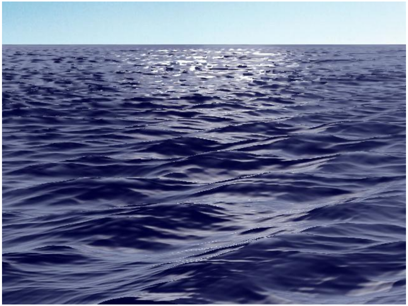
Quasi-periodic solutions in fluid equations

Massimiliano Berti, SISSA,
Long Time Behavior and Singularity Formation in PDEs, part 2,
NYU-AD 13 January 2021



Quasi-periodic traveling water waves, M. Berti, L. Franzoi, A. Maspero,

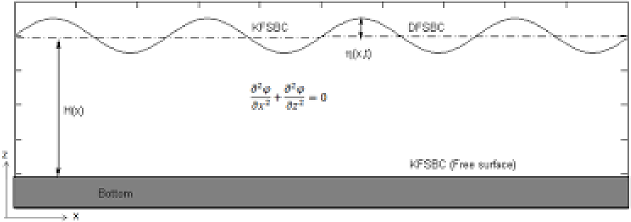
- Gravity-Capillary water waves, part 1, 2020 (Conference in May)
- Pure gravity water waves, part 2, 2021, today

in progress, M. Berti, Z. Hassaina, N. Masmoudi

- Quasi-periodic vortex patches

Time evolution of space periodic water waves:

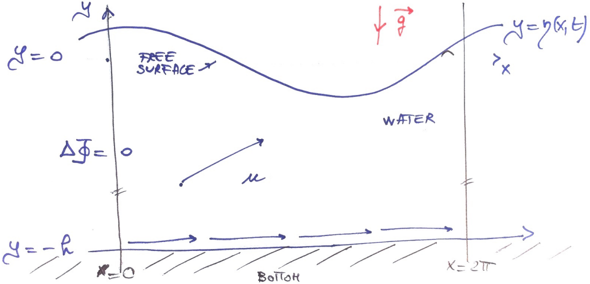
In a vertical section it is described by a bi-dimensional fluid, periodic in x



Water Waves : Euler equations for an incompressible fluid with constant vorticity γ in $\mathcal{D}_\eta(t) = \{-h < y < \eta(t,x)\}$ under gravity

Equation of motions for $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ in $-h < y < \eta(t,x)$

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla P - g e_y \\ \mathrm{div}\,\vec{u} = u_x + v_y = 0 \\ \mathrm{rot}\,\vec{u} = v_x - u_y = \gamma \end{cases}$$



Boundary conditions:

$$\begin{cases} \eta_t = v - u\eta_x & \text{at } y = \eta(t,x) \\ P = P_0 & \text{at } y = \eta(t,x) \\ v = 0 & \text{at } y = -h \end{cases}$$

g = gravity, γ = vorticity, P = pressure of fluid, P_0 = atmospheric pressure,

Unknowns:

free surface $y = \eta(t,x)$ and the velocity field $\vec{u}(t,x,y)$

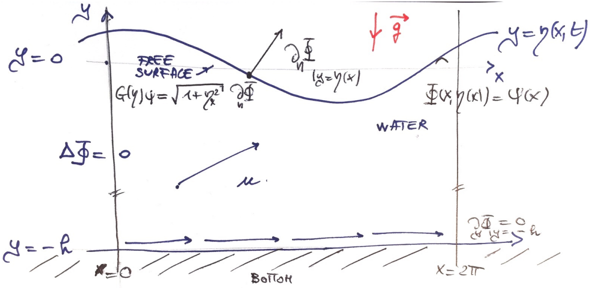
- In case of constant vorticity γ, $\vec{u}$ is the sum of a *Couette flow* and of an irrotational flow

$$\vec{u}(t,x,y) = \begin{pmatrix} -\gamma y \\ 0 \end{pmatrix} + \nabla \Phi, \qquad \Phi(t,x,y) = \text{ velocity potential}$$

$\vec{u}(t,x,y)$ is completely determined by $\eta(t,x)$ and $\psi(t,x) = \Phi(t,x,\eta(t,x))$

$$\begin{cases} -\Delta \Phi = 0 & \text{in } -h < y < \eta(t,x) \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \partial_y \Phi = 0 & \text{at } y = -h \end{cases}$$

Elliptic problem
Dirichlet-Neumann bc



Reformulate the equations in terms of (η, ψ): e.g.

$$\eta_t = v - u\eta_x \quad \rightsquigarrow \quad \eta_t = (\Phi_y - \eta_x \Phi_x + \gamma \eta_x y)|_{y=\eta(t,x)} = G(\eta)\psi + \gamma \eta_x \eta$$

Zakharov-Constantin-Wahlen formulation of WW with constant vorticity

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \end{cases}$$

Dirichlet–Neumann operator

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\Phi_y - \eta_x\Phi_x)|_{y=\eta(x)}$$

1. $G(\eta)$ is linear in ψ, non-local,
2. self-adjoint with respect to $L^2(\mathbb{T}_x)$
3. $G(\eta) \geq 0$, $G(1) = 0$
4. $\eta \mapsto G(\eta)$ nonlinear, smooth,
5. $G(\eta)$ is pseudo-differential, $G(\eta) = D\tanh(hD) + OPS^{-\infty}$, $D := \frac{1}{i}\partial_x$

Calderon, Craig, Lannes, Metivier, Alazard, Burq, Zuily, Delort...

Fourier multiplier notation

$$m(D)u := \sum_{j\in\mathbb{Z}} m_j u_j e^{\mathrm{i}jx}\,, \quad u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}\,, \; (\partial_x^{-1}u)(x) := \sum_{j\neq 0} \frac{1}{\mathrm{i}j}\, u_j\, e^{\mathrm{i}jx} \quad \text{on functions with } u_0 = 0$$

$$\partial_t u = J(\gamma)\nabla_u^{L^2} H(u)\,, \quad u := \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad J(\gamma) := \begin{pmatrix} 0 & Id \\ -Id & \gamma\partial_x^{-1} \end{pmatrix}$$

$$\eta_t = \nabla_\psi^{L^2} H(\eta,\psi)\,, \quad \psi_t = -\nabla_\eta^{L^2} H(\eta,\psi) + \gamma\partial_x^{-1}\nabla_\psi^{L^2} H(\eta,\psi)$$

Hamiltonian (Constantin, Wahlen)

$$H(\eta,\psi) = \tfrac{1}{2}\int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx + \tfrac{1}{2}\int_{\mathbb{T}} g\eta^2\, dx + \tfrac{\gamma}{2}\int_{\mathbb{T}} (-\psi_x\eta^2 + \tfrac{\gamma}{3}\eta^3)\, dx$$

Wahlen coordinates (η,ζ) are Darboux coordinates:

$$\zeta := \psi - \frac{\gamma}{2}\partial_x^{-1}\eta$$

$$\partial_t \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = J\nabla_{(\eta,\zeta)}^{L^2} H(\eta,\zeta)\,, \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

$$\eta_t = \nabla_\zeta^{L^2} H(\eta,\zeta)\,, \quad \zeta_t = -\nabla_\eta^{L^2} H(\eta,\zeta)$$

Reversibility

$$H \circ S = H, \qquad \text{Involution: } S : (\eta, \zeta)(x) \mapsto (\eta, -\zeta)(-x)$$

$\Leftrightarrow$ if $u(t) = (\eta, \psi)(t)$ is a solution then $Su(-t)$ is a solution

Reversible solutions

$$u(t) = Su(-t) \quad \Longleftrightarrow \quad \eta(t,x) = \eta(-t,-x)\,, \;\; \psi(t,x) = -\psi(-t,-x)$$

Translation invariance

$$H \circ \tau_\varsigma = H, \qquad \tau_\varsigma : (\eta, \zeta)(x) \mapsto (\eta, \zeta)(x + \varsigma)$$

Momentum

$$\int_{\mathbb{T}} \zeta_x(x) \eta(x)\, dx$$

Mass

$$\int_{\mathbb{T}} \eta(x)\, dx = \mathrm{const}.$$

Phase space

$$\eta \in H^s_0(\mathbb{T}) := \Big\{ \eta \in H^s(\mathbb{T}) \, : \, \int_{\mathbb{T}} \eta(x) dx = 0 \Big\}$$

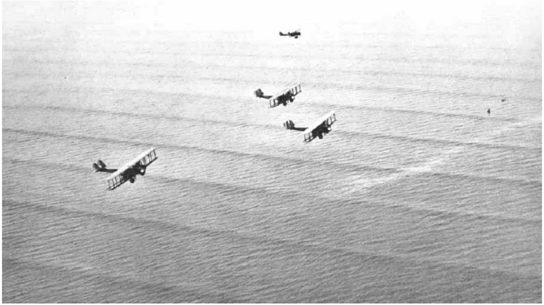
$$u \in H^s(\mathbb{T}) \ \Leftrightarrow \ u(x) = \sum_{k \in \mathbb{Z}} u_k e^{\mathrm{i}kx} \, , \ \ \sum_{k \in \mathbb{Z}} |u_k|^2 \langle k \rangle^{2s} =: \|u\|^2_{H^s} < +\infty$$

The variable ζ is defined modulo constants: only the velocity field $\nabla_{x,y}\Phi$ has physical meaning:

$$\zeta \in \dot{H}^s(\mathbb{T}) = H^s(\mathbb{T}) / \sim \qquad u(x) \sim v(x) \quad \Longleftrightarrow \quad u(x) - v(x) = c$$

Main question:

- Are there **time** quasi-periodic **traveling** solutions of the water waves equations?



Periodic traveling wave solution

$\exists\ \breve{\eta}(\varphi), \breve{\psi}(\varphi)$, 2π-periodic in φ, and $c \in \mathbb{R}$ such that

$$\begin{pmatrix} \eta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \breve{\eta}(x-ct) \\ \breve{\psi}(x-ct) \end{pmatrix} \qquad \Big(\frac{2\pi}{c}-\textit{periodic in time and } 2\pi-\textit{periodic in space}\Big)$$

Levi-Civita, Struik, Nekrasov, Toland, Amick, Craig-Nicholls, Constantin, Whalen, Martin, ..

Remark: they are steady -stationary- in a moving frame with speed c

Time quasiperiodic traveling wave solution

$\exists$ functions $\breve{\eta}(\varphi_1, \ldots, \varphi_\nu)$, $\breve{\psi}(\varphi_1, \ldots, \varphi_\nu)$, 2π-periodic in each variable
frequencies $\omega_1, \ldots, \omega_\nu \in \mathbb{R}$, $\omega_1 \ell_1 + \ldots + \omega_\nu \ell_\nu \neq 0$, $\forall (\ell_1, \ldots, \ell_\nu) \in \mathbb{Z}^\nu \setminus \{0\}$
wave vectors $j_1, \ldots, j_\nu \in \mathbb{Z}$

$$\begin{pmatrix} \eta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \breve{\eta}(j_1 x - \omega_1 t, \ldots, j_\nu x - \omega_\nu t) \\ \breve{\psi}(j_1 x - \omega_1 t, \ldots, j_\nu x - \omega_\nu t) \end{pmatrix}$$

Remark: not steady in any moving frame

Literature: Periodic/quasi-periodic 2D water waves

- **Periodic traveling waves**: (steady in a moving frame)
  - Nekrasov '21, Levi-Civita '26, Struik '37, Zeidler '71
    2D, small amplitude, *irrotational*
  - Dubreil-Jacotin '34, Goyon '58, Zeidler '73, Wahlen '09, Martin '13 :
    2D, small amplitude with *vorticity*
  - Krasovskii '71, Keady-Norbury '78, Toland '78, McLeod '97, Constantin-Strauss '04 , Constantin -Strauss -Varvaruca '18
    2D, large amplitude, irrotational or with vorticity

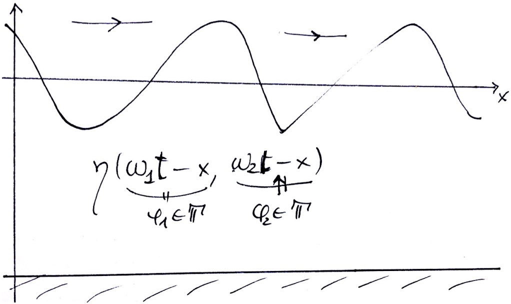
Proof: Crandall-Rabinowitz (not a small divisors problem) + global bifurcation analysis

- **Time periodic standing waves:** even in x
  - Plotnikov-Toland '01, Iooss-Plotnikov-Toland '05, Alazard-Baldi '15
    2D small amplitude, small divisors problem
- **Time quasiperiodic standing waves:**
  - Berti-Montalto '16 (gravity-capillary), Baldi-Berti-Haus-Montalto '17 (gravity)
    2D small amplitude , small divisors problem

"Take away Theorem" (Berti, Franzoi, Maspero, '21)

$\forall$ depth h (also infinity), for γ (also zero) in a Cantor set of large measure
$\exists$ small amplitude time quasi-periodic traveling solutions of 2D- pure gravity Water Waves.
If $\gamma = 0$ we prove time quasi-periodic traveling waves for most h

- Berti, Franzoi, Maspero '20, gravity-capillary. We used the surface tension κ as a parameter,
- Feola-Giuliani 2020, irrotational, $h = +\infty$, pure gravity, completely resonant PDE



Linearized system at $(\eta, \zeta) = (0, 0)$

$$\begin{cases} \partial_t \eta = G(0)\zeta + \frac{\gamma}{2} G(0)\partial_x^{-1}\eta, \\ \partial_t \zeta = -g\eta + \frac{\gamma}{2}\partial_x^{-1} G(0)\zeta + \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0)\partial_x^{-1}\eta \end{cases}$$

Dirichlet-Neumann operator at the flat surface $\eta = 0$ is

$$G(0) = D \tanh(hD) \,, \quad D = \frac{\partial_x}{\mathrm{i}} = \mathrm{Op}(\xi)_{\xi \in \mathbb{R}}$$

Complex variable

$$z = \frac{1}{\sqrt{2}}(M^{-1}(D)\eta + \mathrm{i}M(D)\zeta)\,, \quad M(D) := \Big(\frac{G(0)}{g - \frac{\gamma^2}{4}\partial_x^{-1}G(0)\partial_x^{-1}}\Big)^{\frac{1}{4}}$$

$\mathrm{i}\dot z = \Omega(\gamma, D)z$ with $\Omega(\gamma, D)$ **real valued** *Fourier multiplier*

Dispersion relation

$$\Omega_j(\gamma) = \sqrt{\Big(g + \frac{\gamma^2}{4}\frac{\tanh(hj)}{j}\Big) j \tanh(hj)} + \frac{\gamma}{2}\tanh(hj), \quad j \in \mathbb{Z}\setminus\{0\}$$

all solutions : $z(t,x) = \sum_{j\in\mathbb{Z}\setminus\{0\}} e^{-\mathrm{i}t\Omega_j(\gamma)} z_j^0 \, e^{\mathrm{i}jx}$

are **periodic, quasi-periodic, almost periodic**

- **Remark:** for $\gamma \neq 0$ then $\Omega_j(\gamma)$ is not even in j $\Longrightarrow$ **no standing waves**

Back in the Wahlen variables (η, ζ), all time reversible linear solutions are

$$\begin{pmatrix} \eta(t,x) \\ \zeta(t,x) \end{pmatrix} = \sum_{n\in\mathbb{N}} \begin{pmatrix} M_n \rho_n \cos(nx - \Omega_n(\gamma)t) \\ M_n^{-1} \rho_n \sin(nx - \Omega_n(\gamma)t) \end{pmatrix} + \sum_{n\in\mathbb{N}} \begin{pmatrix} M_n \rho_{-n} \cos(nx + \Omega_{-n}(\gamma)t) \\ -M_n^{-1} \rho_{-n} \sin(nx + \Omega_{-n}(\gamma)t) \end{pmatrix}$$

for $\rho_n \geq 0$ **amplitudes**

Linear combination of waves traveling to the right or to the left:

- **GOAL: AVOID SUPERPOSITION OF IDENTICAL WAVES TRAVELING IN OPPOSITE DIRECTIONS**

e.g. $\gamma = 0 \Rightarrow \Omega_n(\gamma) = \Omega_{-n}(\gamma)$, $\rho_n = \rho_{-n} \ \Rightarrow \begin{pmatrix} \eta(t,x) \\ \zeta(t,x) \end{pmatrix}$ are even in x (standing waves)

Bifurcate from time quasi-periodic traveling waves of the linear WW system

- Select finitely many "wave vectors"

$$\mathbb{S}^+ := \{\bar{n}_1, \ldots, \bar{n}_\nu\} \subset \mathbb{N} \, , \quad 1 \leq \bar{n}_1 < \ldots < \bar{n}_\nu \, ,$$

- "directions"

$$\Sigma = \{\sigma_1, \ldots, \sigma_\nu\}, \quad \sigma_a \in \{\pm 1\}$$

- amplitudes $\xi_{\bar{n}_1}, \ldots, \xi_{\bar{n}_\nu} > 0$

$$\begin{pmatrix} \eta(t,x) \\ \zeta(t,x) \end{pmatrix} = \sum_{a \in \{1,\ldots,\nu : \, \sigma_a = +1\}} \begin{pmatrix} M_{\bar{n}_a} \sqrt{\xi_{\bar{n}_a}} \cos(\bar{n}_a x - \Omega_{\bar{n}_a}(\gamma) t) \\ M_{\bar{n}_a}^{-1} \sqrt{\xi_{\bar{n}_a}} \sin(\bar{n}_a x - \Omega_{\bar{n}_a}(\gamma) t) \end{pmatrix} + \sum_{a \in \{1,\ldots,\nu : \, \sigma_a = -1\}} \begin{pmatrix} M_{\bar{n}_a} \sqrt{\xi_{-\bar{n}_a}} \cos(\bar{n}_a x + \Omega_{-\bar{n}_a}(\gamma) t) \\ -M_{\bar{n}_a}^{-1} \sqrt{\xi_{-\bar{n}_a}} \sin(\bar{n}_a x + \Omega_{-\bar{n}_a}(\gamma) t) \end{pmatrix}$$

If $\sigma_a = 1$, then pick the wave with wave vector n_a traveling *to the right*
If $\sigma_a = -1$, then pick the wave with wave vector n_a traveling *to the left*

Question: do they persist in the nonlinear water waves?

Major difficulties:

1) Gravity WW with vorticity are fully non-linear PDEs

$$z_t + \mathrm{i}\Omega(D)z = N(z, \bar{z})\,, \quad \Omega(D) \sim |D|^{1/2}$$

$N =$ quadratic nonlinearity with first order derivatives $N(\partial_x z)$: a transport term

- Verify all non-resonance conditions, using the vorticity as a parameter

$$\Omega_j(\gamma) = \omega_j(\gamma) + \frac{\gamma}{2}\tanh(hj), \quad \omega_j(\gamma) := \sqrt{\Big(g + \frac{\gamma^2}{4}\frac{\tanh(hj)}{j}\Big) j \tanh(hj)}$$

- There are linear resonances: for example, if $h = +\infty$, $\bar{\jmath}_1, \bar{\jmath}_2 > 0$,

$$-\Omega_{\bar{\jmath}_1}(\gamma) + \Omega_{\bar{\jmath}_2}(\gamma) + \Omega_{-\bar{\jmath}_1}(\gamma) - \Omega_{-\bar{\jmath}_2}(\gamma) = \gamma\big[-\operatorname{sign}(\bar{\jmath}_1) + \operatorname{sign}(\bar{\jmath}_2)\big] \equiv 0, \ \forall \gamma$$

These resonance can be avoided for traveling waves

Theorem (Berti, Franzoi, Maspero, 2021)

For any choice of finitely many tangential sites $\mathbb{S}^+ \subset \mathbb{N} \setminus \{0\}$ and signs Σ, $\exists \bar{s} > \frac{\nu+1}{2}$, $\varepsilon_0 \in (0,1)$ such that: $\forall \xi_{\sigma_a \bar{n}_a} \in (0, \varepsilon_0^2)$, $\exists$ a Cantor-like set $\mathcal{G}_\xi \subset [\gamma_1, \gamma_2]$ with asymptotically full measure as $\xi \to 0$, i.e. $\lim_{\xi \to 0} |\mathcal{G}_\xi| = \gamma_2 - \gamma_1$, such that, for any $\gamma \in \mathcal{G}_\xi$, the *gravity water waves equations* have a time reversible quasi-periodic traveling wave solution of the form

$$\begin{pmatrix} \eta(t,x) \\ \zeta(t,x) \end{pmatrix} = \sum_{a \in \{1,\dots,\nu:\ \sigma_a = +1\}} \begin{pmatrix} M_{\bar{n}_a} \sqrt{\xi_{\bar{n}_a}} \cos(\bar{n}_a x - \widetilde{\Omega}_{\bar{n}_a} t) \\ M_{\bar{n}_a}^{-1} \sqrt{\xi_{\bar{n}_a}} \sin(\bar{n}_a x - \widetilde{\Omega}_{\bar{n}_a} t) \end{pmatrix}$$

$$+ \sum_{a \in \{1,\dots,\nu:\ \sigma_a = -1\}} \begin{pmatrix} M_{\bar{n}_a} \sqrt{\xi_{-\bar{n}_a}} \cos(\bar{n}_a x + \widetilde{\Omega}_{-\bar{n}_a} t) \\ -M_{\bar{n}_a}^{-1} \sqrt{\xi_{-\bar{n}_a}} \sin(\bar{n}_a x + \widetilde{\Omega}_{-\bar{n}_a} t) \end{pmatrix} + r(t,x)$$

where $r(t,x) = \breve{r}(\widetilde{\Omega}_{\sigma_1 \bar{n}_1} t - \sigma_1 \bar{n}_1 x, \dots, \widetilde{\Omega}_{\sigma_\nu \bar{n}_\nu} t - \sigma_\nu \bar{n}_\nu x)$ is a quasi-periodic traveling wave with

$$\breve{r} \in H^{\bar{s}}(\mathbb{T}^\nu, \mathbb{R}^2), \quad \lim_{\xi \to 0} \frac{\|\breve{r}\|_{\bar{s}}}{\sqrt{|\xi|}} = 0,$$

and with a Diophantine frequency vector $\widetilde{\Omega} := (\widetilde{\Omega}_{\sigma_a \bar{n}_a})_{a=1,\dots,\nu} \in \mathbb{R}^\nu$ satisfying

$$\widetilde{\Omega} \to \vec{\Omega}(\gamma) := (\Omega_{\sigma_a \bar{n}_a}(\gamma))_{a=1,\dots,\nu} \quad \text{as} \quad \xi \to 0.$$

In addition these quasi-periodic solutions are linearly stable.

Remarks:

1. There are no global existence results for WW with vorticity on $\mathbb{T}$.
   the previous theorem selects initial conditions giving rise to smooth solutions defined for all times
2. The restriction $\gamma \in \mathcal{G}_\xi$ is not technical: otherwise there could be "Arnold diffusion", growth of Sobolev norms, chaotic dynamics, ... (Wilkening)
3. This result does not reduce to [*BBHM*] for irrotational fluids ($\gamma = 0$), because we construct traveling waves, whereas [BBHM] construct standing waves, i.e. even in x.
4. In case $\gamma \neq 0$ there are no standing wave solutions, since the WW vector field does not leave invariant the subspace of functions even in x.
5. we obtain quasiperiodic solutions of the Euler equation which are small perturbations of the Couette flow:
$$\vec{u} = \begin{pmatrix} -\gamma y \\ 0 \end{pmatrix} + \nabla \Phi$$
6. For $\nu = 1$ these solutions are time periodic traveling Stokes waves (Crandall-Rabinowitz)

There exist coordinates (around the torus)

$$(\phi, y, v) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times (H^s_x \cap L^2_{\mathbb{S}^c}),$$

in which the quasi-periodic solution $(\eta, \zeta)(\omega t)$ reads $t \mapsto (\omega t, 0, 0)$ and the linearized equation $\partial_t h = J \nabla H((\eta, \zeta)(\omega t))h$ reads

$$\begin{cases} \dot\phi = by \\ \dot y = 0 \\ v_t = \mathrm{i} D_\infty v\,, \quad v = \sum_{j \notin \mathbb{S}} v_j e^{\mathrm{i}jx}\,,\ D_\infty := \mathrm{Op}(\mu_j)\,,\ \mu_j \in \mathbb{R}\,,\ \mathbb{S} := (n_1\sigma_1, \ldots, n_\nu\sigma_\nu) \end{cases}$$

$y(t) = y_0, v_j(t) = v_j(0) e^{\mathrm{i}\mu_j t} \Longrightarrow \|v(t)\|_{H^s_x} = \|v(0)\|_{H^s_x}$: stability

$0, \{\mathrm{i}\mu_j\}_{j \in \mathbb{S}^c}$ **= Floquet exponents**

1. Sharp **asymptotic expansion** of the **Floquet exponents**

$$\mu_j(\gamma) = \mathtt{m}_1 j + \mathtt{m}_{\frac{1}{2}} \Omega_j(\gamma) - m_0 \mathrm{sign}(j) + \mathtt{r}_j$$

where $\mathtt{m}_1, \mathtt{m}_{\frac{1}{2}}, \mathtt{m}_0, r_j \in \mathbb{R}$ are constants satisfying

$$|\mathtt{m}_{\frac{1}{2}} - 1| + |\mathtt{m}_1| + |\mathtt{m}_0| + \sup_{j \in \mathbb{Z}} |\mathtt{r}_j||j|^{\frac{1}{2}} = O(|\xi|^a), \quad a > 0,$$

2. The change of variables $\Phi(\varphi)$ satisfies **tame estimates** in Sobolev spaces:

$$\|\Phi h\|_s, \|\Phi^{-1} h\|_s \leq \|h\|_s + \|u\|_{s+\sigma} \|h\|_{s_0}, \ \forall s \geq s_0$$

MAIN STEPS

- Nash-Moser iteration
- Pseudodifferential normal form
- KAM reducibility

Look for embedded torus $\mathbb{T}^\nu \ni \varphi \mapsto u(\varphi, x) = (\eta(\varphi, x), \zeta(\varphi, x)) \in H^s_0 \times \dot{H}^s$ zero of

$$\mathcal{F}(\omega, \gamma, u) := \begin{pmatrix} \omega \cdot \partial_\varphi \eta - \nabla_\zeta^{L^2} H(\eta, \zeta) \\ \omega \cdot \partial_\varphi \zeta + \nabla_\eta^{L^2} H(\eta, \zeta) \end{pmatrix}, \qquad \partial_t \rightsquigarrow \omega \cdot \partial_\varphi$$

Quasiperiodic traveling solution

$$u(\varphi, x) = U(\varphi - \vec{\jmath}x) \iff \tau_\varsigma u(\varphi, x) = u(\varphi - \varsigma \vec{\jmath}, x) \quad \forall \varsigma \in \mathbb{R}$$

where $\vec{\jmath} := (\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu)$, $\quad \bar{\jmath}_a := \sigma_a \bar{n}_a \in \mathbb{Z} \setminus \{0\}$

Small amplitude solutions

$$\mathcal{F}(\omega, \gamma, 0) = 0,$$

$$D_u \mathcal{F}(\omega, \gamma, 0) = \begin{pmatrix} \omega \cdot \partial_\varphi - \frac{\gamma}{2} G(0) \partial_x^{-1} & -G(0) \\ g - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0) \partial_x^{-1} & \omega \cdot \partial_\varphi - \frac{\gamma}{2} \partial_x^{-1} G(0) \end{pmatrix}$$

$$\simeq \mathrm{diag}_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} \begin{pmatrix} \mathrm{i}(\omega \cdot \ell + \Omega_j(\gamma)) & 0 \\ 0 & \mathrm{i}(\omega \cdot \ell + \Omega_{-j}(\gamma)) \end{pmatrix}$$

Question: Is $D_u\mathcal{F}(\omega,\gamma,0)$ invertible?

Non-resonance condition

$$|\omega\cdot\ell+\Omega_{\pm j}(\gamma)|\geq\frac{\gamma}{\langle\ell\rangle^{\tau}},\quad\forall(\ell,j)\in\mathbb{Z}^{\nu}\times\mathbb{Z}\setminus\{0\}\,,\quad\tau>0$$

for "most" (ω,γ)

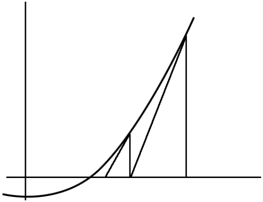
$\Longrightarrow$ $D_u\mathcal{F}(\omega,\gamma,0)$ is invertible, but loses derivatives

$$D_u\mathcal{F}(\omega,\gamma,0)^{-1}\colon H^s\to H^{s-\tau},\qquad\tau:=\text{“loss of derivatives”}$$

Newton tangent method for zeros of $\mathcal{F}(u)+$ smoothing:

$$u_{n+1} := u_n - S_n (D_u \mathcal{F})^{-1}(u_n)[\mathcal{F}(u_n)]$$

where S_n is a regularizing operator



Problem: invert $D_u\mathcal{F}(u)$ in H^s with tame estimates

$$\|(D_u\mathcal{F})(u)^{-1} h\|_s \leq \|h\|_{s+\sigma} + \|u\|_{s+\sigma}\|h\|_{s_0}, \quad \forall s \geq s_0$$

$$(D_u\mathcal{F})(u) = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} + G(\eta)B & -G(\eta) \\ g + B\widetilde{V}_x + BG(\eta)B & \widetilde{V}\partial_x - BG(\eta) \end{pmatrix}$$
$$+ \frac{\gamma}{2} \begin{pmatrix} -G(\eta)\partial_x^{-1} & 0 \\ \partial_x^{-1} G(\eta) B - BG(\eta)\partial_x^{-1} - \frac{\gamma}{2}\partial_x^{-1} G(\eta)\partial_x^{-1} & -\partial_x^{-1} G(\eta) \end{pmatrix}$$

$$(V, B) := (\Phi_x, \Phi_y)|_{y=\eta(\varphi,x)}, \qquad \widetilde{V} := V - \gamma\eta,$$

Diagonalization of $D_u\mathcal{F}(u)$

- **Step 0:** Symplectic decoupling of "action-angle" (tangential dynamics) and "normal" variables to the torus
- **Step 1:** Reduction in order: pseudodifferential normal form

$$D_u\mathcal{F}(u) \sim \omega \cdot \partial_\varphi + \underbrace{A(D)}_{\text{diag. Fourier multiplier, constant in } \varphi} + \underbrace{\varepsilon R(\varphi)}_{\text{small bounded}}$$

- **Step 2:** Reduction in size of $\varepsilon R(\varphi)$: KAM reducibility

Always: "preserve momentum"

$u(\varphi, x)$ traveling wave $+\,\mathcal{F}$ transl. invariant $\Longrightarrow$ $\tau_\varsigma \circ L(\varphi) = L(\varphi - \vec{\jmath}\varsigma) \circ \tau_\varsigma, \ \forall \varsigma \in \mathbb{R}$

The linearized operator looks after the linear "Alinhac good-unknown" transformation

Water waves linearized Hamiltonian operator

$$\omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} & 0 \\ 0 & \widetilde{V}\partial_x \end{pmatrix} + \begin{pmatrix} -\frac{\gamma}{2} G(0)\partial_x^{-1} & -G(0) \\ a - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0)\partial_x^{-1} & -\frac{\gamma}{2}\partial_x^{-1} G(0) \end{pmatrix} + \ldots$$

terms in red are singular perturbations

We conjugate the Hamiltonian transport operator

$$\mathcal{L} := \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V & 0 \\ 0 & V\partial_x \end{pmatrix}$$

via the symplectic map

$$\mathcal{E} := \begin{pmatrix} (1+\beta_x(\varphi,x)) \circ \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{pmatrix},$$

where

$$(\mathcal{B}u)(\varphi,x) := u\left(\varphi, x + \beta(\varphi,x)\right)$$

Proposition:

There exist a constant

$$\mathtt{m}_1 := \mathtt{m}_1(\omega,\gamma) \in \mathbb{R}$$

defined for any $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$, *and a quasi-periodic traveling wave* $\beta(\varphi,x)$ *such that, for any* (ω,γ) *satisfying*

$$|(\omega - \mathtt{m}_1 \vec{\jmath}) \cdot \ell| \geq \upsilon \langle \ell \rangle^{-\tau} , \ \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}$$

we have

$$\mathcal{E}^{-1}\Big(\omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V & 0 \\ 0 & V\partial_x \end{pmatrix}\Big)\mathcal{E} = \omega \cdot \partial_\varphi + \begin{pmatrix} \mathtt{m}_1 \, \partial_x & 0 \\ 0 & \mathtt{m}_1 \, \partial_x \end{pmatrix}$$

Remark: for the standing waves [Baldi,Berti,Haus, Montalto] it results $\mathtt{m}_1 = 0$

Reduction of lower orders: find map Φ such that $\mathcal{L} := \Phi^{-1} D_u \mathcal{F}(u) \Phi$ equals

$$\omega \cdot \partial_\varphi + \mathtt{m}_1 \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} + \mathtt{m}_{\frac{1}{2}} \begin{pmatrix} \mathrm{i}\Omega(\gamma, D) & 0 \\ 0 & -\mathrm{i}\overline{\Omega(\gamma, D)} \end{pmatrix} + \mathtt{m}_0 \begin{pmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{H} \end{pmatrix} + R_{-\frac{1}{2}}$$

- $R_{-\frac{1}{2}}(\varphi, x) \in OPS^{-1/2}$
- $\mathtt{m}_1, \mathtt{m}_{\frac{1}{2}}, \mathtt{m}_0 \in \mathbb{R}$, constants

Perform the reduction

- preserving the Hamiltonian structure, otherwise unstable operators compatible with reversibility
  1. Symplectic Egorov
  2. Symplectic pseudodiff transformation
  3. Preserve "momentum"

Reduction in SIZE: $\forall n$ find map Φ_n such that

$$\mathcal{L}_n := \Phi_n^{-1} \mathcal{L} \Phi_n = \omega \cdot \partial_\varphi + \mathrm{i} \mathcal{D}_n + \mathcal{R}_n$$

- KAM quadratic scheme + loss of derivatives: $\mathcal{R}_n = O(\varepsilon^{(\frac{3}{2})^n})$
- $\mathcal{D}_n = \begin{pmatrix} \mathrm{diag}_{j \in \mathbb{S}^c}(\mu_j^{(n)}) & 0 \\ 0 & -\mathrm{diag}_{-j \in \mathbb{S}^c}(\mu_{-j}^{(n)}) \end{pmatrix}$

  Asymptotic expansion:

$$\mu_j^{(n)} := \mathtt{m}_1 j + \mathtt{m}_{\frac{1}{2}} \Omega_j(\gamma) + \mathtt{m}_0 \mathrm{sign}(j) + \mathtt{r}_j^{(n)}(\gamma, \omega)$$

with $\mathtt{m}_1, \mathtt{m}_{\frac{1}{2}}, \mathtt{m}_0, r_j^{(n)} \in \mathbb{R}$ and

$$|\mathtt{m}_1| + \left| \mathtt{m}_{\frac{1}{2}} - 1 \right| + |\mathtt{m}_0| + \sup_{j \in \mathbb{Z}} |j|^{\frac{1}{2}} |\mathtt{r}_j^{(n)}| = O(|\xi|^a), \quad a > 0$$

Preserve the momentum

$$\tau_\varsigma \circ \mathcal{L}_n(\varphi) = \mathcal{L}_n(\varphi - \vec{\jmath}\varsigma) \circ \tau_\varsigma, \qquad \forall \varsigma \in \mathbb{R}.$$

Second order Melnikov conditions with momentum

$$\begin{cases} \left|\omega\cdot\ell+\mu_j+\mu_{-j'}\right|\geq\frac{\gamma}{\langle\ell\rangle^\tau}, & \forall\ell\in\mathbb{Z}^n,\ \ \forall j,-j'\in\mathbb{S}_0^c,\ \ \vec{\jmath}\cdot\ell+j-j'=0\,,\ \vec{\jmath}:=(j_1,\ldots,j_\nu) \\ \left|\omega\cdot\ell+\mu_j-\mu_{j'}\right|\geq\frac{\gamma}{\langle\ell\rangle^\tau}, & \forall\ell\in\mathbb{Z}^n,\ \ \forall j,j'\in\mathbb{S}_0^c,\ \ (\ell,j,j')\neq(0,j,j'),\ \ \vec{\jmath}\cdot\ell+j-j'=0 \end{cases}$$

Question: how to verify? Develop "Degenerate KAM Theory with momentum"

Since $\omega\sim\vec{\Omega}(\gamma):=(\Omega_{j_1}(\gamma),\ldots,\Omega_{j_\nu}(\gamma))$ and $\mu_j\sim\Omega_j(\gamma)$

Lemma (Transversality)

There exist $m_0\in\mathbb{N}$ *and* $\rho_0>0$ *such that, for any* $\gamma\in[\gamma_1,\gamma_2]$,

$$\max_{0\leq n\leq m_0}\left|\partial_\gamma^n\left(\vec{\Omega}(\gamma)\cdot\ell+\Omega_j(\gamma)-\Omega_{j'}(\gamma)\right)\right|\geq\rho_0\langle\ell\rangle\,,\qquad \text{for } \vec{\jmath}\cdot\ell+j-j'=0\quad(\ell,j,j')\neq(0,j,j)$$

Van der Monde determinant + analyticity + asymptotics of $\vec{\Omega}(\gamma)$

Second order Melnikov conditions

$$\vec{\Omega}(\gamma) \cdot \ell + \Omega_j(\gamma) - \Omega_{j'}(\gamma) \equiv 0\,, \quad j, j' \notin \mathbb{S}\,, \; j \neq j'\,,$$

$$\vec{\Omega}(\gamma) = (\Omega_{\bar{\jmath}_1}(\gamma), \ldots, \Omega_{\bar{\jmath}_\nu}(\gamma)), \quad \Omega_j(\gamma) = \omega_j(\gamma) + \tfrac{\gamma}{2}\mathrm{sign}(j) \quad (\text{for } h = +\infty),\; j \mapsto \omega_j(\gamma) \text{ is even in } j$$

Resonances: $\ell = (\ell_1, \ell_2, 0, \ldots, 0) = (-1, 1, 0, \ldots, 0)$, $j = -\bar{\jmath}_1$, $j' = -\bar{\jmath}_2$, $\bar{\jmath}_1 \neq \bar{\jmath}_2$, $\mathrm{sign}(\bar{\jmath}_1) = \mathrm{sign}(\bar{\jmath}_2)$

$$\begin{aligned}
&\vec{\Omega}(\gamma) \cdot \ell + \Omega_j(\gamma) - \Omega_{j'}(\gamma) = \\
&\Omega_{\bar{\jmath}_1}(\gamma)\ell_1 + \Omega_{\bar{\jmath}_2}(\gamma)\ell_2 + \Omega_{-\bar{\jmath}_1}(\gamma) - \Omega_{-\bar{\jmath}_2}(\gamma) = \\
&\omega_{\bar{\jmath}_1}(\gamma)(\ell_1 + 1) + \omega_{\bar{\jmath}_2}(\gamma)(\ell_2 - 1) + \frac{\gamma}{2}\big[(\ell_1 - 1)\mathrm{sign}(\bar{\jmath}_1)) + (\ell_2 + 1)\mathrm{sign}(\bar{\jmath}_2)) = \\
&\gamma\big[- \mathrm{sign}(\bar{\jmath}_1) + \mathrm{sign}(\bar{\jmath}_2)) \equiv 0, \qquad \forall \gamma
\end{aligned}$$

But there is the momentum restriction on the indices

$$\vec{\jmath} \cdot \ell + j - j' = \bar{\jmath}_1 \ell_1 + \bar{\jmath}_2 \ell_2 - \bar{\jmath}_1 + \bar{\jmath}_2 = -2\bar{\jmath}_1 + 2\bar{\jmath}_2 = 0 \quad \Rightarrow \quad \bar{\jmath}_1 = \bar{\jmath}_2$$

contradiction with $\bar{\jmath}_1 \neq \bar{\jmath}_2$

In the limit we have diagonalized the linearized operator $D_u \mathcal{F}(u)_{|\mathcal{H}_{\mathbb{S}^c}}$: there is $\Phi : H^s \to H^s$ such that

Diagonal operator

$$\Phi^{-1} D_u \mathcal{F}(u) \Phi = \mathrm{diag}_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{S}^c} \begin{pmatrix} \mathrm{i}(\omega \cdot \ell + \mu_j^\infty) & 0 \\ 0 & \mathrm{i}(\omega \cdot \ell - \mu_{-j}^\infty) \end{pmatrix}$$

$$\mu_j^\infty = \mathtt{m}_1(\gamma) j + \mathtt{m}_{\frac{1}{2}} \Omega_j(\gamma) + \mathtt{m}_0 \mathrm{sign}(j) + \mathtt{r}_j^\infty(\gamma)$$

It is invertible, with an inverse which satisfies tame estimates, with loss of derivatives, provided

Non-resonance condition

$$|\omega \cdot \ell \pm \mu_j^\infty(\gamma)| \geq \frac{\gamma}{\langle \ell \rangle^\tau}, \quad \forall (\ell, j) \in \mathbb{Z}^\nu \times \mathbb{S}^c, \qquad \tau > 0$$

2D-Euler equations on $\mathbb{R}^2$: vorticity formulation

$$\partial_t \omega + \vec{u} \cdot \nabla \omega = 0$$

$$\vec{u} := \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\psi_y \\ \psi_x \end{pmatrix}, \qquad \Delta \psi = \psi_{xx} + \psi_{yy} = v_x - u_y = \omega$$

velocity field , $\mathrm{div} \vec{u} = 0$, $\psi =$ stream function, vorticity

vortex patches:

$$\omega(t)_{|t=0} = \chi_{D(0)} \quad \Longrightarrow \quad \omega(t) = \chi_{D(t)}$$

domain $D(t) = \phi^t_\psi(D(0))$,

ϕ^t_ψ flow in $\mathbb{R}^2$ generated by the Hamiltonian vector field $\vec{u}$

Identify $\mathbb{C} \equiv \mathbb{R}^2$ via $z = x + iy \in \mathbb{C}$

Contour dynamics equation

A parametrization $\theta \mapsto z(t,\theta)$ of the boundary $\partial D(t)$ evolves according to

$$\big(\partial_t z(t,\theta) - \vec{u}(t, z(t,\theta))\big) \cdot \vec{n}(t) = 0$$

where $\vec{n}(t)$ is the outer normal to $\partial D(t)$

Bifurcation of uniformly rotating time periodic vortex patches, stationary in a rotating frame:

Hmidi '15, Castro-Cordoba-Gomez Serrano '16, Hmidi-Hassaina '15,

Global Bifurcation of Rotating Vortex Patches, Hassaina, Masmoudi, Wheeler '19

Not stationary in any rotating frame

Kirkhoff ellipses

$$D_\gamma := \left\{ (x,y) \in \mathbb{R}^2 \, : \, \frac{x^2}{\gamma} + \gamma y^2 = 1 \right\}, \ \gamma > 1 \,,$$

rotate with an angular velocity $\Omega_\gamma = \frac{\gamma}{(1+\gamma)^2}$

Equation for the radial deformation in rotating frame with angular velocity Ω

$$z(t,\theta) = e^{i\Omega t}(1 + 2\xi(t,\theta))^{\frac{1}{2}} (\gamma^{\frac{1}{2}} \cos(\theta) + i\gamma^{-\frac{1}{2}} \sin(\theta))$$

assume the Hamiltonian form

$$\partial_t \xi = \partial_\theta \nabla H(\xi) \,,$$

where ∇H denotes the L^2-gradient of a suitable Hamiltonian energy

For $\Omega = \Omega_\gamma$ there is an equilibrium $\nabla H(0) = 0$, for any γ

$$\partial_t \xi = \partial_\theta A \xi \,, \quad A := (D^2 H)(0)\,, \quad \Omega_\gamma = \frac{\gamma}{(1+\gamma)^2}$$

Normal mode coordinates: $\xi(\theta) = \sum_{n \in \mathbb{N}} M_n \alpha_n \cos(n\theta) - M_n^{-1} \beta_n \sin(n\theta)$

$$\frac{1}{2}((D^2 H)(0)\xi, \xi) = -\frac{\Omega_1}{2}(\alpha_1^2 + \beta_1^2) + \frac{\Omega_2}{2}\alpha_2^2 + \frac{1}{2}\sum_{n \geq 3} \Omega_n (\alpha_n^2 + \beta_n^2)\,, \quad 1 < \gamma < 3\,,$$

$$\Omega_1(\gamma) = \Omega_\gamma = \frac{\gamma}{(1+\gamma)^2}\,, \quad \Omega_n(\gamma) = \sqrt{\Big(\frac{n\gamma}{(1+\gamma)^2} - \frac{1}{2}\Big)^2 - \frac{1}{4}\Big(\frac{\gamma-1}{\gamma+1}\Big)^{2n}}\,, \ \forall n \geq 3\,,$$

Mode 2 is *degenerate*. Angular momentum is a prime integral with a linear term

$$J = \alpha_2 + Q_2(\alpha, \beta)\,, \quad Q_2 = O(|\alpha| + |\beta|)^2$$

Remark: for $\gamma > 3$ appears hyperbolic directions

Expected Theorem: Berti, Hassaina, Masmoudi, '21 in progress

Consider an interval of eccentricities $[\gamma_1, \gamma_2] \subset [1, 3]$. Let $\mathbb{S}$ be any finite subset of integers in $\{3, 4, 5, \ldots\}$ and fix a vector $(a_n)_{n \in \mathbb{S}}$ with $a_n > 0$, for all $n \in \mathbb{S}$. Then there exist $\bar{s} > (|\mathbb{S}| + 1)/2$, $\varepsilon_0 \in (0, 1)$ so that, for any $\varepsilon \in (0, \varepsilon_0)$, the following holds

1. there exists a subset $\mathcal{G}_\varepsilon \subset [\gamma_1, \gamma_2]$ with asymptotically full measure, i.e.

$$\lim_{\varepsilon \to 0} |[\gamma_1, \gamma_2] \setminus \mathcal{G}_\varepsilon| = 0\,;$$

2. there exists a diophantine frequency vector $\vec{\omega}_\varepsilon(\gamma) = (\omega_{\varepsilon,n}(\gamma))_{n \in \mathbb{S}}$ with $\vec{\omega}_\varepsilon(\gamma) \to (\Omega_n(\gamma))_{n \in \mathbb{S}}$ as $\varepsilon \to 0$;
3. for any $\gamma \in \mathcal{G}_\varepsilon$, there is a quasi-periodic vortex patch $\tilde{z}(\vec{\omega}_\varepsilon(\gamma)t, \theta)$ of the form

$$\tilde{z}(\varphi, \theta) = \varepsilon \sum_{n \in \mathbb{S}} \Lambda_n^2 a_n \cos(\varphi_n) \cos(n\theta) - \Lambda_n^{-2} a_n \sin(\varphi_n) \sin(n\theta) + r_\varepsilon(\varphi, \theta)$$

   with a function $r_\varepsilon(\varphi, \theta) \in H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$, $\|r_\varepsilon\|_{\bar{s}} = o(\varepsilon)$ as $\varepsilon \to 0$.

For γ in any interval $[\gamma_1, \gamma_2] \subset \mathbb{R}$ it is expected a similar result.

1. Transversality/Non-degeneracy of frequencies $(\Omega_n(\gamma))_{n\geq 3}$
2. Serious difficulty: degeneracy of mode 2

Linearized operator on the normal components

$$\omega \cdot \partial_\varphi + \partial_\theta \circ V + \partial_\theta A + R_\varepsilon$$

where $\partial_\theta A = \mathcal{H} + \ldots$, and $R_\varepsilon \in OPS^{-\infty}$.
Conjugate it via a symplectic transformation to

$$\omega \cdot \partial_\varphi + m_1 \partial_\theta + \partial_\theta A + R'_\varepsilon$$

$+$ KAM diagonalization to prove invertibility.

Thanks for your attention!