Global dynamics around two-solitons for the damped nonlinear Klein-Gordon equation

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(joint work with Kenjiro Ishizuka)

Introduction (summary)

Damped nonlinear Klein-Gordon equation

Consider the equation for $u(t,x)$, $t \geq 0$, $x \in \mathbb{R}$,

$$\text{(DNKG)} \quad \ddot{u} + \alpha\dot{u} - u_{xx} + u = u^p, \quad (\alpha > 0,\ p > 2: \text{ constants})$$

Ultimate goal: global behavior of solutions $\leftrightarrow$ initial data

Structure and relations of solution sets of different global behavior.

Main result

Description for all initial data in a small neighborhood of any 2-soliton, in terms of the 1-soliton manifold and the 2-soliton manifold.

- around 1-soliton: known (easy)
- around 2-soliton: energy transfer between 2 solitons (non-trivial)
- around 3-soliton: soliton merger (much harder)

Types of solutions

(DNKG) is well-posed in the energy space

$$\vec{u}(t) := (u(t), \dot{u}(t)) \in \mathcal{H} := H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

where the energy is decreasing by

$$E(\vec{u}) := \int_{\mathbb{R}} \frac{|\dot{u}|^2 + |u_x|^2 + |u|^2}{2} - \frac{|u|^{p+1}}{p+1} dx, \quad \partial_t E(\vec{u}) = -\alpha \|\dot{u}\|_2^2.$$

The focusing nonlinearity produces various types of solutions.

1. Global and decaying solutions $\|\vec{u}(t)\|_{\mathcal{H}} \to 0 \ (t \to \infty)$
2. Blow-up solutions $\exists T \in (0, \infty)$, $\|\vec{u}(t)\|_{\mathcal{H}} \to \infty \ (t \to T - 0)$
3. Stationary solution $u = Q(x) := \left\{ \frac{p+1}{2} \operatorname{sech}^2(\frac{p-1}{2} x) \right\}^{\frac{1}{p-1}}$
4. (Asymptotic) multi-solitons (N-solitons)

$$\vec{u}(t) = \sum_{j=0}^{N-1} (-1)^j \vec{Q}(x - c_j(t)) + o(1) \text{ in } \mathcal{H} \quad (t \to \infty)$$

Soliton resolution by Côte-Martel-Yuan (ARMA 2021)

established that the 3 types (decaying, blow-up and multi-solitons) exhaust all solutions, as well as existence and detailed behavior of N-solitons

$$c_{j+1}(t) - c_j(t) \sim \log t \quad (t \to \infty).$$

Soliton resolution along a time sequence was by Feireisl ('98). In the radial 3D case, the resolution for all time was by Burq-Schlag-Rougel ('17). The soliton resolution conjecture is for (undamped) nonlinear dispersive equations. So far, there are essentially only two types of equations where the resolution is known without size restriction or sequence in time:

1. Completely integrable systems: needs decay in x and generic spectra.
2. Energy-critical wave equation (Duyckaerts-Kenig-Merle '13), etc.: needs symmetry or obstacle, fixing soliton positions.

Côte-Martel-Yuan [CMY] is apparently the only result with moving solitons and for all initial data in the energy space.

Instability of the solitons

The two types of soliton resolutions have distinctive character: either the solitons are very stable (integrable case) or unstable (energy-critical wave). In this respect, (DNKG) belongs to the latter: Q is unstable, so are N-solitons. Indeed, Q has exactly one unstable direction, namely the unique negative eigenvalue of the linearized operator

$$L := -\partial_x^2 + 1 - pQ^{p-1}, \quad L\rho = -\kappa\rho, \quad (\kappa := \frac{(p+3)(p-1)}{4}, \quad \rho := Q^{\frac{p+1}{2}})$$

which produces a growing solution to the linearized evolution:

$$\boxed{u := e^{\pm\nu_\pm t}\rho,} \quad \nu_\pm := \frac{\sqrt{4\kappa + \alpha^2} \pm \alpha}{2} > 0$$
$$\implies \ddot{u} + \alpha\dot{u} - u_{xx} + u = pQ^{p-1}u.$$

Q_x is the unique kernel of L coming from the translation invariance. All the other directions are damped for $\alpha > 0$. Hence N-solitons have exactly N unstable directions.

Manifold of 2-solitons by Côte-Martel-Yuan-Zhao (arxiv)

One can expect that the set of N-solitons forms a manifold of codim$=N$.
This was proven for $N=2$ by Côte-Martel-Yuan-Zhao (arxiv'19) in general dimensions.
Let $\vec{u}_* \in C([0,\infty);\mathcal{H})$ be a 2-soliton of (DKG) with

$$\vec{u}_*(t) = \sum_{j=0}^{1}(-1)^j \vec{Q}(x-c_j(t)) + \gamma(t), \quad \|\gamma(0)\|_{\mathcal{H}} \ll 1 \ll |c_0(0)-c_1(0)|.$$

Denote the unstable directions and the orthogonal compliment by

$$Y_j := (-1)^j(1,\nu_+)\rho(x-c_j(0)), \quad Y_j^\dagger := (-1)^j(\nu_-,1)\rho(x-c_j(0)),$$
$$\mathcal{Y}_\delta^\perp := \{\varphi \in \mathcal{H} \mid \varphi \perp Y_0^\dagger, Y_1^\dagger,\ \|\varphi\|_{\mathcal{H}} < \delta\},$$

where the orthogonality is in L^2_x. Then $\exists\delta>0$ and $\exists G : \mathcal{Y}_\delta^\perp \to [-\delta,\delta]^2$ Lipschitz continuous, s.t. the solution u of (DKG) with

$$\vec{u}(0) = \vec{u}_*(0) + \varphi + a_0 Y_0 + a_1 Y_1, \quad \varphi \in \mathcal{Y}_\delta^\perp,\ a_j \in [-\delta,\delta]$$

is a 2-soliton iff $a = G(\varphi)$. Our question: What happens for $a \neq G(\varphi)$?

Main result: Global dynamics around 2-solitons

As above, let u_* be the 2-soliton, Y_j be the unstable directions, $\mathcal{Y}_\delta^\perp$ be the compliment, and $G = (G_0, G_1)$ be the graph of the 2-soliton manifold. Then $\exists F = (F_0, F_1) : \mathcal{Y}_\delta^\perp \times [-\delta, \delta] \to [-\delta, \delta]$ Lipschitz continuous s.t. the solution u of (DKG) with

$$\vec{u}(0) = \vec{u}_*(0) + \varphi + a_0 Y_0 + a_1 Y_1, \quad \varphi \in \mathcal{Y}_\delta^\perp, \ a_j \in [-\delta, \delta]$$

1. is a 2-soliton iff $\boxed{a = G(\varphi)}$ [CMYZ].
2. is a 1-soliton of the form $(-1)^j \vec{Q}(x - c_j(t)) + o(1)$ $(t \to \infty)$, iff $a_j = F_j(\varphi, a_{1-j})$ and $a_{1-j} < G_{1-j}(\varphi)$.
3. is global decaying iff $a_j < F_j(\varphi, a_{1-j})$ for both $j = 0, 1$.
4. blows up otherwise.

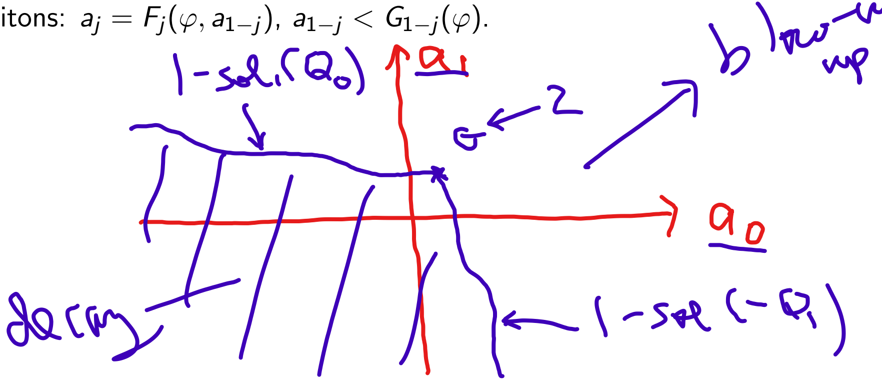
In short, F_j is the graph of the 1-soliton manifold of $(-1)^j Q$, and the 2-soliton graph G is the corner joining them. The rest is separated by them into two open sets corresponding to the decay and the blow-up.

Graphs of soliton manifolds

Around a given 2-soliton u_* (with sufficient separation), consider

$$\vec{u}(0) = \vec{u}_*(0) + \varphi + a_0 Y_0 + a_1 Y_1$$
$$= \sum_{j=0}^{1} (-1)^j (\vec{Q} + a_j Y)(x - c_j(0)) + \gamma(0) + \varphi$$

2-solitons: $(a_0, a_1) = (G_0(\varphi), G_1(\varphi))$ [CMYZ].
1-solitons: $a_j = F_j(\varphi, a_{1-j})$, $a_{1-j} < G_{1-j}(\varphi)$.



Comparison among N-solitons (1,2,3)

One may expect similar structures around N-solitons for every $N \in \mathbb{N}$. However, the difficulty of proof, as well as complication of dynamics, is quite different among $N = (0,)1, 2, 3$.

Below solitons $E(u) < E(Q)$ In this case, there are only decaying and blow-up solutions. Splitting into those two regions is general and classical (cf. Payne-Sattinger '75). Even without damping for nonlinear dispersive equations, the splitting into scattering and blow-up is standard since Kenig-Merle '06 (at least for high powers $p > 5$).

Around 1-solitons The codim-1 instability of the ground state is also classical. The key observation for classification is that the solution loses energy to $E(u) < E(Q)$ when it gets away from the soliton Q, so that we can reduce it to the above case. Even without the damping, as soon as one can preclude orbits returning close to Q, a similar classification is proved, e.g. for $p > 5$ in the even symmetric case (Krieger-N.-Schlag '12).

2-soliton: energy transfer between the 2 solitons

This is more difficult than a mere superposition of the 1-soliton instability, due to soliton interactions of $O(1/t)$ that is not integrable in time.
The issue is possible energy transfer between the two growing modes. The solution may be decomposed as long as it stays close to 2-solitons

$$\vec{u}(t) = \sum_{j=0}^{1} (-1)^j (\vec{Q} + a_j(t) Y)(x - c_j(t)) + \gamma(t)$$

so that $|a(t)| + \|\gamma(t)\| \sim$ the distance of $\vec{u}(t)$ and 2-solitons. For generic choice of $a(0)$, it grows exponentially by

$$\frac{d}{dt}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \nu_+ \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} M_{00}(t) & M_{01}(t) \\ M_{10}(t) & M_{11}(t) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \cdots$$

where $M(t) \sim 1/t \notin L^1(1,\infty)$ due to the soliton interactions.
It changes the growth rate from $e^{\nu_+ t}$ such as $t^\delta e^{\nu_+ t}$, even for a single mode. In the case of two modes, it can even change "the direction".

2-soliton: energy transfer between the 2 solitons

To prove the corner-like structure, we need: if one of the unstable modes is bigger (in the difference of two solutions), it will grow dominantly.
But the $O(1/t)$ interactions could possibly transfer the growth from one to the other. A simple example:

$$M(t) = \delta \begin{bmatrix} 0 & e^{-t} \\ e^{-t} & \frac{1}{1+t} \end{bmatrix}, \quad \begin{bmatrix} a_0(0) \\ a_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} a_0(t) \\ a_1(t) \end{bmatrix} \sim t^{\delta} e^{\nu_+ t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (t \to \infty)$$

The key observation to preclude such phenomena is

$$\exists \mu(t) \sim 1/t, \quad \begin{bmatrix} M_{00}(t) & M_{01}(t) \\ M_{10}(t) & M_{11}(t) \end{bmatrix} = \mu(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + L^1_t(1,\infty).$$

$\mu(t)$ depends entirely on the remainder component of u, whose behavior differs within the neighborhood of 2-soliton u_*. This is a difference from the explicit ODE system for the motion of centers $c_j(t)$ given by [CMY].
Still, $M_{00}(t) - M_{11}(t)$ is controlled thank to the symmetry of (DKG).

3-soliton: soliton merger

One may expect the same result and proof for $N \geq 3$, but actually one needs to treat a much harder phenomenon: soliton merger.
This comes from the fact that the soliton interaction is repulsive because of the sign alternating structure. Indeed, [CMY] also prove that solitons of the same sign cannot survive because of attractive interactions.
In the proof around 2-solitons, it is essential that the solitons stay away from each other. This is violated if we start near 3-solitons and the middle soliton is destroyed by instability, since the other two have the same sign.
So we must consider initial data around 2-solitons with the same sign, e.g.

$$\vec{u}(0) = (\vec{Q} + aY)(x + c) + (\vec{Q} + aY)(x - c), \quad |c| \gg 1 \gg |a|$$

By the soliton resolution [CMY], as well as stability of decay and blow-up, we deduce that u must become 1-soliton for some $a \in \mathbb{R}$. Then the symmetry implies that such $\vec{u}$ has to stay away from both 2-solitons and 1-solitons for some time, where the behavior must be something like 2 solitons merging into 1, but describing it seems difficult.

Difference estimate around 1-solitons

The main ingredient of proof is the difference estimate for two solutions starting from the neighborhood of the 2-soliton. In the construction of 2-soliton manifold, we may assume that one of them is 2-soliton [CMYZ]. For our result, we need to extend it to the case of 1-solitons.
Let $u^{(0)}, u^{(1)}$ be two solutions starting near the 2-soliton u_*:

$$\vec{u}^{(k)}(0) = \vec{u}_*(0) + \varphi^{(k)} + a_0^{(k)} Y_0 + a_1^{(k)} Y_1,$$

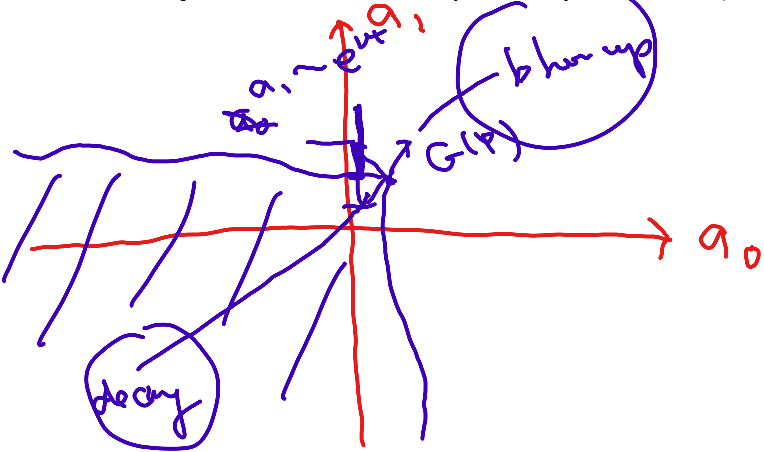
such that $\vec{u}^{(0)}(t) = \vec{Q}(x - c(t)) + o(1)$ as $t \to \infty$. Then

$$|a_0^{(0)} - a_0^{(1)}| \sim \|\vec{u}^{(0)}(0) - \vec{u}^{(1)}(0)\|_{\mathcal{H}}$$
$$\implies \langle \vec{u}^{(0)}(t) - \vec{u}^{(1)}(t) | Y^\dagger(x - c(t))\rangle \gtrsim (a_0^{(0)} - a_0^{(1)}) e^{\nu_+ t/2},$$

as long as the difference remains small (for appropriate $c(t)$).
This implies the uniqueness as well as Lipschitz continuity of the Q_0-soliton graph F_0. The existence, as well as the remaining dynamics, follows from the stability of decay and blow-up around Q, and superposition of them.

Construction of the 1-soliton manifolds (near 2-solitons)

The graphs $a_j = F_j(\varphi, a_{1-j})$ of 1-solitons are constructed using the above difference estimate, together with the stability of decay and blow-up.



All-time dynamics around 2-solitons

For the difference estimate around the solution $\vec{u}^{(0)}$, which goes from 2-soliton to 1-soliton, we need to know its behavior for all $t > 0$. This description may be of independent interest. It is more involved than starting with 1-solitons, due to the soliton interactions.

Let u be any solution starting near the 2-soliton. Then we can decompose

$$\vec{u}(t) = \sum_{j=0}^{1} (-1)^j (\vec{Q} + a_j(t)Y)(x - c_j(t)) + \gamma(t)$$

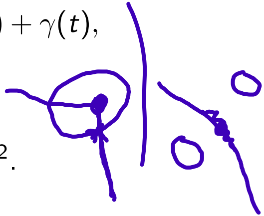
and $0 \le \exists T_1 \le \exists T_2 \le \exists T_3 \le T_*$: maximal existence, with $F(t) := \exp(-|c_0(t) - c_1(t)|)$ measuring the soliton interactions,

1. $0 \le t < T_1 \implies |a| \lesssim \|\gamma\|_{\mathcal{H}}^2 + F$ (unstable mode a is sleeping)
2. $T_1 \le t < T_2 \implies \|\gamma\|_{\mathcal{H}}^2 + F \lesssim |a| \lesssim \|\gamma\|_{\mathcal{H}} + F^{1/2}$ (rise of a)
3. $T_2 \le t < T_3 \implies \|\gamma\|_{\mathcal{H}} + F^{1/2} \lesssim |a| < \delta_*$ (dominance by a)

for some absolute $\delta_* > 0$ (determining the neighborhood of 2-solitons), provided that $|a(0)| + \|\gamma(0)\| + F(0)^{1/2} \ll \delta_*$.

All-time dynamics around 2-solitons (continued)

$$\vec{u}(t) = \sum_{j=0}^{1} (-1)^j (\vec{Q} + a_j(t) Y)(x - c_j(t)) + \gamma(t),$$



with $F(t) := \exp(-|c_0(t) - c_1(t)|)$,

1. $0 \le t < T_1 \implies |a| \lesssim \|\gamma\|_{\mathcal{H}}^2 + F$.
2. $T_1 \le t < T_2 \implies \|\gamma\|_{\mathcal{H}}^2 + F \lesssim |a| \lesssim \|\gamma\|_{\mathcal{H}} + F^{1/2}$.
3. $T_2 \le t < T_3 \implies \|\gamma\|_{\mathcal{H}} + F^{1/2} \lesssim |a| < \delta_*$.

Moreover, $\exists T_s \in [0, T_2]$ s.t. for $0 < t < T_s$, we have *separate dynamics*

$$|\partial_t c| + |(\partial_t - \nu_+)a| \lesssim \|\gamma\|_{\mathcal{H}}^2, \quad \|\gamma\|_{\mathcal{H}} \lesssim e^{-\mu t}\|\gamma(0)\|_{\mathcal{H}},$$

for some constant $\mu > 0$. For $T_s < t < T_2$, we have *soliton interactions*

$$|(\partial_t - \nu_+)a_j - F(t)C_0| \ll F \sim [t - T_s + 1/F(T_s)]^{-1},$$
$$\|\gamma\|_{\mathcal{H}} \lesssim e^{-\mu(t - T_s)}\|\gamma(T_s)\|_{\mathcal{H}} + F,$$

for some constant $C_0 > 0$. They are not bounded in $L^1_t(T_s, T_2)$.
(For $T_2 < t < T_3$, exponential growth implies L^1_t bound.)

Key estimate to preclude growth transfer

In the difference estimate, the key point is to preclude the possibility of growth transfer from a_0 to a_1 through $O(1/t)$ soliton interactions. The main estimate is on the linearized equation around the solution $u^{(0)}$ during it is still around 2-solitons. For the (difference of) unstable modes it is

v_t

$$\dot{a}_j(t) = \sum_{k=0}^{1} M_{jk}(t) a_k(t) + \cdots$$

with $M(t)$ depending on $u^{(0)}(t)$. By symmetry of (DKG), we obtain

$$|M_{00}(t) - M_{11}(t)| \lesssim \|v(t) + v^\dagger(t)\|_{\mathcal{H}} + \|v(t)\|_{\mathcal{H}}^2 + F(t)^2 \in L_t^1,$$

where $F(t) = O(1/t)$ contribution is cancelled in the "even" part $v + v^\dagger$,

$$v(t) := \vec{u}^{(0)}(t) - \sum_{j=0}^{1} (-1)^j \vec{Q}(x - c_j(t)) =: v^\dagger(t, -x + c_0(T_2) + c_1(T_2)).$$

The proof relies on the complete description of $u^{(0)}$ for $0 < t < T_3$.