Dyadic models for MHD

SITE Conference

Mimi Dai, University of Illinois at Chicago

June 17, 2021

Long Time Behaviour and Singularity Formation in PDEs

MHD system

Incompressible magnetohydrodynamics (MHD) with Hall effect:

$$
\begin{aligned}
u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla p &= \nu \Delta u, \\
B_t + u \cdot \nabla B - B \cdot \nabla u + d_i \nabla \times ((\nabla \times B) \times B) &= \mu \Delta B, \\
\nabla \cdot u &= 0,
\end{aligned} \tag{1}
$$

on $\Omega \times [0, \infty)$ with $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$.

$u : \Omega \times [0, \infty) \to \mathbb{R}^3$, fluid velocity,
$B : \Omega \times [0, \infty) \to \mathbb{R}^3$, magnetic field,
$p : \Omega \times [0, \infty) \to \mathbb{R}$, fluid pressure,
ν : kinematic viscosity,
μ : resistivity $\sim 1/\text{conductivity}$,
d_i : ion inertial length.

Subsystems and scalings

- $B \equiv 0$: (1) $\Longrightarrow$ Navier-Stokes equation

$$\text{scaling:} \quad u_\lambda = \lambda u(\lambda x, \lambda^2 t)$$

- $u \equiv 0$: (1) $\Longrightarrow$ Electron MHD:

$$B_t + d_i \nabla \times ((\nabla \times B) \times B) = \mu \Delta B, \quad \nabla \cdot B = 0 \qquad (2)$$

$$\text{scaling:} \quad B_\lambda = B(\lambda x, \lambda^2 t)$$

- $d_i \equiv 0$: (1) $\Longrightarrow$ Usual MHD

$$\text{scaling:} \quad u_\lambda = \lambda u(\lambda x, \lambda^2 t), \quad B_\lambda = \lambda B(\lambda x, \lambda^2 t)$$

Subsystems and scalings

- $d_i > 0$: Hall MHD, no natural scaling
- Two nonlinear structures:

$$\nabla \times ((\nabla \times B) \times B) = \nabla \times \nabla \cdot (B \otimes B)$$

$$(u \cdot \nabla) \cdot u = \nabla \cdot (u \otimes u)$$

  different scalings; different "degrees of singular effect"; different geometry properties
- MHD and Hall MHD obey the same energy law:

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|B\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \mu\|\nabla B\|_{L^2}^2 = 0$$

Unanswered Questions (perspective of mathematics)

- (i) Global regularity / finite time singularity
  (ii) Uniqueness / non-uniqueness of Leray-Hopf solution
  (iii) Stability / instability
  (iv) Turbulence related questions: anomalous dissipation...

Unanswered Questions (perspective of mathematics)

- (i) Global regularity / finite time singularity
  (ii) Uniqueness / non-uniqueness of Leray-Hopf solution
  (iii) Stability / instability
  (iv) Turbulence related questions: anomalous dissipation...
- Pure fluid VS MHD: similarity + complexity

$\downarrow$

interactions of u and $B+$ Hall nonlinearity

Unanswered Questions (perspective of mathematics)

- (i) Global regularity / finite time singularity
  (ii) Uniqueness / non-uniqueness of Leray-Hopf solution
  (iii) Stability / instability
  (iv) Turbulence related questions: anomalous dissipation...
- Pure fluid VS MHD: similarity + complexity

$\downarrow$

interactions of u and $B+$ Hall nonlinearity

- Toy models to gain insights towards understanding the questions above: 1D models, dyadic models, ...

1D models for Euler

- Constantin-Lax-Majda model, De Gregorio model, Cordoba-Cordoba-Fontelos model, Okamoto-Sakajo-Wunsch model
- Hou-Li-Shi-Wang-Yu model
- Elgindi-Jeong (2017): "On the Effects of Advection and Vortex Stretching"
- Elgindi-Ghoul-Masmoudi (2019): "Stable self-similar blowup for a family of nonlocal transport equations"

Dyadic Euler/NSE models

- Gledzer, Ohkitani-Yamada, Desnyanskiy-Novikov, Obukhov, Dinaburg-Sinai, Katz-Pavlović, Kiselev-Zlatoš, etc.
- Cheskidov, Friedlander, Pavlović, 2005-2008: well-posedness, smooth solutions, blow-up, anomalous dissipation, etc.
- Barbato, Flandoli, Romito, etc: dyadic models, stochastic dyadic models, ...

Dyadic Euler/NSE models

Very well-understood! It did give some insights for the real dynamics, for instance, on the problem of Onsager's conjecture.

"Susan Friedlander's contributions in mathematical fluid dynamics" - Cheskidov-Glatt-Holtz-Pavlović-Shvydkoy-Vicol

Legacy of Kolmogorov

Highlights of Kolmogorov's classical phenomenological theory of turbulence for hydrodynamics (1941):

- Assumptions on the flow: homogeneity, isotropy, self-similarity
- Conjecture on dissipation wavenumber: There exists a critical wavenumber

$$\kappa_{\mathrm{d}} = \left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{4}}, \quad \varepsilon = \nu \left\langle \|\nabla u\|_2^2 \right\rangle$$

  such that the dynamics above the wavenumber κ_{d} is dominated by the linear dissipative term.
- Energy spectrum below κ_{d} (the inertial range):

$$\mathcal{E}(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

- $\liminf_{\nu \to 0} \varepsilon > 0$

Legacy of Kolmogorov

Deviation from 1941's classical theory:

- Landau, 1942: fully developed turbulent flow may be spatially and temporally inhomogeneous
- Experimental evidences show discrepancy from the $-5/3$ law, small scales have fractal properties
- Kolmogorov, 1962: concept of intermittency was introduced to describe the deviation; K41 was updated to K62, in which intermittency was studied via a fractal dimension parameter D and included in κ_{d} and $\mathcal{E}(k)$ (statistical tools and scaling analysis)

Intermittency dimension: towards more precisely mathematical characterization

- Cheskidov-Shvydkoy, 2012: intermittency dimension based on Littlewood-Paley theory; active volume, region, eddy, etc, reformulated in mathematical language
- Cheskidov-Dai, 2015: intermittency dimension through the saturation level of Bernstein's inequality
- Cheskidov-Dai, 2015-2016: wavenumber splitting approach, low modes regularity criteria for dissipative systems, determining wavenumber for supercritical systems, number of degrees of freedom

Intermittency dimension parameter: saturation of Bernstein's inequality

- Bernstein's inequality in 3D: $\|v_j\|_{L^\infty} \leq c\lambda_j^{3/2}\|v_j\|_{L^2}$, $\lambda_j = 2^j$
- Intermittency dimension (Cheskidov-D., 2015):

$$\delta_v := \sup\left\{ s \in \mathbb{R} : \left\langle \sum_j \lambda_j^{-1+s}\|v_j\|_{L^\infty}^2 \right\rangle \leq c \left\langle \sum_j \lambda_j^2 \|v_j\|_{L^2}^2 \right\rangle \right\}$$

- $\delta_v \in [0, 3]$
- Extreme intermittency: $\delta_v = 0$, e.g., Dirac delta function;
- Kolmogorov's regime: $\delta_v = 3$, e.g., $\sin(\lambda x)$;
- Bernstein's relationship with correction of δ_v:

$$\|v_j\|_{L^q} \sim \lambda_j^{(3-\delta_v)\left(\frac{1}{p}-\frac{1}{q}\right)}\|v_j\|_{L^p}, \quad q \geq p$$

Principles of proposing dyadic models

- Assume local interactions: only the nearest shells interact with each other
- Preserve invariant quantities: energy, helicity, ...
- Energy balance through each shell

Remarks

- PDE $\to$ ODE with infinitely many equations
- Spatial structure is over simplified
- Geometry features are not preserved

Dyadic MHD

- δ_u: intermittency dimension for the velocity field u
- δ_b: intermittency dimension for the magnetic field B
- Energy balance at j-th shell

$$\frac{1}{2}\frac{d}{dt}\|u_j\|_{L^2}^2 + \int_{\mathbb{R}^3} (u \cdot \nabla u)_j \cdot u_j \, dx - \int_{\mathbb{R}^3} (B \cdot \nabla B)_j \cdot u_j \, dx + \nu \|\nabla u_j\|_{L^2}^2$$

$$\frac{1}{2}\frac{d}{dt}\|B_j\|_{L^2}^2 + \int_{\mathbb{R}^3} (u \cdot \nabla B)_j \cdot B_j \, dx - \int_{\mathbb{R}^3} (B \cdot \nabla u)_j \cdot B_j \, dx$$

$$+ d_i \int_{\mathbb{R}^3} ((\nabla \times B) \times B)_j \cdot \nabla \times B_j \, dx + \mu \|\nabla B_j\|_{L^2}^2 = 0.$$

Dyadic MHD: $a_j = \|u_j\|_{L^2}$, $b_j = \|B_j\|_{L^2}$

$$\frac{d}{dt}a_j + \alpha_1\left(\lambda_j^{\frac{5-\delta_u}{2}} a_j a_{j+1} - \lambda_{j-1}^{\frac{5-\delta_u}{2}} a_{j-1}^2\right) + \beta_1\left(\lambda_j^{\frac{5-\delta_u}{2}} a_{j+1}^2 - \lambda_{j-1}^{\frac{5-\delta_u}{2}} a_{j-1}a_j\right)$$
$$+\alpha_3\left(\lambda_j^{\frac{5-\delta_b}{2}} b_j b_{j+1} - \lambda_{j-1}^{\frac{5-\delta_b}{2}} b_{j-1}^2\right) + \beta_3\left(\lambda_{j+1}^{\frac{5-\delta_b}{2}} b_{j+1}^2 - \lambda_j^{\frac{5-\delta_b}{2}} b_{j-1}b_j\right)$$
$$+\nu\lambda_j^2 a_j = 0,$$

$$\frac{d}{dt}b_j + \alpha_2\left(\lambda_j^{\frac{5-\delta_b}{2}} a_j b_{j+1} - \lambda_{j-1}^{\frac{5-\delta_b}{2}} a_{j-1}b_{j-1}\right) + \beta_2\left(\lambda_{j+1}^{\frac{5-\delta_b}{2}} a_{j+1}b_{j+1} - \lambda_j^{5}\right.$$
$$+\alpha_3\left(\lambda_j^{\frac{5-\delta_b}{2}} b_j a_{j+1} - \lambda_{j-1}^{\frac{5-\delta_b}{2}} a_{j-1}b_{j-1}\right) + \beta_3\left(\lambda_{j+1}^{\frac{5-\delta_b}{2}} b_{j+1}a_{j+1} - \lambda_j^{5}\right.$$
$$+d_i\alpha_4\left(\lambda_j^{\frac{7-\delta_b}{2}} b_j b_{j+1} - \lambda_{j-1}^{\frac{7-\delta_b}{2}} b_{j-1}^2\right) + d_i\beta_4\left(\lambda_j^{\frac{7-\delta_b}{2}} b_{j+1}^2 - \lambda_{j-1}^{\frac{7-\delta_b}{2}} b_j b_{j-1}\right) +$$

Remark

The sign of the parameters α's and β's indicates the direction of energy transfer:

- Positive sign: forward energy cascade
- Negative sign: backward energy cascade

Remarks:

- Consistent with dyadic models introduced by physicists
- Energy is conserved for any coefficient parameters α's and β's
- With particular choice of parameters, cross helicity is conserved

Special case I: both forward and backward energy cascade

$\alpha_1 = \alpha_2 = \alpha_4 = 1$, $\alpha_3 = -1$; $\beta_k = 0$ for $1 \le k \le 4$; $\delta_u = \delta_b =: \delta$; $\theta = \frac{5-\delta}{2}$:

$$\frac{d}{dt} a_j = -\nu\lambda_j^2 a_j - \lambda_j^\theta a_j a_{j+1} + \lambda_{j-1}^\theta a_{j-1}^2 + \lambda_j^\theta b_j b_{j+1} - \lambda_{j-1}^\theta b_{j-1}^2,$$
$$\frac{d}{dt} b_j = -\mu\lambda_j^2 b_j - \lambda_j^\theta a_j b_{j+1} + \lambda_j^\theta b_j a_{j+1} - d_i \left(\lambda_j^{\theta+1} b_j b_{j+1} - \lambda_{j-1}^{\theta+1} b_{j-1}^2\right) \tag{3}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & a_{j-1} & \longrightarrow & a_j & \longrightarrow & a_{j+1} & \longrightarrow \cdots \\
& \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \\
\cdots \longrightarrow & b_{j-1} & \longrightarrow & b_j & \longrightarrow & b_{j+1} & \longrightarrow \cdots
\end{array}$$

Total energy and cross helicity are conserved if $\nu = \mu = d_i = 0$.

Special case II: only forward energy cascade

$\alpha_k = 1$ for $1 \leq 4$; $\beta_k = 0$ for $1 \leq k \leq 4$; $\delta_u = \delta_b =: \delta$; $\theta = \frac{5-\delta}{2}$:

$$
\begin{aligned}
\frac{d}{dt} a_j &= -\nu \lambda_j^2 a_j - \lambda_j^\theta a_j a_{j+1} + \lambda_{j-1}^\theta a_{j-1}^2 - \lambda_j^\theta b_j b_{j+1} + \lambda_{j-1}^\theta b_{j-1}^2, \\
\frac{d}{dt} b_j &= -\mu \lambda_j^2 b_j + \lambda_j^\theta a_j b_{j+1} - \lambda_j^\theta b_j a_{j+1} - d_i \left(\lambda_j^{\theta+1} b_j b_{j+1} - \lambda_{j-1}^{\theta+1} b_{j-1}^2 \right)
\end{aligned}
\tag{4}
$$

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & a_{j-1} & \longrightarrow & a_j & \longrightarrow & a_{j+1} & \longrightarrow \cdots \\
& \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \\
\cdots \longrightarrow & b_{j-1} & \longrightarrow & b_j & \longrightarrow & b_{j+1} & \longrightarrow \cdots
\end{array}
$$

Total energy is conserved; cross helicity is not conserved.

Notions of solutions

- Weak solution: $(a_j(t), b_j(t))$ satisfies (3), $\forall j \geq 0$; $(a(t), b(t)) \in \ell^2 \times \ell^2$; $(a_j, b_j) \in C^1([t_0, \infty))$, $\forall j \geq 0$.
- Strong solution: $(a_j(t), b_j(t))$ is a (weak) solution; $\|a(t)\|_{H^1}$ and $\|b(t)\|_{H^1}$ are bounded.

Main results: viscous case

Model (3) with $d_i > 0$, both forward and backward energy cascade, (M.D. 2020):

- $\delta = 3 \Leftrightarrow \theta = \frac{5-\delta}{2} = 1$: global strong solution
- $\delta \in (1,3) \Leftrightarrow \theta \in (1,2)$: local strong solution
- $\delta \in [-1,1] \Leftrightarrow \theta \in [2,3]$: Not much is known; anything could happen...
- $\delta < -1 \Leftrightarrow \theta > 3$: positive solutions with large initial data develops blow-up

larger $\delta \sim$ more regular u and $B \sim$ weaker nonlinearity $\sim$ smaller θ

Main results: viscous case

Model (3) with $d_i = 0$, i.e. dyadic model of usual MHD with forward and backward energy cascade (M.D. 2020):

- $\delta \in [1,3] \Leftrightarrow \theta \in [1,2]$: global strong solution
- $\delta \in [0,1) \Leftrightarrow \theta \in (2,\frac{5}{2}]$: local strong solution
- $\delta < 0 \Leftrightarrow \theta > \frac{5}{2}$: gap...

Model (4) with $d_i = 0$, i.e. dyadic model of usual MHD with forward energy cascade, (M.D. 2021):

- $\delta < -1 \Leftrightarrow \theta > 3$: blow-up for positive solutions with large initial data

Main results: inviscid case

Model (4) with $\nu = \mu = d_i = 0$ and forcing f_0 on the first model of a_0, i.e. dyadic model of usual MHD with forward energy cascade: (M.D. - S. Friedlander, 2021):

- Fixed points: $\bar{a}_j^2 + \bar{b}_j^2 = \lambda^{\frac{1}{3}\theta} f_0 \lambda_j^{-\frac{2}{3}\theta}$, "circle", Onsager scaling
- Stability of fixed points
- Finite-time blow-up
- Solutions attracted by a fixed point dissipates energy

Contrasts

- Fixed point of dyadic Euler: a unique fixed point $\rightarrow$ strong global attractor, Cheskidov-Friedlander-Pavlović

Contrasts

- Fixed point of dyadic Euler: a unique fixed point $\rightarrow$ strong global attractor, Cheskidov-Friedlander-Pavlović
- Stability/instability of the original MHD: interesting + challenging, vast literature in both mathematics + physics communities: Liu-Masmoudi-Zhai-Zhao, Ren-Wei-Zhang, Ren-Zhao, ...

Ongoing and future work

- Energy transfer from fluid to magnetic field and vice versa
- Attractor
- Vanishing viscosity limit
- Anomalous dissipation
- Models with different energy cascade scenarios
- Models with $\delta_u \neq \delta_b$, ...

THANK YOU!