p-3}\varphi^2 - \frac{1}{p-1}\right), \\
B(q,y,s) & = & (1+i\delta)\left(|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1} q\right. \\
 & & \left. -\frac{p-1}{2}|\varphi|^{p-3}\varphi(\varphi\bar{q} + \bar{\varphi} q\right), \\
R^*(\theta', y, s) & = & R(y,s) - i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)\varphi, \\
R(y,s) & = & -\partial_s\varphi + \partial_y^2\varphi - \frac{1}{2} y \cdot \partial_y \varphi - \frac{(1+i\delta)}{p-1}\varphi + (1+i\delta)|\varphi|^{p-1}\varphi
\end{array}$$

The linear operator $\mathcal{L}_{\beta,\delta}$

Note that $\mathcal{L}_{\beta,\delta}$ is not self-adjoint and is not diagonalisable.

$\mathrm{L}^2_{|\rho_\beta|} = \{g \in \mathrm{L}^2_{loc}(\mathbb{R},\mathbb{C}) | \int_{\mathbb{R}} |g(y)|^2 |\rho_\beta(y)| dy < \infty\}$and

$$\rho_\beta(y) = \frac{e^{-\frac{|y|^2}{4(1+i\beta)}}}{(4\pi(1+i\beta))^{1/2}}.$$

The spectrum of $\mathcal{L}_{\beta,\delta}$ is given by

$$spec(\tilde{\mathcal{L}}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\}$$

Jordan block's decomposition of $\mathcal{L}_{\beta,\delta}$

For all $n \in$, there exists two polynomials

$$
\begin{aligned}
h_n &= if_n + \textstyle\sum_{j=0}^{n-1} d_{j,n} f_j, \text{ where } d_{j,n} \in \mathbb{C} \\
\tilde{h}_n &= (1+i\delta) f_n + \textstyle\sum_{j=0}^{n-1} \tilde{d}_{j,n} f_j, \text{ where } \tilde{d}_{j,n} \in \mathbb{C},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{L}_{\beta,\delta} h_n = -\frac{n}{2} h_n, \\
&\mathcal{L}_{\beta,\delta} \tilde{h}_n = \left(1 - \frac{n}{2}\right) \tilde{h}_n + c_n h_{n-2}, \text{ with } c_n = n(n-1)\beta(1+\delta^2),
\end{aligned}
$$

$$
f_i(y) = c_i H_i\left(\frac{y}{2\sqrt{1+i\beta}}\right) \text{ where } H_i \text{ is Hermite polynomial.}
$$

f_i are the eigenfunctions of the linear operator

$$
\mathcal{L}_\beta v = (1+i\beta)\partial_y^2 v - \frac{1}{2} y \cdot \partial_y v.
$$

Some examples of polynomials h_n and $\tilde{h}_n$.

$$h_0(y) = i, \qquad \tilde{h}_0 = (1+i\delta),$$
$$h_1(y) = iy, \qquad \tilde{h}_1 = (1+i\delta)y,$$
$$h_2(y) = iy^2 + \beta - i(2+\delta\beta), \quad \tilde{h}_2 = (1+i\delta)(y^2 - 2 + 2\beta\delta),$$

$$\begin{aligned}
\mathcal{L}_{\beta,\delta}\tilde{h}_0 &= \tilde{h}_0,\\
\mathcal{L}_{\beta,\delta}\tilde{h}_1 &= \tfrac{1}{2}\tilde{h}_1,\\
\mathcal{L}_{\beta,\delta}\tilde{h}_2 &= 2\beta(1+\delta^2)h_0 = 2i\beta(1+\delta^2),\\
\mathcal{L}_{\beta,\delta}h_6 &= -\tilde{h}_4 + 12\beta(1+\delta^2)h_2,\\
\mathcal{L}_{\beta,\delta}\tilde{h}_6 &= -2\tilde{h}_6 + 30\beta(1+\delta^2)h_4.
\end{aligned}$$

effect of the different terms

- The potential terms V_1 and V_2: $V_1 + V_2 \to 0$ in $L^2_{|\rho_\beta|}(\mathbb{R})$. The effect of $V_1 + V_2$ in the blow-up area is regarded as a perturbation of the effect of $\mathcal{L}_{\beta,\delta}$.
- The nonlinear term B: It is quadratic $|B(q)| \leq C|q|^2$
- The rest term R^*: It is small $\|R^*(.,s)\|_{L^\infty} \leq \frac{C}{\sqrt{s}} + |\theta'(s)|$.

Decomposition of the inner part

We decompose q_{inner} according to the sign of the eigenvalues of $\tilde{\mathcal{L}}$

$$\begin{aligned} q_{inner} &= \sum_{n \le M} \mathcal{Q}_n(s) f_n(y) + q_-(y,s), \\ &= \sum_0^1 \tilde{q}_n \tilde{h}_n + \tilde{\mathbf{q}}_\mathbf{2} \tilde{\mathbf{h}}_\mathbf{2} + \left(\sum_3^M \tilde{q}_n \tilde{h}_n + \sum_0^M q_n h_n \right) + q_-(y,s) \end{aligned}$$

- We choose $M \ge 4(\sqrt{1+\delta^2} + 1 + 2 \max_{y \in \mathbb{R}, s \ge 1, i=1,2} |V_i(y,s)|)$, then q_- is easily controlled.
- Smalness of the modulation parameter $\theta(s)$

$$|\theta'(s)| \le \frac{C}{s^{\frac{5}{4}}}, \text{ where } C > 0.$$

- It remains then to control $\tilde{q}_0$, $\tilde{q}_1$ and $\tilde{\mathbf{q}}_\mathbf{2}$.

Control of $\tilde{q}_2$

This is delicate, because it corresponds to the direction $(1+i\delta)h_2(y)$, the null mode of the linear operator $\mathcal{L}_{\beta,\delta}$.
We need to refine the contribition of the potentials V_1+V_2, the nonlinear term B and the rest term R^*, this is delicate because we are studying the critical problem. we obtain

$$\tilde{\mathbf{q}}_2' = \tilde{\mathbf{H}}_0 \frac{\tilde{\mathbf{q}}_2}{\sqrt{\mathbf{s}}} + \tilde{\mathbf{H}}_1 \frac{\tilde{\mathbf{q}}_2}{\mathbf{s}} + \frac{\tilde{\mathbf{K}}_1}{\mathbf{s}^{3/2}} + \frac{\tilde{\mathbf{K}}_2}{\mathbf{s}^2} + \mathbf{O}(\frac{\mathbf{1}}{\mathbf{s}^{9/4}})$$

Cancelation of $\tilde{H}_0$ give us the value of ν and a.
Cancelation of $\tilde{K}_1$ give us the value of b_{cri}.

$$\tilde{H}_1 \leq -\frac{3}{2}.$$

- In the subcritical case, $\tilde{H}_1 = -2$.
- In the critical case $\beta = 0$, $\tilde{H}_1 = -\frac{3}{2}$.

Control of $\tilde{q}_2$

$$\tilde{q}_2' = \tilde{H}_1 \frac{\tilde{q}_2}{s} + \frac{\tilde{K}_2}{s^2} + O(\frac{1}{s^{9/4}})$$

We introduce $\tilde{Q}_2 = \tilde{q}_2 - \frac{\tilde{\mathcal{A}}_2}{s}$, then we have

$$\tilde{Q}_2' = \tilde{H}_1 \frac{\tilde{Q}_2}{s} + \frac{\tilde{\mathcal{A}}_2(\tilde{H}_1 + 1) + \tilde{K}_2}{s^2} + O(\frac{1}{s^{9/4}})$$

Cancelation of order $\frac{1}{s^2}$ gives us μ.

Conclusion $\tilde{Q}_2$ and $\tilde{q}_2$ can be controlled as well.

The finite dimentional problem $\tilde{q}_0$ and $\tilde{q}_1$

The remaining components correspond respectively to the projection along $\tilde{h}_0 = (1 + i\delta)$ and $\tilde{h}_1 = (1 + i\delta)y$, the positive direction of $\mathcal{L}_{\beta,\delta}$.
Projecting the equation

$$\partial_s q = \mathcal{L}_{\beta,\delta} q - i\left(\frac{\nu}{2s^{1/2}} + \frac{\mu}{s} + \theta'(s)\right) q + V_1 q + V_2 \bar{q} + B(q, y, s) + R^*(\theta', y, s),$$

we obtain

$$\tilde{Q}_0' = \tilde{Q}_0 + O\left(\frac{1}{s^{7/4}}\right), \text{ where} \tilde{Q}_0 = \tilde{q}_0 - \frac{\tilde{\mathcal{A}}_0}{s}$$

$$\tilde{q}_1' = \frac{1}{2}\tilde{q}_1 + O\left(\frac{1}{s^{3/2}}\right).$$

with given initial data at s_0 by $\tilde{Q}_0 = d_0, \;\; \tilde{q}_1 = d_1$.
This problem can be easily solved by contradiction, using **index Theory**.
There exist a particular $(d_0, d_1) \in \mathbb{R}^2$ such that the problem has a solution $(\tilde{Q}_0(s), \tilde{q}_1(s))$ which converges to $(0, 0)$ as $s \to \infty$.

Open problem

How a collapse of the solutions of the CGLE may be suppressed for suitable parameters?

- Kaplan, Kuznetsov and V. Steinberg 94.
- Kramer, Kuznetsov, Popp and Turitsyn 95.
- Popp, Stiller, Kuznetsov and Kramer 98.

Consider the CGL equation ($N = 1,\ p = 3,\ \beta = 0,\ \delta >> 15$) in modulus and phase $u = Ae^{i\phi}$ and let $k := \partial_x \phi$.
How to explain that the phase gradien k supress the explosive growth of amplitude A?

Thank you !