Classification of GBU and RBC behaviors in the viscous Hamilton-Jacobi equation

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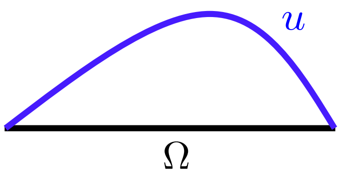
Consider the viscous Hamilton-Jacobi eq.

$$(P)\quad \begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \end{cases}$$



where $p > 2$, $\Omega \subset \mathbb{R}^N$ and

$$\phi \in \mathcal{W} = \left\{ u_0 \in W^{1,\infty}(\Omega);\ u_0 \geq 0,\ u_0 = 0 \text{ on } \partial\Omega \right\}.$$

Since $\|u(t)\|_\infty \leq \|\phi\|_\infty$, $t > 0$ by the maximum principle,

$$u \text{ blows up at } t = T < \infty \quad \Leftrightarrow \quad \limsup_{t \to T} \|\nabla u(t)\|_\infty = \infty.$$

$u_t - \Delta u = u^q$

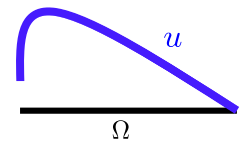
gradient blowup (GBU)

[Barles - Da Lio (2004)]

(P) admits a unique global (generalized) viscosity solution u s.t.

- u coincides with the classical sol. in $(0,T)$,
- $u \in C(\overline{\Omega} \times [0,\infty)) \cap C^{1,2}(\Omega \times (0,\infty))$, $u \geq 0$
- u solves the PDE in $\Omega \times (0,\infty)$.

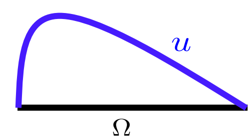
[Porretta - Souplet(2017), Quaas - Rodríguez(2018)]



ϕ is suitably large,

$\Rightarrow u$ exhibits a loss of boundary condition (LBC) at $\exists t > T(\phi)$.

[Porretta - Souplet(2017, 2020)]



there is a GBU sol. without LBC,

which is a separatrix between global sols. and GBU sols. with LBC.

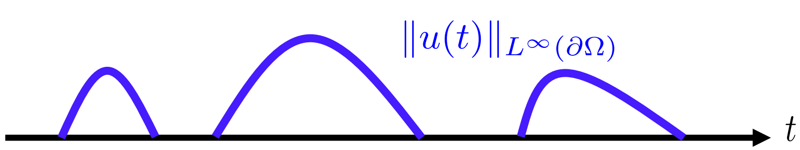
[Porretta - Zuazua(2012)]

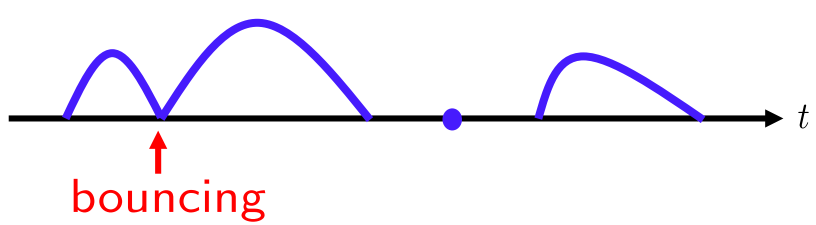
$u \in C^{2,1}(\overline{\Omega} \times (\tilde{T}, \infty))$ with $u = 0$ on $\partial\Omega \times [\tilde{T}, \infty)$ for $\exists \tilde{T} \geq T$

and $u \to 0$ in $C^1(\overline{\Omega})$ as $t \to \infty$.

[M.-Souplet(2020)]$(N = 1)$

There is a viscosity sol. with arbitrary combination of GBU and RBC.





Rate of gadient blow-up (GBU)

[Porretta - Souplet(2020)]

$$\|\nabla u(t)\|_\infty \geq C(T-t)^{-\frac{1}{p-2}}, \quad t \in (0,T)$$ type II

faster than $(T-t)^{-\frac{1}{2(p-1)}}$ (type I)

[Attouchi - Souplet(2020)]

If a sol. is increasing in time in a nbhd. of $\partial\Omega$, then

$$C_1(T-t)^{-\frac{1}{p-2}} \leq \|\nabla u(t)\|_\infty \leq C_2(T-t)^{-\frac{1}{p-2}}, \quad t \in (0,T)$$

for some $C_1, C_2 > 0$.

[Porretta - Souplet(2020)] $(N = 1)$

If u is a separatrix between global sol. and GBU sol. with LBC,

$$\|u_x(t)\|_\infty \geq C(T-t)^{-\frac{2}{p-2}}, \quad t \in (0,T)$$

for some $C > 0$.

Denote by z the number of sign changes on $[0, R]$, i.e.,

$$z(v : [0,R]) = \sup\{m \in \mathbb{N} : \exists x_0 < \cdots < x_m \in (0,R) \text{ s.t. } v(x_{i-1})v(x_i) < 0, i = 1, \ldots, m\}$$

Rate of rocovery of boundary condition (RBC)

[Porretta - Souplet(2020)] $(N = 1)$

Under $z(u_t : [0,1]) = 2$ and some assumptions,

if u recoveries BC at $(x,t) = (0,\tau)$, then

$$C_1(\tau - t) \leq u(0,t) \leq C_2(\tau - t), \quad t \in (0,\tau)$$

for some $C_1, C_2 > 0$.

Let $0 < R \leq \infty$ and Consider

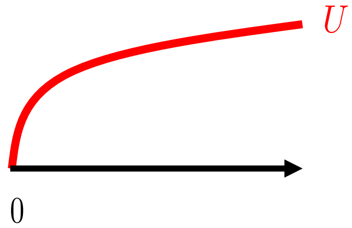
$$\begin{cases} u_t = u_{xx} + |u_x|^p, & x \in \Omega := (0, R),\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

Let

$$\beta := \tfrac{1}{p-1} \in (0,1), \quad k := \tfrac{p-2}{2(p-1)}$$

The singular and regular steady-states are respectively given by

$$U(x) := c_p x^{1-\beta}, \quad x > 0, \qquad \text{where } c_p := (1-\beta)^{-1}\beta^{\beta}$$



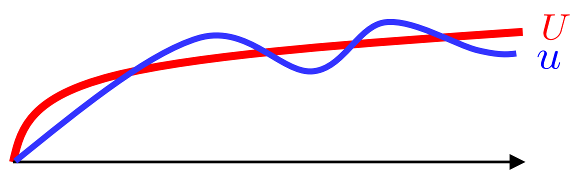
and for $a > 0$,

$$U_a(x) := U(a+x) - U(a), \quad x > 0.$$

Theorem 1

(i) Suppose that u undergoes GBU at $(x,t)=(0,T)$.
Let n be the number of vanishing intersetions between u and U at $(x,t)=(0,T)$.
Then there exists $C>0$ s.t.

$$\lim_{t\to T^-}(T-t)^{\frac{n}{p-2}}u_x(0,t)=C$$



and

$$u(x,t)=U_{a(t)}(x)+O(x^2),\quad u_x(x,t)=U'_{a(t)}(x)+O(x)$$

as $t\to T_-$ with $a(t):=\beta C^{1-p}(T-t)^{\frac{p-1}{p-2}n}$.

(ii) For each $n\geq 1$, there exists a sol. that behaves as above with some $T<\infty$, $C>0$.

Theorem 2

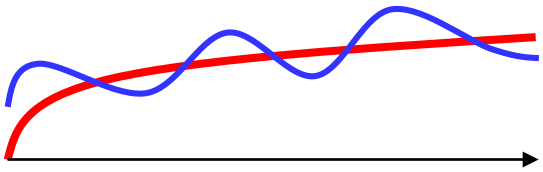
(i) Assume that a sol. u undergoes RBC at $(x,t)=(0,\tau)$, i.e.,

$$u(0,t)>0 \text{ for } t<\tau \text{ close to } \tau \quad \text{and} \quad u(0,\tau)=0.$$

Let n be the number of vanishing intersections at $(x,t)=(0,\tau)$.

Then there exists $C>0$ s.t.

$$\lim_{t\to\tau^-}(\tau-t)^{-n}u(0,t)=C$$



and

$$u(x,t)=C(\tau-t)^n\phi_n((\tau-t)^{-1/2}x)+o((\tau-t)^n) \text{ as } t\to\tau_-,$$

where ϕ_n is the eigenft.

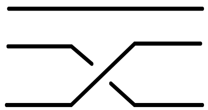
$$\phi_{yy}+\left(\frac{p}{p-1}\frac{1}{y}-\frac{y}{2}\right)\phi_y+k\phi=-\lambda\phi$$

with $\phi_n(0)=1$ associated with the nth eigenvalue $n-k$.

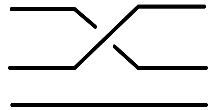
(ii) For each $n\geq 1$, there exists a sol. that behaves as above with some $\tau<\infty$, $C>0$.

$\mathbf{G}$: the braid group of three strands

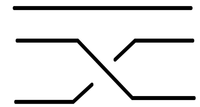
X, Y : generators of $\mathbf{G}$



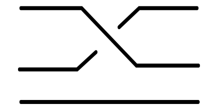
X



Y

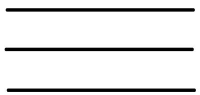


X^{-1}



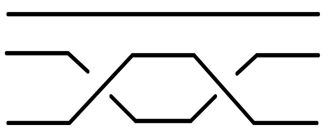
Y^{-1}

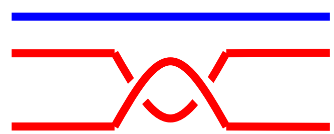
I : the trivial braid.



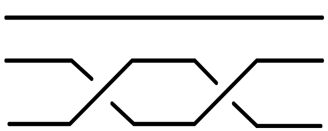
I

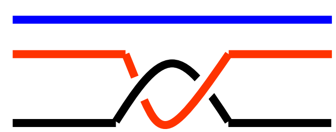
$$XX^{-1} = X^{-1}X = YY^{-1} = Y^{-1}Y = I$$





$XX^{-1} = I$

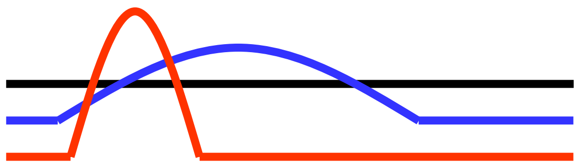




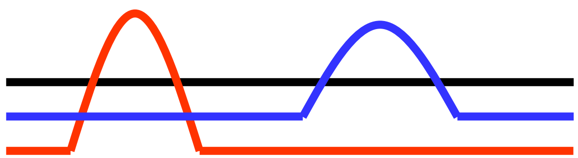
X^2

$A \in \mathbf{G}$ is a positive braid $\Leftrightarrow$ A contains neither X^{-1} nor Y^{-1}. Denote by $\mathbf{G}^+$ the semigroup of positive braids in $\mathbf{G}$.

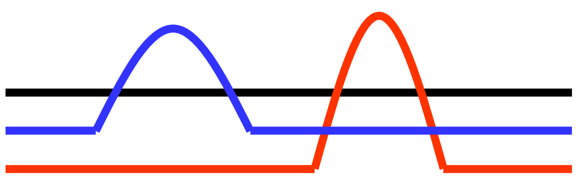
Artin's formula : $XYX = YXY$



$XYXYXY$



$XYXYXY = XY^2XY^2$



$$\begin{aligned} XYXYXY &= YXYXYX \\ &= Y^2XY^2X \end{aligned}$$

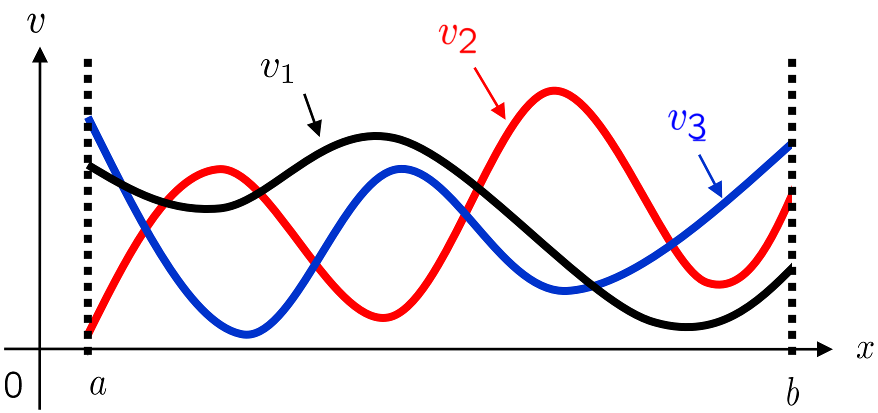
For $A, B \in \mathbf{G}^+$, A is topologically equivalent to B $\Leftrightarrow$ A is modified to B applying Artin's formula at most finitely many times.

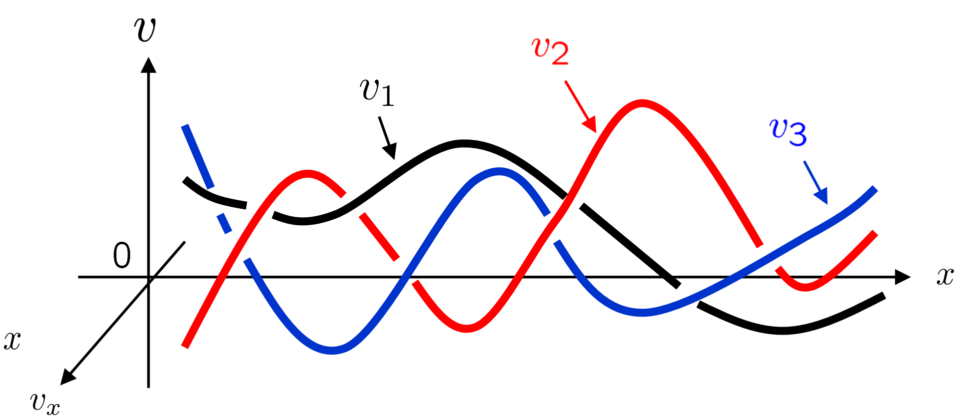
[Ghrist, Van den Berg, Vandervorst (2003)]

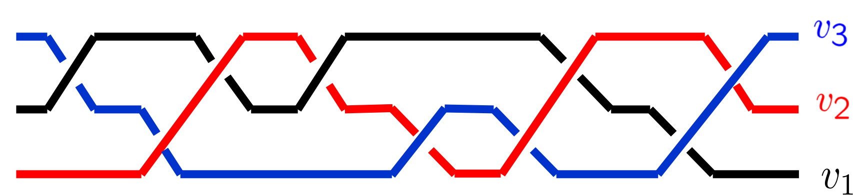
Let v_1, v_2, v_3 be solutions of a unif. parabolic eq.

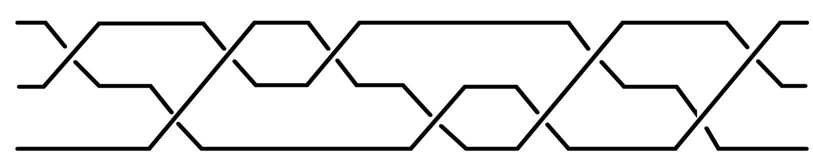
$$v_t = \alpha(x)v_{xx} + \beta(x)v_x + f(x, v, v_x) \quad \text{in } (a,b) \times (T_1, T_2)$$

s.t. $v_i(t)$ and $v_j(t)$ $(i, j = 1, 2, 3)$ transversally intersect.







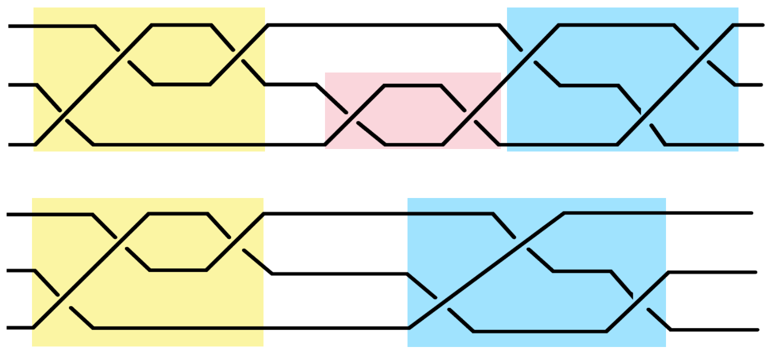


For $A, B \in \mathbf{G}^+$, defined by Matano

B is a simple parabolic reduction of A ($A \Rrightarrow_1 B$)

$\Leftrightarrow$ there exist $C, D \in \mathbf{G}^+$ s.t.

$$A = CX^2D,\ B = CD \quad \text{or} \quad A = CY^2D,\ B = CD,$$



B is a parabolic reduction of A ($A \Rrightarrow B$)

$\Leftrightarrow {}^{\exists}A_1, A_2, \cdots, A_k \in \mathbf{G}^+$ s.t. $A \Rrightarrow_1 A_1 \Rrightarrow_1 \cdots \Rrightarrow_1 A_k \Rrightarrow_1 B$

Proposition 1 (Matano (2007), M. (2011))
Let $A, B, H \in \mathbf{G}^+$,
If $HA \Rrightarrow HB$, then $A \Rrightarrow B$.
If $AH \Rrightarrow BH$, then $A \Rrightarrow B$.

Lemma 1
For positive integer k, let

$$\tilde{A}_{2k} = (XY^2X)^k Y^{2k},\ \tilde{A}_{2k+1} = (XY^2X)^k XYX^{2k+1},$$
$$\tilde{B}_{2k} = X^2Y^{2k}XY^{2k}X,\ \tilde{B}_{2k+1} = X^2Y^{2k+1}X^{2k+1}Y.$$

Then $\tilde{A}_{2k} \not\Rrightarrow \tilde{B}_{2k}$ and $\tilde{A}_{2k+1} \not\Rrightarrow \tilde{B}_{2k+1}$.

Lemma 2
For positive integer k, let

$$\hat{A}_{2k} = (YX^2Y)^k X^{2k},\ \hat{A}_{2k+1} = (YX^2Y)^k YXY^{2k+1},$$
$$\hat{B}_{2k} = Y^2X^{2k}YX^{2k}Y,\ \hat{B}_{2k+1} = Y^2X^{2k+1}Y^{2k+1}X.$$

Then $\hat{A}_{2k} \not\Rrightarrow \hat{B}_{2k}$ and $\hat{A}_{2k+1} \not\Rrightarrow \hat{B}_{2k+1}$.

Lemma 3

Under the hypotheses of Theorem 1,

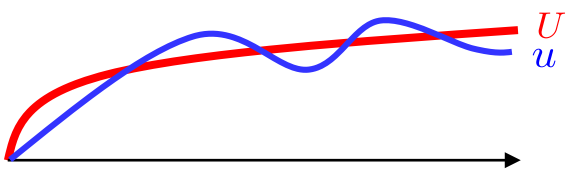
$$0 < \liminf_{t \to T_-} (T - t)^{\frac{n}{p-2}} u_x(0,t) \le \limsup_{t \to T_-} (T - t)^{\frac{n}{p-2}} u_x(0,t) < \infty.$$

Proof.

Let $0 < X_1(t) < X_2(t) < \cdots < X_n(t)$ be the vanishing intersections beween $u(t)$ and U at $(x,t) = (0,T)$.

Then we derive

$$\lim_{t \to T_-} X_n(t) = 0.$$



For $0 < D \ll 1$, there exists $t_0 < T$ s.t.

$$X_n(t) < D \quad \text{in } [t_0, T).$$

For $a > 0$, define a solution u_a by

$$u_a(x,t) := a^k u(a^{-1/2}x, T + a^{-1}t) \quad \text{in } (0, a^{1/2}R) \times (-aT, 0)$$

GBU time of u_a is $t = 0$

$u_a \to U$ in C^1_{loc} as $a \to \infty$

We construct a special sol. v with n vanishing intersections with U at $(x,t)=(0,T)$ satisfying

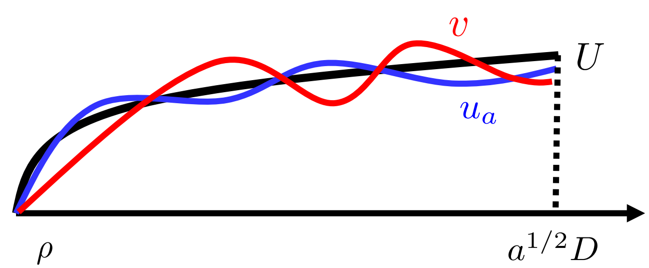
$$\lim_{t\to T_-}(T-t)^{n/(p-2)}v_x(0,t)=C \quad \text{for some } C>0$$

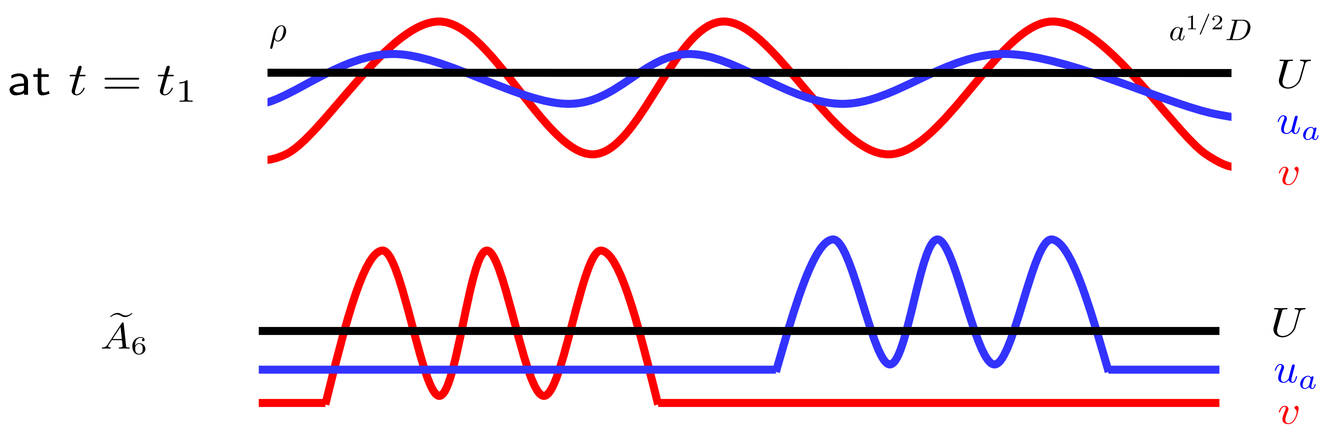
and there exist $a\gg 1$, $t_1<0$ s.t.

$$|U(a^{1/2}D)-v(a^{1/2}D,T+t)|>|U(a^{1/2}D)-u_a(a^{1/2}D,t)|$$

in $[t_1,0)$ and

$$z\big(v(T+t_1)-U:[0,a^{1/2}D]\big)$$
$$=z\big(v(T+t_1)-u_a(t_1):[0,a^{1/2}D]\big)=n.$$





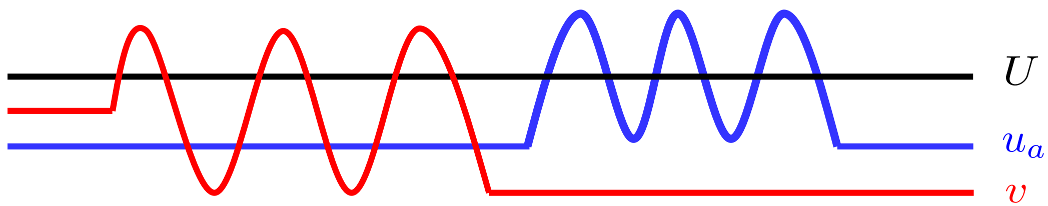
Assume that the first ineq. does not hold. $0 < \liminf_{t \to T_-} (T-t)^{\frac{n}{p-2}} u_x(0,t)$

Then there exists $t_2 \in (t_1, 0)$ s.t.

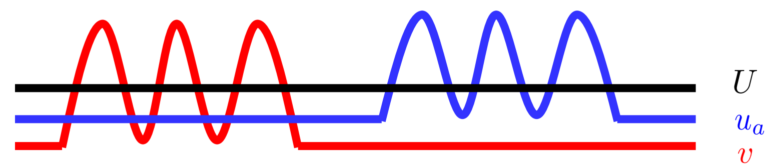
$$v(x, T+t_2) > u_a(x, t_2) \quad \text{for } 0 < x \ll 1.$$

$v(T+t_2) - u_a(t_2)$ loses one zero
(or odd number of zeros) at $x = 0$

at $t = t_2$



at $t = t_1$



For $0 < \lambda < 1$, let

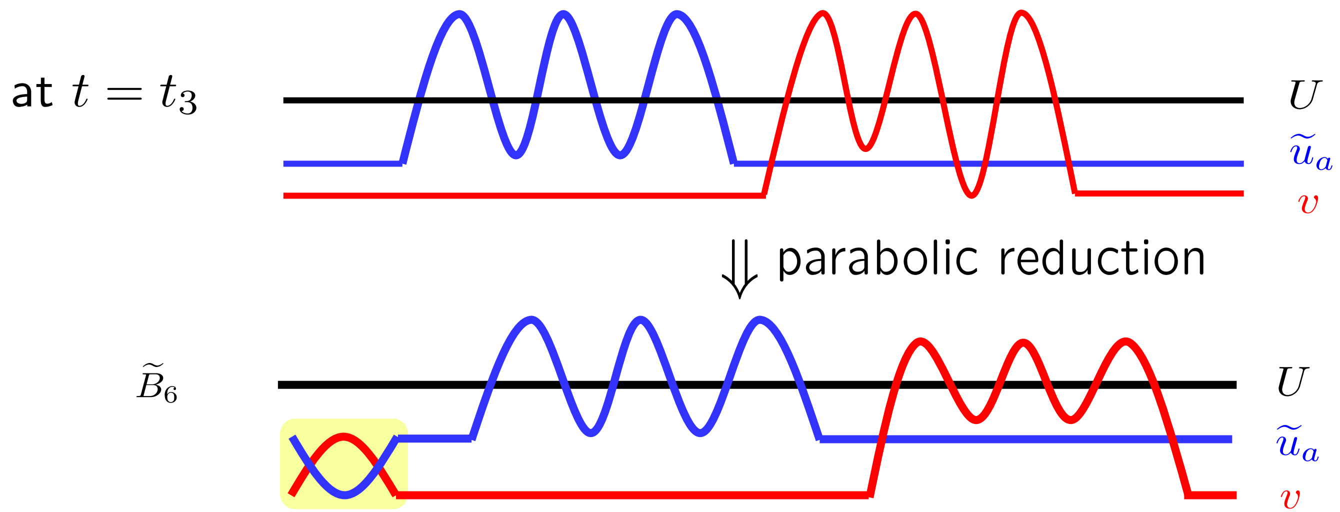
$$\tilde{u}_a(x,t) := \lambda^k u_a(\lambda^{-1/2} x, t_1 + \lambda^{-1}(t - t_1))$$

in $(0, \lambda^{1/2} a^{1/2} R) \times (t_1, \tilde{T})$ with $\tilde{T} := (1-\lambda)t_1 < 0$.

For λ close to 1, the above statements at $t = t_1, t_2$ hold with u_a replaced by $\tilde{u}_a$.

Since $\tilde{u}_a$ undergoes GBU at $t = \tilde{T}$, there is $t_3 \in (t_2, \tilde{T})$ s.t.

$$v(x, t_3) < \tilde{u}_a(x, t_3) \quad \text{for } 0 < x \ll 1.$$



The process from $t = t_1$ to $t = t_3$ means $\tilde{A}_n \Rrightarrow \tilde{B}_n$.
On the other hand, we have $\tilde{A}_n \not\Rrightarrow \tilde{B}_n$ by Lemma 1.
The contradiction implies the first ineq.

$$\limsup_{t \to T_-} (T - t)^{\frac{n}{p-2}} u_x(0, t) < \infty.$$

As for the last ineq., there is $C_1 > 0$ s.t. for $a \gg 1$
all zeros of $v_a(t) - U$ locate in $(0, C_1(-t)^{1/2})$ for $t \in [t_0, 0)$.
It suffices to take $u, v_a, [\rho, D]$ instead of $v, u_a, [\rho, a^{1/2} D]$. $\square$

Lemma 4

If a sol. u undergoes GBU at $(x,t)=(0,T)$, then for $\ell \in \mathbb{N}$,

$$\liminf_{t\to T_-} (T-t)^{\frac{\ell}{p-2}} u_x(0,t) = \limsup_{t\to T_-} (T-t)^{\frac{\ell}{p-2}} u_x(0,t).$$

Proof.

Assume for contradiction that there exist $0 < L_1 < L_2 < \infty$ s.t.

$$\liminf_{t\to T_-}(T-t)^{\frac{\ell}{p-2}} u_x(0,t) < L_1 < L_2 < \limsup_{t\to T_-}(T-t)^{\frac{\ell}{p-2}} u_x(0,t).$$

We construct a special sol. $\hat{u}$ s.t.

$$L_1 < \liminf_{t\to T_-}(T-t)^{\frac{\ell}{p-2}} \hat{u}_x(0,t) \le \limsup_{t\to T_-}(T-t)^{\frac{\ell}{p-2}} \hat{u}_x(0,t) < L_2$$

and

$$\hat{u}(x_0,t) \neq u(x_0,t) \ \text{ in } (t_0,T)$$

for some $x_0 \in (0,R), t_0 < T$.

Then

$$z(u(t) - \hat{u}(t) : [0, x_0]) \leq z(u(t_0) - \hat{u}(t_0) : [0, x_0]) \text{ in } [t_0, T).$$

On the other hand,

$$u_x(0, t_i) = \hat{u}_x(0, t_i) \quad \text{for some } t_i \nearrow T.$$

and hence

$$z(u(t) - \hat{u}(t) : [0, x_0]) = \infty.$$

This contradicts above. □

Proof of Theorem 1

It is known that for $t < T$ close to T,

$$u_x(x, t) = \left[m^{1-p}(t) + (p-1)x\right]^{-\frac{1}{p-1}} + O(x) \text{ for } 0 < x \ll 1,$$

where $m(t) = u_x(0, t)$.

This and Lemmas 3,4 imply the claim for u_x.

Integrating the formula of u_x yields the claim for u. □

Thank you for your attention!