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Displacement convexity inequalities in MFG

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The measure-potential framework

Many models for large populations (Chemotaxis, mean-field games, Hughes model,...) fit the following framework:

- A probability density m gives the population distribution;
- A "potential", "pressure", or "value function" u that encodes population effects on the environment;
- A PDE for u that depends on m (typically, nonlinear elliptic or parabolic equation)
- An evolution PDE for m driven by the potential u.

Mean-field games

- Mean-field games (MFGs) model systems with a large number of rational agents who seek to minimize a cost functional that depends on statistical or aggregated quantities.
- These models were introduced in the engineering community by Caines, Huang and Malhamé and in the mathematical community by Lasry and Lions.

Mean-field models

A canonical MFG comprises:

- a Hamilton-Jacobi (HJ) equation
- a transport or Fokker-Planck (FP) equation
- The HJ and the FP equations are fully coupled and the FP equation is the adjoint of the linearization of the HJ equation.

The PDEs

The workhorse of MFG theory is the system:

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(x) = g(m), \\ m_t - \operatorname{div}(mDu) = 0 \end{cases}$$

with initial and terminal conditions

$$\begin{cases} u(x,T) = u_T(x) \\ m(x,0) = m_0(x). \end{cases}$$

Here, m_0 and u_T are given, $m_0 \geq 0$ with $\int_{\mathbb{R}^d} m_0 dx = 1$.
Often, we take the domain of u and m to be $\mathbb{T}^d \times [0,T]$.

Optimal control and Hamilton-Jacobi equations

- We fix $T > 0$ and consider an agent whose state is $\mathbf{x}(t) \in \mathbb{R}^d$ for $0 \leq t \leq T$.
- Agents can change their state by choosing a control in $\mathbf{v} \in \mathcal{W} = L^\infty([t, T], \mathbb{R}^d)$.
- The state of an agent evolves according to

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t).$$

- We fix a Lagrangian $\tilde{L} : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$, with $v \mapsto L(x, v, t)$ uniformly convex.
- Agents have preferences encoded by the functional,

$$J(\mathbf{v}; x, t) = \int_t^T \tilde{L}(\mathbf{x}(s), \mathbf{v}(s), s)ds + u_T(\mathbf{x}(T)),$$

where $\dot{\mathbf{x}} = \mathbf{v}$ with $\mathbf{x}(t) = x$.

- Each agent seeks to minimize J in $\mathcal{W}$. The value function is

$$u(x, t) = \inf_{\mathbf{v} \in \mathcal{W}} J(\mathbf{v}; x, t).$$

The Legendre transform, $\tilde{H}$, of $\tilde{L}$ is the Hamiltonian

$$\tilde{H}(x,p,t) = \sup_{v \in \mathbb{R}^d} \left[-p \cdot v - \tilde{L}(x,v,t) \right].$$

By the uniform convexity of $\tilde{L}$ in the second coordinate, the maximum is achieved at a unique point, v^* given by

$$v^* = -D_p \tilde{H}(x,p,t).$$

Theorem (Verification Theorem)

- *Let* $\tilde{u} \in C^1(\mathbb{R}^d \times [t_0, T])$ *solve the Hamilton–Jacobi equation with the terminal condition* $u_T(x)$.
- *Let*

$$\mathbf{v}^*(t) = -D_p\tilde{H}(\mathbf{x}^*(t), D_x\tilde{u}(\mathbf{x}^*(t), t), t)$$

  and $\mathbf{x}^*(t)$ *be the corresponding trajectory.*

Then,

- $\mathbf{v}^*(t)$ *is an optimal control*
- $\tilde{u}(x, t)$ *is the value function,* u.

Transport equation

Let $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be a Lipschitz vector field. The ODE

$$\begin{cases} \dot{\mathbf{x}}(t) = b(\mathbf{x}(t), t) & t > 0, \\ \mathbf{x}(0) = x \end{cases}$$

induces a flow, Φ^t, in $\mathbb{R}^d$ that maps the initial condition, $x \in \mathbb{R}^d$, at $t = 0$ to the solution at time $t > 0$.

Fix a probability measure, $m_0 \in \mathcal{P}(\mathbb{R}^d)$. For $0 \leq t \leq T$, let $m(\cdot, t)$ be the push-forward, $\Phi^t \sharp m_0$, by Φ^t of m_0 given by

$$\int_{\mathbb{R}^d} \phi(x) m(x,t) dx = \int_{\mathbb{R}^d} \phi\left(\Phi^t(x)\right) m_0 dx.$$

For $0 \leq t \leq T$, $m(\cdot, t)$ is a probability measure.

Proposition

Assume that $b(x,t)$ is Lipschitz continuous in x. Let Φ^t be the corresponding flow and $m = \Phi^t \sharp m_0$. Then, $m \in C(\mathbb{R}_0^+, \mathcal{P}(\mathbb{R}^d))$ and

$$\begin{cases} m_t(x,t) + \operatorname{div}(b(x,t)m(x,t)) = 0, & (x,t) \in \mathbb{R}^d \times [0,T], \\ m(x,0) = m_0(x), & x \in \mathbb{R}^d, \end{cases}$$

in the distributional sense.

Mean-field models I

- The mean-field game framework studies systems with infinitely many competing rational agents.
- Each agent seeks to optimize an individual control problem that depends on statistical information about the whole population.
- The only information available to the agents is the probability distribution of the agents' states.

- For each time t, $m(x,t)$ is a probability density in $\mathbb{R}^d$ that gives the distribution of the agents
- We set

$$\tilde{L}(x,v,t) = L(x,v,m(\cdot,t)).$$

  and denote the Legendre transform of L by H.
- Each agent seeks to minimize a control problem whose value function solves

$$-u_t + H(x, D_x u, m) = 0.$$

By the Verification Theorem, if u is a solution, $b = -D_p H(x, D_x u(x,t), m)$, is the optimal strategy. Because all agents are rational, they use this strategy.

Hence, u and m are determined by

$$\begin{cases} -u_t + H(x, D_x u, m) = 0 \\ m_t - \operatorname{div}(D_p H m) = 0. \end{cases}$$

We supplement this system with terminal data for the value function $u_T : \mathbb{R}^d \to \mathbb{R}$ and the initial distribution of agents $m_0 : \mathbb{R}^d \to \mathbb{R}_0^+$.

Example

Consider the Hamiltonian

$$H(p, x, m) = \frac{p^2}{2} + V(x) - g(m)$$

- The first term corresponds to the moving cost $\frac{v^2}{2}$ in the Lagrangian;
- V encodes the spatial preferences of the agents (agents prefer large values of V)
- typically, g is increasing and reflects crowd aversion of the agents.

Optimal transport problem

Given two probability measures $m_0, m_1 \in \mathcal{P}(\mathbb{R}^d)$, and we seek to transport m_0 into m_1 while minimizing a transport cost.
More precisely, we seek a map $T : \mathbb{R}^d \to \mathbb{R}$ such that $T^\sharp m_0 = m_1$ and minimizes

$$\int_{\mathbb{R}^d} |T(x) - x|^2 d\nu(x).$$

This problem has a long story: Kantorowich, McCann, Brenier, Villani ....

Benamou-Brenier formulation

The Benamou-Brenier formulation of optimal transport consists of minimizing

$$\int_{\mathbb{R}^d} \int_0^1 m(x,t)|v(x,t)|^2 dxdt,$$

over all smooth velocity fields $v(x,t)$, with trajectories $T^t(x)$, and densities $m(x,t) = T^t_\# m_0$, such that $m(x,0) = m_0$ and $m(x,1) = m_1$.

The optimality conditions of this variational problem are

$$\begin{cases} -u_t + \frac{|Du|^2}{2} = 0 \\ m_t - \operatorname{div}(mDu) = 0 \\ m(x,t) \in \mathcal{P}_{ac}(\mathbb{R}^d) \quad \forall\, t \in [0,1] \\ m(x,0) = m_0, m(x,1) = m_1. \end{cases}$$

The optimal velocity field is $v(x,t) = -Du(x,t)$.

Planning problem for MFGs

The Benamou-Brenier formulation of optimal transport is a particular case of the planning problem for MFGs

$$\begin{cases} -u_t + \frac{|Du|^2}{2} = g(m) \\ m_t - \operatorname{div}(mDu) = 0 \\ m(x,t) \in \mathcal{P}_{ac}(\mathbb{R}^d) \quad \forall\, t \in [0,1] \\ m(x,0) = \mu, m(x,1) = \nu. \end{cases}$$

Previous work

- The planning problem was introduced by P. L. Lions
- Using variational methods, several authors studied the planning problem: Achdou, Camilli, Dolcetta; Graber, Mészáros, Silva and Tonon;
- The parabolic case was studied by Porretta;
- Using different methods, Lavenant and Santambrogio established related estimates to the ones we will discuss here;
- Following the ideas of Lions, S. Muñoz established the existence of solutions when $V = 0$.

A key question

Consider the planning problem

$$\begin{cases} -u_t + \frac{u_x^2}{2} + V(t,x) = m \\ m_t - (u_x m)_x = 0 \qquad (t,x) \in [0,T] \times \mathbb{T} \\ m(0,x) = m_0, \quad m(T,x) = m_T. \end{cases}$$

Is $m > 0$ if $m_0 > 0$ and $m_T > 0$?

Relevance of lower bounds

When $V = 0$, we have $m = -u_t + \frac{u_x^2}{2}$, using this in the second equation

$$-u_{tt} - u_x u_{xt} + u_x u_{xt} - u_x^2 u_{xx} - m u_{xx} = 0;$$

that is,

$$-u_{tt} - (m + u_x^2) u_{xx} = 0,$$

Thus, if $m > 0$ the previous equation is uniformly elliptic.

What do we expect?

- Yes, if $V = 0$ (Lions + details in S. Muñoz paper)
- Maybe if V has small oscillation
- No, if V has large oscillation

High oscillation of V means that there exist regions that are undesirable, and hence we expect that m can vanish.

A stationary example

The following stationary example illustrates the role of the oscillation of V for this question:

$$\begin{cases} \frac{|Du|^2}{2} + V(x) = m + \overline{H} \\ -\operatorname{div}(mDu) = 0. \end{cases}$$

Let $(u, m, \overline{H})$ be solution with $m > 0$ and $\int_{\mathbb{T}^d} m\, dx = 1$.
Multiplying the second equation by u and integrating, we have

$$\int_{\mathbb{T}^d} m|Du|^2\, dx = 0.$$

Hence, because m does not vanish, u is constant.

Accordingly,

$$m = -\overline{H} + V(x).$$

Using $\int_{\mathbb{T}^d} m\,dx = 1$ and assuming without loss of generality

$$\int_{\mathbb{T}^d} V dx = 0,$$

we obtain

$$m = 1 + V(x).$$

A time-dependent example

Let

$$m(t,x) = 1 + \sin 2\pi x \sin 2\pi t.$$

m is a probability density that vanishes at $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{3}{4}, \frac{1}{4})$.
Replacing m into the second equation gives

$$u(t,x) = -\frac{1}{2\pi} \cot(2\pi t) \log[1 + \sin(2\pi t)\sin(2\pi x)].$$

Finally, we set

$$V(t,x) = m + u_t - \frac{u_x^2}{2}.$$

Displacement convexity

- The *displacement interpolant* between μ and ν is the minimizer of the Benamou-Brenier problem.
- A functional, $\mathcal{F} : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, is *displacement convex* if $t \mapsto \mathcal{F}(\rho^t)$ is convex for all displacement interpolants ρ^t.
- Formally, we can differentiate twice $\mathcal{F}(\rho^t)$ to study displacement convexity.

McCann introduced displacement convexity to study a gas model with a density $\rho \in P_{ac}(\mathbb{R}^d)$.

- Particles have an interaction potential

$$\mathcal{W}(\rho) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) d\rho(x) d\rho(y),$$

  and an *internal energy*

$$\mathcal{U}(\rho) = \int_{\mathbb{R}^d} U(\rho(x)) dx.$$

- The configuration of the gas minimizes

$$E(\rho) = \mathcal{U}(\rho) + \mathcal{W}(\rho).$$

Our goal is to identify functions $U : \mathbb{R}_0^+ \to \mathbb{R}$ such that *internal energy*

$$t \mapsto \int U(m(x,t))dx \quad \text{is convex,}$$

when $m(x,t)$ solves the first-order MFG.

For optimal transport, the *internal energy* is displacement convex if the *McCann condition* below holds:

$$\begin{cases} P(z) = U'(z)z - U(z), \\ P \in C^1(\mathbb{R}_0^+), \ \ P(z) \geqslant 0, \\ \frac{P(z)}{z^{1-\frac{1}{d}}} \quad \text{non-decreasing.} \end{cases}$$

For example, if $d = 1$, and $U(z) = z^q$,

$$P(z) = (q-1)z^q$$

So McCann condition holds if $q < 0$ or $q > 1$.

The convexity of the *internal energy* gives

$$\int U(m(x,t))dx \leqslant \frac{t}{T}\int U(m(x,T))dx + \left(1-\frac{t}{T}\right)\int U(m(x,0))dx.$$

Hence, if U is bounded by below $\left|\int U(m(x,t))dx\right|$ is bounded by the initial and terminal data.

Displacement convexity without potential

Theorem

Let $m, u \in C^\infty(\mathbb{T}^d \times [0, T])$, $m \geqslant 0$, *solve*

$$\begin{cases} -u_t + H(Du) = g(m) \\ m_t - \operatorname{div}(mD_pH(Du)) = 0 \end{cases}$$

with $g : \mathbb{R}_0^+ \to \mathbb{R}$, $H : \mathbb{R}^d \to \mathbb{R}$ *smooth,* g *non-decreasing, and* H *convex. If* $U : \mathbb{R}_0^+ \to \mathbb{R}$ *is such that the* McCann condition *holds, then*

$$t \mapsto \int_{\mathbb{T}^d} U(m(x,t))dx \quad \text{is convex.}$$

Proof of displacement convexity for first order MFGs

We have

$$\frac{d}{dt}\int U(m) = \ldots = \int P(m)\,\mathrm{div}(D_pH),$$

and

$$\frac{d^2}{dt^2}\int U(m) = \int \overbrace{P'(m)m\,\mathrm{div}(D_pH)^2}^{A} + \overbrace{P'(m)DmD_pH\,\mathrm{div}(D_pH)}^{B}$$
$$+ \overbrace{P(m)\,\mathrm{div}(D_{pp}^2HD(H))}^{C} - \overbrace{P(m)\,\mathrm{div}(g'(m)D_{pp}^2HDm)}^{D}.$$

The key point is in estimating the terms A-D and establishing their positivity.

Proof of displacement convexity for first-order MFGs

In particular, we get

$$\frac{d^2}{dt^2}\int U(m) \geqslant \int \left(P'(m)m - P(m) + \frac{1}{d}P(m)\right)\operatorname{div}(D_pH)^2 + P'(m)g'(m)DmD^2_{pp}HDm \geqslant 0.$$

Application to congestion models

MFGs with congestion model correspond to the system

$$\begin{cases} -u_t + m^\alpha H\left(\frac{Du}{m^\alpha}\right) = g(m) \\ m_t - \operatorname{div}\left(m D_p H\left(\frac{Du}{m^\alpha}\right)\right) = 0 \qquad\qquad \forall\ (x,t) \in \mathbb{T}^d \times (0,T) \\ m(\cdot,0) = m^0(\cdot),\ m(\cdot,T) = m^T(\cdot) \end{cases}$$

for $\alpha > 0$.
For $H(p) = \frac{|p|^\beta}{\beta}$, $t \mapsto \int_{\mathbb{T}^d} m(x,t)^p dx$ is convex for p depending on α and β. As an application, we obtain L^∞ bounds for the density.

L^q estimates

Proposition

Let $u, m \in C^\infty(\mathbb{T}^d \times [0,T])$ solve the first order MFG with g, H smooth, g non-decreasing, and H convex. Then, for all $1 \leqslant q \leqslant \infty$,

$$\|m(\cdot,t)\|_{L^q(\mathbb{T}^d)} \leqslant \|m^0(\cdot)\|_{L^q(\mathbb{T}^d)}^{1-\frac{t}{T}} \|m^T(\cdot)\|_{L^q(\mathbb{T}^d)}^{\frac{t}{T}}, \quad \forall\, t \in [0,T].$$

Proof

If f is smooth and positive, then $\ln f$ is convex if and only if

$$(\ln f)'' = \left(\frac{f'}{f}\right)' = \frac{f''f - (f')^2}{f^2} \geqslant 0;$$

that is,

$$f''f \geqslant (f')^2.$$

Proof

First, we consider the case $1 \leqslant q < \infty$. Then,

$$\frac{d^2}{dt^2}\int m(x,t)^q \geqslant \ldots \geqslant (q-1)^2 \int m^q \operatorname{div}(D_pH)^2.$$

Thus,

$$\begin{aligned}\left(\frac{d}{dt}\int m^q\right)^2 &= \left((q-1)\int m^q \operatorname{div}(D_pH)\right)^2 \\ &\leqslant \left(\int m^q\right)\left(\frac{d^2}{dt^2}\int m^q\right).\end{aligned}$$

Thus, ln $\left(\int m^q\right)$ is convex.

Proof

Therefore,

$$\ln\left(\int m(x,t)^q\right) \leqslant \left(1-\frac{t}{T}\right)\ln\left(\int m^0(x)^q\right) + \frac{t}{T}\ln\left(\int m^T(x)^q\right)$$
$$= \ln\left(\left(\int m^0(x)^q\right)^{1-\frac{t}{T}}\left(\int m^T(x)^q\right)^{\frac{t}{T}}\right).$$

Therefore,

$$\int m(x,t)^q \leqslant \left(\int m^0(x)^q\right)^{1-\frac{t}{T}}\left(\int m^T(x)^q\right)^{\frac{t}{T}}.$$

Exponentiating the previous inequality to $\frac{1}{q}$ to obtain the result.

Proof

For $q = \infty$, we can pass to the limit as $q \to \infty$ to derive the estimate for the supremum.

Finally, we address the one-dimensional case, $d = 1$. A direct computation shows that the convexity of U implies the convexity of $t \mapsto \int_0^1 U(m(x,t))dx$.

Accordingly, convexity holds for functions of the form $U(z) = (z+\varepsilon)^{-q}$, $q \geqslant 0, \varepsilon > 0$; that is,

$$\int_0^1 \frac{1}{(m(x,t)+\varepsilon)^q}dx \leqslant \left(1-\frac{t}{T}\right)\int_0^1 \frac{1}{(m^0(x)+\varepsilon)^q}dx + \frac{t}{T}\int_0^1 \frac{1}{(m^T(x)+\varepsilon)^q}dx.$$

Now, raising both sides to the power $\frac{1}{q}$ and bounding the r.h.s, we get

$$\|(m(\cdot,t)+\varepsilon)^{-1}\|_{L^q} \leqslant$$
$$\left(\left(1-\frac{t}{T}\right)\int_0^1 \frac{1}{(m^0(x)+\varepsilon)^q}dx + \frac{t}{T}\int_0^1 \frac{1}{(m^T(x)+\varepsilon)^q}dx\right)^{\frac{1}{q}}$$
$$\leqslant \max\left\{\int_0^1 \frac{1}{(m^0(x)+\varepsilon)^q}dx, \int_0^1 \frac{1}{(m^T(x)+\varepsilon)^q}dx\right\}^{\frac{1}{q}}$$
$$= \max\{\|(m^0(\cdot)+\varepsilon)^{-1}\|_{L^q}, \|(m^T(\cdot)+\varepsilon)^{-1}\|_{L^q}\}.$$

By letting $\varepsilon \to 0$ and then $q \to \infty$, we get

$$\|m(\cdot,t)^{-1}\|_{L^\infty} \leqslant \max\{\|m^0(\cdot)^{-1}\|_{L^\infty}, \|m^T(\cdot)^{-1}\|_{L^\infty}\}.$$

Theorem

Let $m, u \in C^\infty(\mathbb{T}^d \times [0,T])$, $m > 0$, *solve*

$$\begin{cases} -u_t + m^{\alpha(1-\beta)} \frac{|Du|^\beta}{\beta} = g(m) \\ m_t - \operatorname{div}(m^{1+\alpha(1-\beta)} Du|Du|^{\beta-2}) = 0 \end{cases}$$

with $g : \mathbb{R}^+ \to \mathbb{R}$ *smooth and non-decreasing. If* $\beta \geqslant 2$ *and*

$$q + 2\alpha(1-\beta) \geqslant 0 \quad \text{and} \quad 1 - \frac{1 - \frac{1}{d}}{q + 2\alpha(1-\beta)} - \frac{\alpha(\beta - 1)}{2} \geqslant 0$$

or $1 < \beta < 2$ *and*

$$q + 2\alpha(1-\beta) \geqslant 0 \quad \text{and} \; 1 - \frac{1 - \frac{1}{d}}{q + 2\alpha(1-\beta)} - \frac{\alpha}{2} \geqslant 0,$$

then $t \mapsto \int_{\mathbb{T}^d} m(x,t)^q dx$ *is convex.*

Corollary

Let (u, m) be solve the MFG with congestion with

$$\beta \geqslant 2 \quad \textit{and} \quad \alpha < \frac{2}{\beta - 1}$$

$$\textit{or}$$

$$1 < \beta < 2 \quad \textit{and} \quad \alpha < 2,$$

then, for every $d \geqslant 1$,

$$\|m(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} \leqslant \max\{\|m^0(\cdot)\|_{L^\infty(\mathbb{T}^d)}, \|m^T(\cdot)\|_{L^\infty(\mathbb{T}^d)}\}.$$

MFG with a potential

Now, we consider

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(t,x) = g(m) \\ m_t - \operatorname{div}(mDu) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ m(0,x) = m_0(x); \quad m(T,x) = m_T(x) \end{cases}$$

Assumptions

- There exists $p > 0$ such that the potential, $V : [0, T] \times \mathbb{T}^d \to \mathbb{R}$, satisfies

$$\|\Delta V\|_{L^\infty([0,T]\times\mathbb{T}^d)} < \frac{2}{T^2}\frac{1}{p}.$$

- There exist positive constants, k_0 and k_1, such that the boundary functions, m_0 and m_T, satisfy

$$0 < k_0 \leqslant m_0(x), m_T(x) \leqslant k_1, \quad x \in \mathbb{T}^d.$$

Theorem

Suppose that $p > 0$. Then, there exists a positive constant, C, depending only on the problem data and on p, such that

$$\max_{t\in[0,T]} \|m\|_{L^{p+1}(\mathbb{T}^d)} \leqslant C.$$

Moreover, if $p \geqslant 2$ and $d = 1$ there exists a positive constant, C, depending only on the problem data and on p, such that

$$\max_{t\in[0,T]} \{\|m\|_{L^{p+1}(\mathbb{T})}, \|m^{-1}\|_{L^{p-1}(\mathbb{T})}\} \leqslant C.$$

If

$$U(z) = z^s, \quad s \geqslant 1,$$

we have the **KEY INEQUALITY**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbb{T}^d} U(m)\,\mathrm{d}x \geqslant -|\alpha - 1| \|\Delta V\|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} U(m)\,\mathrm{d}x.$$

In one-dimensional case, $d = 1$, a direct computation shows that the preceding estimate holds for $s \in \mathbb{R} \setminus (0, 1)$.

Definition

Suppose that $a, b \geqslant 0$ *and* $c > 0$. *Let* $\mathcal{F}_a^b(c)$ *be the set of all* $f \in C^2[0, T]$, *with* f *non-negative and satisfying*

$$\begin{cases} f''(t) + cf(t) \geqslant 0 & \text{for all } t \in [0, T], \\ f(0) = a, \ \ f(T) = b. \end{cases}$$

Lemma

Suppose that $0 < \varepsilon \leqslant 2$ *and* $a, b \geqslant 0$. *Let* $c \leqslant \frac{2-\varepsilon}{T^2}$. *Then, the family of functions* $\mathcal{F}_a^b(c)$ *is uniformly bounded; more precisely, for any* $f \in \mathcal{F}_a^b(c)$, *we have*

$$0 \leqslant f(t) \leqslant \frac{2(a+b)}{\varepsilon} \quad \textit{for all } t \in [0, T].$$

Proof I

We prove only the first estimate, the second one is similar.
Let $s = q + 1$. There exists $\varepsilon > 0$ such that

$$s\|\Delta V\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leqslant \frac{2-\varepsilon}{T^2}.$$

Combining the Lemma and with the **KEY INEQUALITY**, we deduce

$$\int_{\mathbb{T}^d} m^s \,\mathrm{d}x \leqslant \frac{2}{\varepsilon}\left(\int_{\mathbb{T}^d} m_0^s \,\mathrm{d}x + \int_{\mathbb{T}^d} m_T^s \,\mathrm{d}x\right).$$

Now consider the MFG:

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(t,x) = m^\alpha & \text{in } (0,T) \times \mathbb{T}^d \\ m_t - \operatorname{div}(mDu) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ m(0,x) = m_0(x); \quad m(T,x) = m_T(x) & \text{in } \mathbb{T}^d. \end{cases}$$

Theorem

Suppose that $p \geqslant 2$. Then there exists a positive constant, C, depending only on the problem data, such that

$$\max_{t\in[0,T]} \left\{ \|m\|_{L^\infty(\mathbb{T}^d)} \right\} \leqslant C.$$

Moreover, if $p > \alpha + 1$ and $d = 1$ there exists a positive constant, C, depending only on the problem data and on p, such that

$$\max_{t\in[0,T]} \left\{ \|m^{-1}\|_{L^\infty(\mathbb{T})}, \|m\|_{L^\infty(\mathbb{T})} \right\} \leqslant C.$$

Proof

The proof of the theorem is divided into the propositions that follow.

Proposition

Suppose that $d = 1$. Suppose that for some $r \geqslant \max\{1, \alpha\}$ there exists a positive constant c, such that

$$\max_{t\in[0,T]} \int_{\mathbb{T}} \frac{1}{m^r}\,\mathrm{d}x < c.$$

There exists a positive constant, C, depending only on the problem data and on c such that

$$\max_{t\in[0,T]} \|m^{-1}\|_{L^\infty(\mathbb{T})} \leqslant C.$$

Proof

For any $s \geqslant 1$, set

$$M_s = \max_{t \in [0,T]} \int_{\mathbb{T}} \frac{1}{m^s} \, \mathrm{d}x.$$

Fix $q > r + \alpha$, and and set $\ell = \frac{2r}{q-\alpha}$.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbb{T}} \frac{1}{m^q} \mathrm{d}x \geq$$

$$= 4\alpha \frac{q(q+1)}{(q-\alpha)^2} \int_{\mathbb{T}} \left| D\left(\frac{1}{m^{\frac{q-\alpha}{2}}} \right) \right|^2 \mathrm{d}x - (q+1) C \int_{\mathbb{T}} \frac{1}{m^q} \, \mathrm{d}x.$$

Proof

After, a few inequalities

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}}\frac{1}{m^q}\mathrm{d}x \geqslant -2\left(qC^{2\gamma+3}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}}.$$

From which we eventually deduce

$$M_q \leqslant 2\left(qC^{2\gamma+3}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}} T^2.$$

Applying Moser's method, we get the iterative estimate

$$M_{q_{n+1}} \leqslant C\left(q_{n+1}C^{2\gamma_n+3}C_{\ell_n}^{2\gamma_n}\right)^{\frac{1}{1-\gamma_n}} M_{q_n}^{\frac{1}{1-\gamma_n}}$$

from which the result follows.

Proposition

There exists a positive constant, C, depending only on the problem data, such that

$$\max_{t\in[0,T]} \|m\|_{L^\infty(\mathbb{T}^d)} \leqslant C.$$

Proof

The proof is similar in this case and holds for $d > 1$.

Existence of solutions

Combining the a-priori estimates with the proof by S. Muñoz for the case $V = 0$, we obtain

Theorem

Let $m_0, m_T, V \in C^4(\mathbb{T})$, and $g(m) = m^\alpha$ for some $\alpha > 0$. Suppose $m_0, m_T > 0$. Then, if T is small enough, there exists a unique (up to constants) classical solution $(u, m) \in C^3([0,T] \times \mathbb{T}) \times C^2([0,T] \times \mathbb{T})$ to the planning problem.

Further credits

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