Global existence and decay of solutions to Prandtl system with small analytic and Gevrey data

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Prandtl system

1904, Prandtl:

$$\begin{cases} \partial_t U + U\partial_x U + V\partial_y U - \partial_y^2 U + \partial_x p = 0, \\ \partial_x U + \partial_y V = 0, \quad (t,x,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\ U|_{y=0} = V|_{y=0} = 0 \quad \text{and} \quad \lim_{y\to+\infty} U(t,x,y) = w(t,x), \\ U|_{t=0} = U_0, \end{cases} \tag{1.1}$$

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- U and V represent the tangential and normal velocities of the boundary layer flow,
- $(w(t,x), p(t,x))$ are the traces of the tangential velocity and pressure of the outflow on the boundary, which satisfy Bernoulli's law:

$$\partial_t w + w\partial_x w + \partial_x p = 0. \tag{1.2}$$

There is no horizontal diffusion in the U equation of (1.1), the nonlinear term $V\partial_y U(\approx -y\partial_x U\partial_y U)$ loses one horizontal derivative in the process of energy estimate, **whether or not the Prandtl system with general data is well-posed in Sobolev spaces is still open.**

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- Gérard-Varet and Nguyen (2012), Guo and Nguyen (2011): The nonlinear ill-posedness in the sense of non-Lipschtiz continuity of the flow.

Positive results for the following classes of data:

Under a monotonic assumption on the tangential velocity of the outflow:

- Oleinik (1963): first introduced Crocco transformation and then proved the local existence and uniqueness of classical solutions to (1.1).
- Xin and Zhang (2004): with the additional “favorable" condition on the pressure, obtained the global existence of weak solutions to this system.
- Alexandre, Wang, Xu and Yang (*JAMS*, 2015), Masmoudi and Wong (*CPAM*, 2015): by ingenious use of the cancelation property of the bad terms containing the tangential derivative, proving the existence of local smooth solution to (1.1) in Sobolev space via performing energy estimates in weighted Sobolev spaces.

Positive results for the following classes of data:

Analytical data:

- Sammartino and Caflisch (*CMP*, 1998): local well-posedness result of (1.1) with data which is analytic in both x and y variables;
- Lombardo, Cannone and Sammartino (*Siam JMA* 2003): removed the analyticity in y variable;
  The main argument used in above papers is to apply the abstract Cauchy-Kowalewskaya (CK) theorem.
- Gérvard-Varet and Masmoudi (*Ann. Sci. École Norm. Sup.* , 2015): for a class of data with Gevrey regularity. W. X. Li and T. Yang (*JEMS* 2020) with non-degenerate critical points. This result was improved to be optimal in sense of by Dietert and Gérvard-Varet in (*Ann. PDE*, 2019);
- Zhang and Zhang (*JFA*: 2016) the long time existence for Prandtl system with small analytic data and an almost global existence result was provided by Ignatova and Vicol in (*ARMA*, 2016).

Oleinik and Samokhin: open problem **4** on page 500 of *Mathematical models in boundary layer theory*, 1999:

“It has been shown in Chapter 4 that under certain assumptions the system of nonstationary two-dimensional boundary layer admits one and only one solution in the domain $D = \{0 < t < T, 0 < x < X, 0 < y < \infty\}$ either for small enough T and any $X > 0$ or small enough X and any $T > 0$. What are the conditions ensuring the existence and uniqueness of a solution of the nonstationary Prandtl system in the domain D with arbitrary X and T?"

It has also pointed out by Grenier, Guo, and Nguyen in a serious papers, in order to make progress towards proving or disproving the inviscid limit of the Navier-Stokes equations, one must understand its behavior on a longer time interval than the one which causes the instability used to prove ill-posedness.

Set up of the problem:

$w(t,x)$ in (1.1) to be $\varepsilon f(t)$ with $f(0)=0$, then (1.2) $\Rightarrow$ $\partial_x p = -\varepsilon f'(t)$.

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Let $\chi \in C^\infty[0,\infty)$ with $\chi(y) = \begin{cases} 1 & \text{if } y \geq 2 \\ 0 & \text{if } y \leq 1, \end{cases}$ we denote $W \overset{\text{def}}{=} U - \varepsilon f(t)\chi(y)$.

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Then W solves

$$\begin{cases} \partial_t W + (W + \varepsilon f(t)\chi(y))\partial_x W + V\partial_y (W + \varepsilon f(t)\chi(y)) - \partial_y^2 W = \varepsilon m, \\ \partial_x W + \partial_y V = 0, \quad (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2_+, \\ W|_{y=0} = V|_{y=0} = 0 \quad \text{and} \quad \lim_{y\to+\infty} W(t,x,y) = 0, \\ W|_{t=0} = U_0, \end{cases} \tag{1.3}$$

where $\mathbb{R}^2_+ \overset{\text{def}}{=} \mathbb{R} \times \mathbb{R}_+$ and $m(t,y) \overset{\text{def}}{=} (1-\chi(y))f'(t) + f(t)\chi''(y)$.

We introduce u^s via

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = \varepsilon m(t,y), \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u^s|_{y=0} = 0 \quad \text{and} \quad \lim_{y\to+\infty} u^s(t,y) = 0, \\ u^s|_{t=0} = 0. \end{cases} \tag{1.4}$$

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We set $u \stackrel{\text{def}}{=} W - u^s$ and $v \stackrel{\text{def}}{=} V$. Then (u,v) verifies

$$\begin{cases} \partial_t u + (u + u^s + \varepsilon f(t)\chi(y))\partial_x u + v\partial_y (u + u^s + \varepsilon f(t)\chi(y)) - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \quad (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}_+^2, \\ u|_{y=0} = v|_{y=0} = 0 \quad \text{and} \quad \lim_{y\to\infty} u(t,x,y) = 0, \\ u|_{t=0} = u_0 \stackrel{\text{def}}{=} U_0. \end{cases} \tag{1.5}$$

Due to $\partial_x u + \partial_y v = 0$, φ so that $u = \partial_y \varphi$ and $v = -\partial_x \varphi$. Then by integrating the u equation of (1.5) with respect to y variable over $[y, \infty)$,

$$\partial_t \varphi + (u + u^s + \varepsilon f(t)\chi(y))\,\partial_x \varphi$$
$$+2\int_y^\infty (\partial_y (u + u^s + \varepsilon f(t)\chi(y'))\,\partial_x \varphi)\, dy' - \partial_y^2 \varphi = Q(t,x),$$

for some function $Q(t,x)$.

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In order to globally control the evolution of the analytic band, we introduce

$$G \stackrel{\text{def}}{=} u + \frac{y}{2\langle t\rangle}\varphi \quad \text{and} \quad g \stackrel{\text{def}}{=} \partial_y G = \partial_y u + \frac{y}{2\langle t\rangle} u + \frac{\varphi}{2\langle t\rangle}. \tag{1.7}$$

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The quantities G and g is inspired by the function $\mathfrak{g} \stackrel{\text{def}}{=} \partial_y u + \frac{y}{2\langle t\rangle} u$, which was introduced by Ignatova and Vicol (*ARMA*, 2016, also by Masmoudi and Wong *CPAM 2015*), where they basically proved that the weighted analytical norm of $\mathfrak{g}(t)$ decays like $\langle t\rangle^{-(\frac{5}{4})_-}$, which decays faster than the weighted analytical norm of u itself. We observe that $g - \mathfrak{g} = \frac{\varphi}{2\langle t\rangle}$. One novelty of this paper is to prove that the analytical norm of g is almost decays like $\langle t\rangle^{-\frac{7}{4}}$.

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$$\begin{cases} \partial_t G - \partial_y^2 G + \langle t\rangle^{-1} G + (u + u^s + \varepsilon f(t)\chi(y))\partial_x G + v\partial_y G \\ \qquad + v\partial_y\left(u^s + \varepsilon f(t)\chi(y)\right) - \frac{1}{2}\langle t\rangle^{-1} v\partial_y(y\varphi) \\ \qquad + \frac{y}{\langle t\rangle}\int_y^\infty \left(\partial_y\left(u + u^s + \varepsilon f(t)\chi(y')\right)\partial_x\varphi\right) dy' = 0, \\ G|_{y=0} = 0 \quad \text{and} \quad \lim_{y\to+\infty} G(t,x,y) = 0, \\ G|_{t=0} = G_0 \stackrel{\text{def}}{=} u_0 + \frac{y}{2}\varphi_0. \end{cases} \tag{1.8}$$

Main results

Theorem (M. Paicu and P. Zhang, *Arch. Ration. Mech. Anal.*, **241**, 2021 403-446.)

Let $\delta > 0$ *and* $f \in H^1(\mathbb{R}_+)$ *which satisfies*

$$\mathcal{C}_f \stackrel{def}{=} \int_0^\infty \langle t\rangle^{\frac{5}{4}} \left(|f(t)| + |f'(t)|\right) dt + \left(\int_0^\infty \langle t\rangle^{\frac{7}{2}} \left(f^2(t) + (f'(t))^2\right) dt\right)^{\frac{1}{2}} < \infty. \tag{1.9}$$

Let $u_0 = \partial_y \varphi_0$ *satisfy* $u_0(x,0) = 0$, $\int_0^\infty u_0\, dy = 0$ *and* $\left\|e^{\frac{y^2}{8}} e^{\delta|D_x|}(\varphi_0, u_0)\right\|_{\mathcal{B}^{\frac{1}{2},0}} < \infty$. *We assume moreover that* $G_0 = u_0 + \frac{y}{2}\varphi_0$ *satisfies*

$$\left\|e^{\frac{y^2}{8}} e^{\delta|D_x|} G_0\right\|_{\mathcal{B}^{\frac{1}{2},0}} \leq c_0 \tag{1.10}$$

for some c_0 *sufficiently small. Then* (1.4) *has a solution* u^s *and there exists* $\varepsilon_0 > 0$ *so that for* $\varepsilon \leq \varepsilon_0$, *the system (1.5) has a unique global solution u which satisfies*

$$\left\|e^{\frac{y^2}{8\langle t\rangle}} e^{\frac{\delta}{2}|D_x|} u\right\|_{L^\infty(\mathbb{R}_+;\mathcal{B}^{\frac{1}{2},0})} + \left\|e^{\frac{y^2}{8\langle t\rangle}} e^{\frac{\delta}{2}|D_x|} \partial_y u\right\|_{L^2(\mathbb{R}_+;\mathcal{B}^{\frac{1}{2},0})} \leq C \left\|e^{\frac{y^2}{8}} e^{\delta|D_x|} u_0\right\|_{\mathcal{B}^{\frac{1}{2},0}}. \tag{1.11}$$

Main result

Furthermore, for any $t>0$, there hold

$$\begin{aligned}\Big\|\langle t\rangle^{\frac34}e^{\frac{y^2}{8\langle t\rangle}}e^{\frac{\delta}{2}|D_x|}u(t)\Big\|_{\mathcal{B}^{\frac12,0}}+\Big\|\langle t'\rangle^{\frac34}e^{\frac{y^2}{8\langle t\rangle}}e^{\frac{\delta}{2}|D_x|}\partial_y u\Big\|_{L^2(t/2,t;\mathcal{B}^{\frac12,0})}&\\ \leq C\big\|e^{\frac{y^2}{8}}e^{\delta|D_x|}(\varphi_0,u_0)\big\|_{\mathcal{B}^{\frac12,0}},&\qquad (2.12)\end{aligned}$$

$$\Big\|\langle t\rangle^{\frac54}e^{\frac{y^2}{8\langle t\rangle}}e^{\frac{\delta}{2}|D_x|}G(t)\Big\|_{\mathcal{B}^{\frac12,0}}+\Big\|\langle t'\rangle^{\frac54}e^{\frac{y^2}{8\langle t\rangle}}e^{\frac{\delta}{2}|D_x|}\partial_y G\Big\|_{L^2(t/2,t;\mathcal{B}^{\frac12,0})}\leq C\big\|e^{\frac{y^2}{8}}e^{\delta|D_x|}G_0\big\|_{\mathcal{B}^{\frac12,0}},$$

and

$$\begin{aligned}\Big\|\langle t\rangle^{\frac54}e^{\frac{\gamma y^2}{8\langle t\rangle}}e^{\frac{\delta}{2}|D_x|}u(t)\Big\|_{\mathcal{B}^{\frac12,0}}+\Big\|\langle t'\rangle^{\frac54}e^{\frac{\gamma y^2}{8\langle t\rangle}}e^{\frac{\delta}{2}|D_x|}\partial_y u\Big\|_{L^2(t/2,t;\mathcal{B}^{\frac12,0})}&\\ \leq C\big\|e^{\frac{y^2}{8}}e^{\delta|D_x|}G_0\big\|_{\mathcal{B}^{\frac12,0}},&\qquad (2.13)\end{aligned}$$

for any $\gamma\in(0,1)$.

- In Zhang-Zhang (*JFA* 2016) and Ignatova-Vicol (*ARMA*, 2016), only a lower bound of the lifespan to the solution was obtained. Xie and Yang (2019) obtained similar result for MHD boundary layer
- E and Enquist (*CPAM* 1998): if $u_0(0,y)=0$ and $a_0(y)=-\partial_x u_0(0,y)$ is nonnegative and of compact support such that

$$E(a_0)<0 \quad \text{with} \quad E(a) \stackrel{\text{def}}{=} \int_0^\infty \Big(\frac{1}{2}(\partial_y a(y))^2-\frac{1}{4}a^3(y)\Big)\,dy<0. \tag{2.14}$$

Then any smooth solution of (1.1) does not exist globally in time.
For small initial data $u_0(x,y)=\eta\phi(x,y)$, we have $a_0(y)=-\eta\partial_x\phi(0,y)$ and

$$E(a_0)=\frac{\eta^2}{2}\int_0^\infty(\partial_x\partial_y\phi(0,y))^2\,dy-\frac{\eta^3}{4}\int_0^\infty(\partial_x\phi(0,y))^3\,dy,$$

which can not satisfy $E(a_0)<0$ for η sufficiently small except that $\partial_x\partial_y\phi(0,y)=0$. However, in the later case, due to the fact that the solution decays to zero as y approaching to $+\infty$, we have $a_0(y)=\partial_x\phi(0,y)=0$, which implies $E(a_0)=0$ so that (2.14) can not be satisfied in both cases.

- The idea of closing the analytic energy estimate, (1.11), for solutions of (1.5) goes back to a paper by Chemin (2004) who introduced a tool to make analytical type estimates and controlling the size of the analytic radius simultaneously. It was used in the context of anisotropic Navier-Stokes system (Chemin-Gallagher-Paicu *Ann of Math.* 2011) which implies the global well-posedness of three dimensional Navier-Stokes system with a class of “ill prepared data", which is slowly varying in the vertical variable, namely of the form εx_3, and the $B^{-1}_{\infty,\infty}(\mathbb{R}^3)$ norm of which blow up as the small parameter goes to zero.
- We mention that in our previous paper with Paicu-Zhang-Zhang (*Adv. Math.* 372 (2020)), we used the weighted analytic norm of $\partial_y u$ to control the analytic band of the solutions, which seems more obvious than the weighted analytic norm of g, which is defined by (1.7). Since there, we worked on Prandt type system in a strip with homogenous boundary condition so that we can use the classical Poincaré inequality to derive the exponential decay estimates for the solutions. Therefore we have a global control for the analytic band. Here in the upper space, by using another type of Poincaré inequality, (3.17), which yields decay of a sort of weighted analytic norm to $\partial_y u$ like $\langle t\rangle^{-\frac{5}{4}}$ as the time t going to ∞. Yet this estimate can not guarantee the quantity: $\int_0^\infty \langle t\rangle^{\frac{1}{4}} \|e^\Psi \partial_y u_\Phi(t)\|_{\mathcal{B}^{\frac{1}{2},0}} dt$, to be finite, which will be crucial to globally control the analytic band of the solutions to (1.5).

Functional spaces

$$\Delta_k^{\mathrm{h}} a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\widehat{a}), \qquad S_k^{\mathrm{h}} a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\widehat{a}), \tag{3.15}$$

where $\widehat{a}(\xi, y) = \mathcal{F}_{x\to\xi}(a)(\xi, y)$, and $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

$$\mathrm{Supp}\ \varphi \subset \Big\{\tau \in \mathbb{R} \ / \ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\Big\} \quad \text{and} \quad \forall \tau > 0, \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\tau) = 1,$$

$$\mathrm{Supp}\ \chi \subset \Big\{\tau \in \mathbb{R} \ / \ |\tau| \leq \frac{4}{3}\Big\} \qquad \text{and} \qquad \chi(\tau) + \sum_{k\geq 0} \varphi(2^{-k}\tau) = 1.$$

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Definition

Let s in $\mathbb{R}$. For u in $\mathcal{S}'_h(\mathbb{R}^2_+)$, which means that u is in $\mathcal{S}'(\mathbb{R}^2_+)$ and satisfies $\lim_{k\to-\infty} \|S_k^{\mathrm{h}} u\|_{L^\infty} = 0$, we set

$$\|u\|_{\mathcal{B}^{s,0}} \stackrel{def}{=} \Big\| \big(2^{ks} \|\Delta_k^{\mathrm{h}} u\|_{L^2_+}\big)_{k\in\mathbb{Z}} \Big\|_{\ell^1(\mathbb{Z})}.$$

- *For $s \leq \frac{1}{2}$, we define $\mathcal{B}^{s,0}(\mathbb{R}^2_+) \stackrel{def}{=} \big\{u \in \mathcal{S}'_h(\mathbb{R}^2_+) \ \big|\ \|u\|_{\mathcal{B}^{s,0}} < \infty\big\}$.*
- *If ℓ is a positive integer and if $\ell - \frac{1}{2} < s \leq \ell + \frac{1}{2}$, then we define $\mathcal{B}^{s,0}(\mathbb{R}^2_+)$ as the subset of distributions u in $\mathcal{S}'_h(\mathbb{R}^2_+)$ such that $\partial_x^\ell u$ belongs to $\mathcal{B}^{s-\ell,0}(\mathbb{R}^2_+)$.*

Definition

Let $p \in [1, +\infty]$ *and* $T_0, T \in [0, +\infty]$. *We define* $\widetilde{L}^p(T_0, T; \mathcal{B}^{s,0}(\mathbb{R}^2_+))$ *as the completion of* $C([T_0, T]; \mathcal{S}(\mathbb{R}^2_+))$ *by the norm*

$$\|a\|_{\widetilde{L}^p(T_0,T;\mathcal{B}^{s,0})} \stackrel{def}{=} \sum_{k\in\mathbb{Z}} 2^{ks} \Big(\int_{T_0}^{T} \|\Delta_k^{\mathrm{h}} a(t)\|_{L^2_+}^p \, dt \Big)^{\frac{1}{p}}$$

with the usual change if $p = \infty$. *In particular, when* $T_0 = 0$, *we shall denote* $\|a\|_{\widetilde{L}^p_T(\mathcal{B}^{s,0})} \stackrel{def}{=} \|a\|_{\widetilde{L}^p(0,T;\mathcal{B}^{s,0})}$ *for simplicity.*

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Definition

Let $f(t) \in L^1_{loc}(\mathbb{R}_+)$ be a nonnegative function and $t_0, t \in [0, \infty]$. We define

$$\|a\|_{\widetilde{L}^p_{t_0,t;f}(\mathcal{B}^{s,0})} \stackrel{def}{=} \sum_{k\in\mathbb{Z}} 2^{ks}\Big(\int_{t_0}^{t} f(t')\|\Delta_k^{\mathrm{h}} a(t')\|_{L^2_+}^p \, dt'\Big)^{\frac{1}{p}}. \qquad (3.16)$$

When $t_0 = 0$, we simplify the notation $\|a\|_{\widetilde{L}^p_{0,t;f}(\mathcal{B}^{s,0})}$ as $\|a\|_{\widetilde{L}^p_{t,f}(\mathcal{B}^{s,0})}$.

Outline of the proof

Lemma

Let $\Psi(t,y) \stackrel{def}{=} \frac{y^2}{8\langle t\rangle}$ *and* d *be a nonnegative integer. Let* u *be a smooth enough function on* $\mathbb{R}^d \times \mathbb{R}_+$ *which decays to zero sufficiently fast as* y *approaching to* $+\infty$*. Then one has*

$$\int_{\mathbb{R}^d\times\mathbb{R}_+} |\partial_y u(X,y)|^2 e^{2\Psi}\,dX\,dy \geq \frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} |u(X,y)|^2 e^{2\Psi}\,dX\,dy. \tag{3.17}$$

Proof.

Similar to Ignatova-Vicol (*ARMA*, 2016):

$$\begin{aligned}\int_{R_+} u^2(X,y)e^{\frac{y^2}{4\langle t\rangle}}\,dy &= \int_{R_+} (\partial_y y)u^2(X,y)e^{\frac{y^2}{4\langle t\rangle}}\,dy \\ &= -2\int_{R_+} yu(X,y)\partial_y u(X,y)e^{\frac{y^2}{4\langle t\rangle}}\,dy - \frac{1}{2\langle t\rangle}\int_{R_+} y^2u^2(X,y)e^{\frac{y^2}{4\langle t\rangle}}\,dy.\end{aligned}$$

□

Proof.

$$\int_{\mathbb{R}^d\times\mathbb{R}_+} u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy+\frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy=-2\int_{\mathbb{R}^d\times\mathbb{R}_+} yu\partial_y u e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy$$

$$\leq 2\left(\frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy\right)^{1/2}\left(2\langle t\rangle\int_{\mathbb{R}^d\times\mathbb{R}_+}(\partial_y u)^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy\right)^{1/2}$$

$$\leq \frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy+2\langle t\rangle\int_{\mathbb{R}^d\times\mathbb{R}_+}(\partial_y u)^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy.$$

□

Proof.

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}_+} u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy+\frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy=-2\int_{\mathbb{R}^d\times\mathbb{R}_+} yu\partial_y u e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy\\
&\leq 2\Big(\frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy\Big)^{1/2}\Big(2\langle t\rangle\int_{\mathbb{R}^d\times\mathbb{R}_+}(\partial_y u)^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy\Big)^{1/2}\\
&\leq \frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy+2\langle t\rangle\int_{\mathbb{R}^d\times\mathbb{R}_+}(\partial_y u)^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy.\end{aligned}$$

□

By virtue of Lemma 3.1, we get, by using a standard argument of energy estimate to the system (1.4), that

$$\|e^{\frac{y^2}{8\langle t\rangle}}\partial_y u^s(t)\|_{L^2_v}\leq C\langle t\rangle^{-\frac{3}{4}}. \tag{3.18}$$

Proof.

$$\int_{\mathbb{R}^d\times\mathbb{R}_+} u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy+\frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy=-2\int_{\mathbb{R}^d\times\mathbb{R}_+} yu\partial_y u e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy$$

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$$\leq \frac{1}{2\langle t\rangle}\int_{\mathbb{R}^d\times\mathbb{R}_+} y^2u^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy+2\langle t\rangle\int_{\mathbb{R}^d\times\mathbb{R}_+}(\partial_y u)^2 e^{\frac{y^2}{4\langle t\rangle}}\,dX\,dy.$$

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By virtue of Lemma 3.1, we get, by using a standard argument of energy estimate to the system (1.4), that

$$\|e^{\frac{y^2}{8\langle t\rangle}}\partial_y u^s(t)\|_{L^2_{\rm v}}\leq C\langle t\rangle^{-\frac{3}{4}}. \tag{3.18}$$

which can not guarantee that the quantity $\int_0^\infty \langle t\rangle^{\frac14}\|e^{\frac{y^2}{8\langle t\rangle}}\partial_y u^s(t)\|_{L^2_{\rm v}}\,dt$ is finite.

Let $u^s(t,y) = \partial_y \psi^s(t,y)$. And we define ψ^s through

$$\begin{cases} \partial_t \psi^s - \partial_y^2 \psi^s = \varepsilon M(t,y), \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \psi^s|_{y=0} = 0 \quad \text{and} \quad \lim_{y\to+\infty} \psi^s = 0, \\ \psi^s|_{t=0} = 0, \end{cases} \tag{3.19}$$

where

$$M(t,y) \overset{\text{def}}{=} -\int_y^\infty (1-\chi(y'))\,dy' f'(t) + f(t)\chi'(y), \tag{3.20}$$

so that m in (1.3) equals to $\partial_y M$, and $M(t,y)$ is supported on the interval $[0,2]$.

Let $u^s(t,y)=\partial_y\psi^s(t,y)$. And we define ψ^s through

$$\begin{cases} \partial_t\psi^s-\partial_y^2\psi^s=\varepsilon M(t,y), \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}_+,\\ \psi^s|_{y=0}=0 \quad \text{and} \quad \lim_{y\to+\infty}\psi^s=0,\\ \psi^s|_{t=0}=0, \end{cases} \tag{3.19}$$

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so that m in (1.3) equals to $\partial_y M$, and $M(t,y)$ is supported on the interval $[0,2]$.

Let

$$G^s\overset{\text{def}}{=}u^s+\frac{y}{2\langle t\rangle}\psi^s \tag{3.21}$$

$$\begin{cases} \partial_t G^s-\partial_y^2 G^s+\langle t\rangle^{-1}G^s=\varepsilon H \quad \text{with} \quad H\overset{\text{def}}{=}m+\frac{y}{2\langle t\rangle}M,\\ G^s|_{y=0}=0 \quad \text{and} \quad \lim_{y\to+\infty}G^s(t,y)=0,\\ G^s|_{t=0}=0, \end{cases} \tag{3.22}$$

With G^s being determined by (3.22), by virtue of (3.21) and $\psi^s|_{y=0}=0$, we obtain

$$\psi^s(t,y)=e^{-\frac{y^2}{4\langle t\rangle}}\int_0^y e^{-\frac{(y')^2}{4\langle t\rangle}}G^s(t,y')\,dy' \quad \text{and} \quad u^s(t,y)\overset{\text{def}}{=}\partial_y\psi^s(t,y). \tag{3.23}$$

As in Chemin (2004); Chemin-Gallagher-Paicu (*Ann of Math.* 2011) etc, for any locally bounded function Φ on $\mathbb{R}^+ \times \mathbb{R}$, we define

$$u_\Phi(t,x,y) = \mathcal{F}^{-1}_{\xi\to x}\big(e^{\Phi(t,\xi)}\widehat{u}(t,\xi,y)\big). \tag{3.24}$$

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Let $\langle t\rangle \overset{\text{def}}{=} 1+t$, the phase function Φ is defined by

$$\Phi(t,\xi) \overset{\text{def}}{=} (\delta - \lambda\theta(t))|\xi|, \tag{3.25}$$

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Let G and G^s be determined respectively by (1.7) and (3.21), we introduce a key quantity $\theta(t)$ to describe the evolution of the analytic band to the solutions of (1.5):

$$\begin{cases} \dot{\theta}(t) = \langle t \rangle^{\frac{1}{4}} \big(\|e^\Psi \partial_y G^s(t)\|_{L^2_v} + \varepsilon f(t)\|e^\Psi \chi'\|_{L^2_v} + \|e^\Psi \partial_y G_\Phi(t)\|_{\mathcal{B}^{\frac{1}{2},0}}\big), \\ \theta|_{t=0} = 0. \end{cases} \tag{3.26}$$

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the weighted function $\Psi(t,y)$ is determined by

$$\Psi(t,y) \overset{\text{def}}{=} \frac{y^2}{8\langle t\rangle}, \tag{3.27}$$

which satisfies

$$\partial_t\Psi(t,y) + 2(\partial_y\Psi(t,y))^2 = 0. \tag{3.28}$$

Theorem

Let Φ *and* Ψ *be defined respectively by* (3.25) *and* (3.27). *Then under the assumptions of Theorem 1.1, there exist positive constants* c_0, ε_0 *and* λ *so that for* u^s *determined by* (3.23) *and* $\varepsilon \le \varepsilon_0$, *the system (1.5) has a unique global solution u which satisfies* $\sup_{t\in[0,\infty)} \theta(t) \le \frac{\delta}{2\lambda}$, *and*

$$\big\|e^{\Psi} u_\Phi\big\|_{\widetilde{L}^\infty(\mathbb{R}_+;\mathcal{B}^{\frac12,0})} + \big\|e^{\Psi}\partial_y u_\Phi\big\|_{\widetilde{L}^2(\mathbb{R}_+;\mathcal{B}^{\frac12,0})} \le C\big\|e^{\frac{y^2}{8}} e^{\delta|D_x|} u_0\big\|_{\mathcal{B}^{\frac12,0}}. \tag{3.29}$$

Moreover, for G given by (1.7), *there exists a positive constant C so that for any* $t>0$ *and* $\gamma \in (0,1)$, *there hold*

$$\begin{aligned}
&\big\|\langle t'\rangle^{\frac34} e^{\Psi} u_\Phi\big\|_{L^\infty(\mathbb{R}^+;\mathcal{B}^{\frac12,0})} + \big\|\langle t'\rangle^{\frac34} e^{\Psi}\partial_y u_\Phi\big\|_{\widetilde{L}^2(t/2,t;\mathcal{B}^{\frac12,0})} \le C\big\|e^{\frac{y^2}{8}} e^{\delta|D_x|}(\varphi_0,u_0)\big\|_{\mathcal{B}^{\frac12,0}},\\
&\big\|\langle t'\rangle^{\frac54} e^{\Psi} G_\Phi\big\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac12,0})} + \big\|\langle t'\rangle^{\frac54} e^{\Psi}\partial_y G_\Phi\big\|_{\widetilde{L}^2(t/2,t;\mathcal{B}^{\frac12,0})} \le C\big\|e^{\frac{y^2}{8}} e^{\delta|D_x|} G_0\big\|_{\mathcal{B}^{\frac12,0}},\\
&\big\|\langle t'\rangle^{\frac54} e^{\gamma\Psi} u_\Phi\big\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac12,0})} + \big\|\langle t'\rangle^{\frac54} e^{\gamma\Psi}\partial_y u_\Phi\big\|_{\widetilde{L}^2(t/2,t;\mathcal{B}^{\frac12,0})} \le C\big\|e^{\frac{y^2}{8}} e^{\delta|D_x|} G_0\big\|_{\mathcal{B}^{\frac12,0}}.
\end{aligned} \tag{3.30}$$

Proposition

Let $f(t) \in H^1(\mathbb{R}_+)$ *and satisfy* (1.9). *Then for* G^s *being determined by* (3.22), *one has*

$$\int_0^\infty \langle t\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t)\|_{L^2_v}\, dt \leq C\mathcal{C}_f\varepsilon, \tag{3.31}$$

for the constant $\mathcal{C}_f$ *given by* (1.9).

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for the constant $\mathcal{C}_f$ given by (1.9)*.*

In what follows, we shall always assume that $t < T^*$ with T^* being determined by

$$T^* \overset{\text{def}}{=} \sup\big\{\, t > 0,\ \ \theta(t) < \delta/\lambda\big\}. \tag{3.32}$$

So that by virtue of (3.25), for any $t < T^*$, there holds the following convex inequality

$$\Phi(t,\xi) \leq \Phi(t,\xi-\eta) + \Phi(t,\eta) \quad \text{for} \quad \forall\, \xi,\eta \in \mathbb{R}. \tag{3.33}$$

Proposition

Let φ *be a smooth enough solution of* (1.6)*. Then there exists a large enough constant* λ *so that for any nonnegative and non-decreasing function* $\hbar \in C^1(\mathbb{R}_+)$ *and any* $t_0 \in [0,t]$ *with* $t < T^*$*, one has*

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{1}{2},0})} \leq C\|e^{\frac{y^2}{8}} e^{\delta|D_x|}\varphi_0\|_{\mathcal{B}^{\frac{1}{2},0}}, \tag{3.34}$$

and

$$\begin{aligned}&\|\hbar^{\frac{1}{2}} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^\infty(t_0,t;\mathcal{B}^{\frac{1}{2},0})} + \|\hbar^{\frac{1}{2}} e^{\Psi}\partial_y\varphi_\Phi\|_{\widetilde{L}^2(t_0,t;\mathcal{B}^{\frac{1}{2},0})}\\ &\qquad \leq \|\hbar^{\frac{1}{2}} e^{\Psi}\varphi_\Phi(t_0)\|_{\mathcal{B}^{\frac{1}{2},0}} + \|\sqrt{\hbar'} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2(t_0,t;\mathcal{B}^{\frac{1}{2},0})}.\end{aligned} \tag{3.35}$$

Proposition

Let φ be a smooth enough solution of (1.6). Then there exists a large enough constant λ so that for any nonnegative and non-decreasing function $\hbar \in C^1(\mathbb{R}_+)$ and any $t_0 \in [0,t]$ with $t < T^*$, one has

$$\|\langle t'\rangle^{\frac14} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac12,0})} \le C\|e^{\frac{y^2}{8}} e^{\delta|D_x|}\varphi_0\|_{\mathcal{B}^{\frac12,0}}, \tag{3.34}$$

and

$$\begin{aligned}&\|\hbar^{\frac12} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^\infty(t_0,t;\mathcal{B}^{\frac12,0})} + \|\hbar^{\frac12} e^{\Psi}\partial_y\varphi_\Phi\|_{\widetilde{L}^2(t_0,t;\mathcal{B}^{\frac12,0})}\\ &\qquad \le \|\hbar^{\frac12} e^{\Psi}\varphi_\Phi(t_0)\|_{\mathcal{B}^{\frac12,0}} + \|\sqrt{\hbar'} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2(t_0,t;\mathcal{B}^{\frac12,0})}.\end{aligned} \tag{3.35}$$

Proposition

Let u be a smooth enough solution of (1.5). Then there exists a large enough constant λ so that for any $t < T^*$, we have

$$\|\langle t'\rangle^{\frac34} e^{\Psi}u_\Phi\|_{L^\infty_t(\mathcal{B}^{\frac12,0})} + \|\langle t'\rangle^{\frac34} e^{\Psi}\partial_y u_\Phi\|_{\widetilde{L}^2(t/2,t;\mathcal{B}^{\frac12,0})} \le C\|e^{\frac{y^2}{8}} e^{\delta|D_x|}(\varphi_0,u_0)\|_{\mathcal{B}^{\frac12,0}}. \tag{3.36}$$

Proposition

Let G be determined by (1.7). Then there exists a large enough constant λ so that for any $t < T^*$, we have

$$\|\langle t'\rangle^{\frac{5}{4}} e^{\Psi} G_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{1}{2},0})} + \|\langle t'\rangle^{\frac{5}{4}} e^{\Psi} \partial_y G_\Phi\|_{\widetilde{L}^2(t/2,t;\mathcal{B}^{\frac{1}{2},0})} + \int_0^t \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2},0}}\, dt' \leq C\|e^{\frac{y^2}{8}} e^{\delta|D_x|} G_0\|_{\mathcal{B}^{\frac{1}{2},0}}. \tag{3.37}$$

Let u and φ be smooth enough solutions of (1.5) and (1.6) respectively on $[0, T^\star)$, For any $t < T^*$ with T^* being defined by (3.32), we deduce that

$$\theta(t) \leq \int_0^t \langle t' \rangle^{\frac{1}{4}} \left(\|e^\Psi \partial_y G^s(t')\|_{L^2_v} + \|e^\Psi \partial_y G_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2},0}} \right) dt' + \varepsilon \int_0^t \|e^{\Psi(t')} \chi'\|_{L^2_v} \langle t' \rangle^{\frac{1}{4}} f(t')\, dt'.$$

Notice that $\mathrm{Supp}\ \chi' \subset [1,2]$, one has

$$\|e^{\Psi(t')} \chi'\|_{L^2_v} \leq e^{\frac{1}{2\langle t' \rangle}} \|\chi'\|_{L^2_v} \leq e^{\frac{1}{2}} \|\chi'\|_{L^2_v},$$

from which, Proposition 3.1 and Proposition 3.4, we infer

$$\theta(t) \leq C\left(\|e^{\frac{y^2}{8}} e^{\delta |D_x|} G_0\|_{\mathcal{B}^{\frac{1}{2},0}} + \varepsilon \mathcal{C}_f \right) \quad \text{for} \quad t < T^*. \tag{3.38}$$

Let u and φ be smooth enough solutions of (1.5) and (1.6) respectively on $[0,T^\star)$, For any $t<T^*$ with T^* being defined by (3.32), we deduce that

$$\theta(t)\leq\int_0^t\langle t'\rangle^{\frac14}\big(\|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\rm v}}+\|e^{\Psi}\partial_y G_\Phi(t')\|_{\mathcal{B}^{\frac12,0}}\big)\,dt'+\varepsilon\int_0^t\|e^{\Psi(t')}\chi'\|_{L^2_{\rm v}}\langle t'\rangle^{\frac14}f(t')\,dt'.$$

Notice that $\operatorname{Supp}\,\chi'\subset[1,2]$, one has

$$\|e^{\Psi(t')}\chi'\|_{L^2_{\rm v}}\leq e^{\frac{1}{2\langle t'\rangle}}\|\chi'\|_{L^2_{\rm v}}\leq e^{\frac12}\|\chi'\|_{L^2_{\rm v}},$$

from which, Proposition 3.1 and Proposition 3.4, we infer

$$\theta(t)\leq C\big(\|e^{\frac{y^2}{8}}e^{\delta|D_x|}G_0\|_{\mathcal{B}^{\frac12,0}}+\varepsilon\mathcal{C}_f\big)\quad\text{for}\quad t<T^*.\tag{3.38}$$

In particular, if we take c_0 in (1.10) and ε_0 so small that

$$C\,(c_0+\varepsilon_0\mathcal{C}_f)\leq\frac{\delta}{2\lambda}.\tag{3.39}$$

Then

$$\sup_{t\in[0,T^*)}\theta(t)\leq\frac{\delta}{2\lambda}\quad\text{for}\quad\varepsilon\leq\varepsilon_0.$$

Lemma

Let $G^s(t,y)$ *and* $\Psi(t,y)$ *be defined respectively by* (3.21) *and* (3.27). *Then for any* $t>0$, *one has*

$$\big\|\langle t'\rangle^{\frac{5}{4}} e^{\Psi} G^s\big\|_{L^\infty(\mathbb{R}^+;L^2_{\rm v})} \leq C\varepsilon \|\langle t'\rangle^{\frac{5}{4}} H\|_{L^1(\mathbb{R}_+;L^2_{\rm v})}, \tag{3.40}$$

and

$$\int_{\frac{t}{2}}^t \|\langle t'\rangle^{\frac{5}{4}} e^{\Psi}\partial_y G^s(t')\|_{L^2_{\rm v}}^2\,dt' \lesssim \varepsilon^2\big(\|\langle t'\rangle^{\frac{5}{4}} H\|_{L^1(\mathbb{R}_+;L^2_{\rm v})}^2 + \|\langle t'\rangle^{\frac{7}{4}} H\|_{L^2(\mathbb{R}_+;L^2_{\rm v})}^2\big), \tag{3.41}$$

for H *given by* (3.22).

By taking L^2_{v} inner product of the G^s equation of (3.22) with $e^{2\Psi}G^s$, we obtain

$$\Big(\partial_t G^s | e^{2\Psi} G^s\Big)_{L^2_{\mathrm{v}}} - \Big(\partial_y^2 G^s | e^{2\Psi} G^s\Big)_{L^2_{\mathrm{v}}} + \langle t\rangle^{-1} \|e^{\Psi} G^s(t)\|^2_{L^2_{\mathrm{v}}} = \varepsilon \Big(H | e^{2\Psi} G^s\Big)_{L^2_{\mathrm{v}}}.$$

By taking $L^2_{\rm v}$ inner product of the G^s equation of (3.22) with $e^{2\Psi}G^s$, we obtain

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$$\left(\partial_t G^s | e^{2\Psi} G^s\right)_{L^2_{\rm v}} = \frac{1}{2}\frac{d}{dt} \|e^{\Psi} G^s(t)\|^2_{L^2_{\rm v}} - \int_{\mathbb{R}^+} e^{2\Psi} \partial_t \Psi |G^s|^2 \, dy.$$

By taking $L^2_{\rm v}$ inner product of the G^s equation of (3.22) with $e^{2\Psi}G^s$, we obtain

$$\left(\partial_t G^s|e^{2\Psi}G^s\right)_{L^2_{\rm v}} - \left(\partial_y^2 G^s|e^{2\Psi}G^s\right)_{L^2_{\rm v}} + \langle t\rangle^{-1}\|e^{\Psi}G^s(t)\|^2_{L^2_{\rm v}} = \varepsilon\left(H|e^{2\Psi}G^s\right)_{L^2_{\rm v}}.$$

$$\left(\partial_t G^s|e^{2\Psi}G^s\right)_{L^2_{\rm v}} = \frac{1}{2}\frac{d}{dt}\|e^{\Psi}G^s(t)\|^2_{L^2_{\rm v}} - \int_{\mathbb{R}^+} e^{2\Psi}\partial_t\Psi|G^s|^2\,dy.$$

Due to $G^s|_{y=0}=0$,

$$\begin{aligned}-\left(\partial_y^2 G^s|e^{2\Psi}G^s\right)_{L^2_{\rm v}} =& \|e^{\Psi}\partial_y G^s\|^2_{L^2_{\rm v}} + 2\int_{\mathbb{R}^+} e^{2\Psi}\partial_y\Psi\partial_y G^s G^s\,dy\\ \geq& \frac{1}{2}\|e^{\Psi}\partial_y G^s\|^2_{L^2_{\rm v}} - 2\int_{\mathbb{R}^+} e^{2\Psi}(\partial_y\Psi)^2|G^s|^2\,dy.\end{aligned}$$

By taking L^2_{v} inner product of the G^s equation of (3.22) with $e^{2\Psi}G^s$, we obtain

$$\left(\partial_t G^s|e^{2\Psi}G^s\right)_{L^2_{\mathrm{v}}} - \left(\partial_y^2 G^s|e^{2\Psi}G^s\right)_{L^2_{\mathrm{v}}} + \langle t\rangle^{-1}\|e^{\Psi}G^s(t)\|_{L^2_{\mathrm{v}}}^2 = \varepsilon\left(H|e^{2\Psi}G^s\right)_{L^2_{\mathrm{v}}}.$$

$$\left(\partial_t G^s|e^{2\Psi}G^s\right)_{L^2_{\mathrm{v}}} = \frac{1}{2}\frac{d}{dt}\|e^{\Psi}G^s(t)\|_{L^2_{\mathrm{v}}}^2 - \int_{\mathbb{R}^+} e^{2\Psi}\partial_t\Psi|G^s|^2\,dy.$$

Due to $G^s|_{y=0} = 0$,

$$\begin{aligned}
-\left(\partial_y^2 G^s|e^{2\Psi}G^s\right)_{L^2_{\mathrm{v}}} &= \|e^{\Psi}\partial_y G^s\|_{L^2_{\mathrm{v}}}^2 + 2\int_{\mathbb{R}^+} e^{2\Psi}\partial_y\Psi\partial_y G^s G^s\,dy \\
&\geq \frac{1}{2}\|e^{\Psi}\partial_y G^s\|_{L^2_{\mathrm{v}}}^2 - 2\int_{\mathbb{R}^+} e^{2\Psi}(\partial_y\Psi)^2|G^s|^2\,dy.
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\|e^{\Psi}G^s(t)\|_{L^2_{\mathrm{v}}}^2 + \frac{1}{2}\|e^{\Psi}\partial_y G^s(t)\|_{L^2_{\mathrm{v}}}^2 + \langle t\rangle^{-1}\|e^{\Psi}G^s(t)\|_{L^2_{\mathrm{v}}}^2 \leq \varepsilon\|e^{\Psi}G^s(t)\|_{L^2_{\mathrm{v}}}\|e^{\Psi}H(t)\|_{L^2_{\mathrm{v}}}. \tag{3.42}$$

Applying Lemma 3.1 for $d = 0$ yields

$$\|e^{\Psi}\partial_y G^s(t)\|_{L^2_{\rm v}}^2 \geq \frac{1}{2\langle t\rangle}\|e^{\Psi} G^s(t)\|_{L^2_{\rm v}}^2,$$

Applying Lemma 3.1 for $d = 0$ yields

$$\|e^{\Psi}\partial_y G^s(t)\|_{L^2_{\rm v}}^2 \geq \frac{1}{2\langle t\rangle}\|e^{\Psi}G^s(t)\|_{L^2_{\rm v}}^2,$$

$$\frac{1}{2}\frac{d}{dt}\|e^{\Psi}G^s(t)\|_{L^2_{\rm v}}^2 + \frac{5}{4\langle t\rangle}\|e^{\Psi}G^s(t)\|_{L^2_{\rm v}}^2 \leq \varepsilon\|e^{\Psi}G^s(t)\|_{L^2_{\rm v}}\|e^{\Psi}H(t)\|_{L^2_{\rm v}},$$

Applying Lemma 3.1 for $d = 0$ yields

$$\left\|e^{\Psi}\partial_y G^s(t)\right\|_{L^2_v}^2 \geq \frac{1}{2\langle t\rangle}\left\|e^{\Psi} G^s(t)\right\|_{L^2_v}^2,$$

$$\frac{1}{2}\frac{d}{dt}\left\|e^{\Psi} G^s(t)\right\|_{L^2_v}^2 + \frac{5}{4\langle t\rangle}\left\|e^{\Psi} G^s(t)\right\|_{L^2_v}^2 \leq \varepsilon\left\|e^{\Psi} G^s(t)\right\|_{L^2_v}\left\|e^{\Psi} H(t)\right\|_{L^2_v},$$

$$\frac{d}{dt}\left\|e^{\Psi} G^s(t)\right\|_{L^2_v} + \frac{5}{4\langle t\rangle}\left\|e^{\Psi} G^s(t)\right\|_{L^2_v} \leq \varepsilon\left\|e^{\Psi} H(t)\right\|_{L^2_v},$$

and

$$\frac{d}{dt}\left(\langle t\rangle^{\frac{5}{4}}\left\|e^{\Psi} G^s(t)\right\|_{L^2_v}\right) \leq \varepsilon\langle t\rangle^{\frac{5}{4}}\left\|e^{\Psi} H(t)\right\|_{L^2_v}.$$

Integrating the above inequality over $[0, t]$ gives rise to (4.77).

$$
\begin{aligned}
\frac{d}{dt}\big\|e^{\Psi}G^s(t)\big\|_{L^2_{\mathrm{v}}}^2 + \big\|e^{\Psi}\partial_y G^s(t)\big\|_{L^2_{\mathrm{v}}}^2 + 2\langle t\rangle^{-1}\big\|e^{\Psi}G^s(t)\big\|_{L^2_{\mathrm{v}}}^2 \\
\leq 2\varepsilon\langle t\rangle^{\frac{1}{2}}\big\|e^{\Psi}H(t)\big\|_{L^2_{\mathrm{v}}}\langle t\rangle^{-\frac{1}{2}}\big\|e^{\Psi}G^s(t)\big\|_{L^2_{\mathrm{v}}} \\
\leq \varepsilon^2\langle t\rangle\big\|e^{\Psi}H(t)\big\|_{L^2_{\mathrm{v}}}^2 + \langle t\rangle^{-1}\big\|e^{\Psi}G^s(t)\big\|_{L^2_{\mathrm{v}}}^2.
\end{aligned} \tag{3.43}
$$

$$
\begin{aligned}
\frac{d}{dt}\left\|e^{\Psi}G^s(t)\right\|_{L^2_{\rm v}}^2+\left\|e^{\Psi}\partial_y G^s(t)\right\|_{L^2_{\rm v}}^2+&2\langle t\rangle^{-1}\left\|e^{\Psi}G^s(t)\right\|_{L^2_{\rm v}}^2\\
&\leq 2\varepsilon\langle t\rangle^{\frac12}\left\|e^{\Psi}H(t)\right\|_{L^2_{\rm v}}\langle t\rangle^{-\frac12}\left\|e^{\Psi}G^s(t)\right\|_{L^2_{\rm v}}\\
&\leq \varepsilon^2\langle t\rangle\left\|e^{\Psi}H(t)\right\|_{L^2_{\rm v}}^2+\langle t\rangle^{-1}\left\|e^{\Psi}G^s(t)\right\|_{L^2_{\rm v}}^2.
\end{aligned}
\tag{3.43}
$$

Multiplying the above inequality by $\langle t\rangle^{\frac52}$ and then integrating the resulting inequality over $[t/2,t]$,

$$
\begin{aligned}
\int_{\frac t2}^t\left\|\langle t'\rangle^{\frac54}e^{\Psi}\partial_y G^s(t')\right\|_{L^2_{\rm v}}^2dt' \leq &\left\|\langle t/2\rangle^{\frac54}e^{\Psi}G^s(t/2)\right\|_{L^2_{\rm v}}^2\\
&+\frac52\int_{\frac t2}^t\langle t'\rangle^{\frac32}\left\|e^{\Psi}G^s(t')\right\|_{L^2_{\rm v}}^2dt'+\varepsilon^2\int_{\frac t2}^t\langle t'\rangle^{\frac72}\left\|e^{\Psi}H(t')\right\|_{L^2_{\rm v}}^2dt'\\
&\leq \max_{t'\in[0,t]}\left\|\langle t'\rangle^{\frac54}e^{\Psi}G^s(t')\right\|_{L^2_{\rm v}}^2\Big(1+\frac{5\ln 2}{2}\Big)+\varepsilon^2\left\|\langle t'\rangle^{\frac74}e^{\Psi}H\right\|_{L^2_t(L^2_{\rm v})}^2.
\end{aligned}
$$

Proof of Proposition 3.1 In view of (1.3) and (3.20), both m and M are supported in $[0,2]$ for any $t \geq 0$, so that we observe from (3.22) that

$$\|e^{\Psi}\partial_y G^s\|^2_{L^2_t(L^2_{\rm v})} + \int_{\frac{t}{2}}^{t} \|\langle t'\rangle^{\frac{5}{4}} e^{\Psi}\partial_y G^s(t')\|^2_{L^2_{\rm v}}\, dt' \leq C C_f^2 \varepsilon^2. \tag{3.44}$$

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While for any $t>1$, we fix an integer N_t so that $2^{N_t-1} \leq t < 2^{N_t}$, which implies $t/2 < 2^{N_t-1}$. Then we deduce from (3.44) that

$$\begin{aligned}\int_{2^{N_t-1}}^t \langle t'\rangle^{\frac{1}{4}}\|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\rm v}}\,dt' \leq &\Big(\int_{2^{N_t-1}}^t \langle t'\rangle^{-2}\,dt'\Big)^{\frac{1}{2}}\Big(\int_{t/2}^t \langle t'\rangle^{\frac{5}{2}}\|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\rm v}}^2\,dt'\Big)^{\frac{1}{2}}\\ \leq & C2^{-\frac{N_t}{2}}\,\mathcal{C}_f\varepsilon.\end{aligned}$$

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$$\begin{aligned}\int_{2^{N_t-1}}^t \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\rm v}} dt' \leq & \Big(\int_{2^{N_t-1}}^t \langle t'\rangle^{-2} dt'\Big)^{\frac{1}{2}} \Big(\int_{t/2}^t \langle t'\rangle^{\frac{5}{2}} \left\|e^{\Psi}\partial_y G^s(t')\right\|^2_{L^2_{\rm v}} dt'\Big)^{\frac{1}{2}} \\ \leq & C2^{-\frac{N_t}{2}} \mathcal{C}_f \varepsilon.\end{aligned}$$

For any $j \in [0, N_t - 2]$, we have

$$\int_{2^j}^{2^{j+1}} \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\rm v}} dt' \leq C2^{-\frac{j}{2}} \mathcal{C}_f \varepsilon.$$

Proof of Proposition 3.1 In view of (1.3) and (3.20), both m and M are supported in $[0,2]$ for any $t \geq 0$, so that we observe from (3.22) that

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While for any $t > 1$, we fix an integer N_t so that $2^{N_t-1} \leq t < 2^{N_t}$, which implies $t/2 < 2^{N_t-1}$. Then we deduce from (3.44) that

$$\begin{aligned}\int_{2^{N_t-1}}^{t} \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\mathrm{v}}}\, dt' \leq & \Big(\int_{2^{N_t-1}}^{t} \langle t'\rangle^{-2}\, dt'\Big)^{\frac{1}{2}} \Big(\int_{t/2}^{t} \langle t'\rangle^{\frac{5}{2}} \|e^{\Psi}\partial_y G^s(t')\|^2_{L^2_{\mathrm{v}}}\, dt'\Big)^{\frac{1}{2}} \\ \leq & C2^{-\frac{N_t}{2}}\, \mathcal{C}_f\varepsilon.\end{aligned}$$

For any $j \in [0, N_t - 2]$, we have

$$\int_{2^j}^{2^{j+1}} \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\mathrm{v}}}\, dt' \leq C2^{-\frac{j}{2}}\, \mathcal{C}_f\varepsilon.$$

$$\begin{aligned}\int_0^t \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\mathrm{v}}}\, dt' \leq & 2^{\frac{1}{4}} \int_0^1 \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\mathrm{v}}}\, dt' \\ & + \int_{2^{N_t-1}}^{t} \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\mathrm{v}}}\, dt' + \sum_{j=0}^{N_t-2} \int_{2^j}^{2^{j+1}} \langle t'\rangle^{\frac{1}{4}} \|e^{\Psi}\partial_y G^s(t')\|_{L^2_{\mathrm{v}}}\, dt' \\ \leq & C\mathcal{C}_f\varepsilon\Big(1 + \sum_{j=0}^{\infty} 2^{-\frac{j}{2}}\Big) \leq C\mathcal{C}_f\varepsilon.\end{aligned}$$

$$\partial_t \varphi_\Phi - \partial_{yy}\varphi_\Phi + \lambda\dot{\theta}(t)|D_h|\varphi_\Phi + [(u+u^s+\varepsilon f(t)\chi(y))\partial_x\varphi]_\Phi + 2\int_y^\infty \left[\partial_y(u+u^s+\varepsilon f(t)\chi(y'))\partial_x\varphi\right]_\Phi dy' = 0. \tag{3.45}$$

$$\partial_t \varphi_\Phi - \partial_{yy}\varphi_\Phi + \lambda\dot{\theta}(t)|D_h|\varphi_\Phi + [(u+u^s+\varepsilon f(t)\chi(y))\partial_x\varphi]_\Phi + 2\int_y^\infty \big[\partial_y(u+u^s+\varepsilon f(t)\chi(y'))\partial_x\varphi\big]_\Phi \, dy' = 0. \tag{3.45}$$

By applying the dyadic operator Δ_k^{h} to (3.45) and then taking the L^2_+ inner product of the resulting equation with $\hbar(t)e^{2\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi$, we find

$$\begin{aligned}
&\hbar(t)\big(e^\Psi\Delta_k^{\mathrm{h}}(\partial_t\varphi_\Phi - \partial_{yy}\varphi_\Phi) \mid e^\Psi\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+} + \lambda\dot{\theta}(t)\hbar(t)\big(e^\Psi|D_h|\Delta_k^{\mathrm{h}}\varphi_\Phi \mid e^\Psi\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\\
&\quad + \hbar(t)\big(e^\Psi\Delta_k^{\mathrm{h}}[(u+u^s+\varepsilon f(t)\chi(y))\partial_x\varphi]_\Phi \mid e^\Psi\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\\
&\quad + 2\hbar(t)\big(e^\Psi\int_y^\infty \Delta_k^{\mathrm{h}}\big[\partial_y(u+u^s+\varepsilon f(t)\chi(y'))\partial_x\varphi\big]_\Phi \, dy' \mid e^\Psi\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+} = 0.
\end{aligned} \tag{3.46}$$

$$
\begin{aligned}
&\int_{t_0}^{t} \hbar(t')\big(e^{\Psi}\Delta_k^{\mathrm{h}}(\partial_t\varphi_\Phi - \partial_{yy}\varphi_\Phi) \mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\,dt' \\
&\geq \frac{1}{2}\Big(\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t)\|_{L^2_+}^2 - \|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t_0)\|_{L^2_+}^2 \\
&\qquad - \int_{t_0}^{t}\hbar'(t')\|e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t')\|_{L^2_+}^2\,dt' + \|e^{\Psi}\Delta_k^{\mathrm{h}}\partial_y\varphi_\Phi\|_{L^2(t_0,t;L^2_+)}^2\Big).
\end{aligned}
\tag{3.47}
$$

$$\begin{aligned}&\int_{t_0}^{t} \hbar(t')\big(e^{\Psi}\Delta_k^{\rm h}(\partial_t\varphi_\Phi-\partial_{yy}\varphi_\Phi)\ |\ e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi\big)_{L^2_+}\,dt'\\ &\geq\frac{1}{2}\Big(\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi(t)\|_{L^2_+}^2-\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi(t_0)\|_{L^2_+}^2\\ &\qquad-\int_{t_0}^{t}\hbar'(t')\|e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi(t')\|_{L^2_+}^2\,dt'+\|e^{\Psi}\Delta_k^{\rm h}\partial_y\varphi_\Phi\|_{L^2(t_0,t;L^2_+)}^2\Big).\end{aligned}\tag{3.47}$$

$$\begin{aligned}\int_{t_0}^{t}\hbar(t')\big|\big(e^{\Psi}\Delta_k^{\rm h}[(u+u^s+\varepsilon f(t')\chi(y))\partial_x\varphi]_\Phi\ |\ e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi\big)_{L^2_+}\big|\,dt'&\\ \lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|^2_{\widetilde{L}^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}&.\end{aligned}\tag{3.48}$$

$$\begin{aligned}&\int_{t_0}^{t}\hbar(t')\big(e^{\Psi}\Delta_k^{\mathrm{h}}(\partial_t\varphi_\Phi-\partial_{yy}\varphi_\Phi)\mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\,dt'\\&\geq\frac{1}{2}\Big(\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t)\|_{L^2_+}^2-\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t_0)\|_{L^2_+}^2\\&\qquad-\int_{t_0}^{t}\hbar'(t')\|e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t')\|_{L^2_+}^2\,dt'+\|e^{\Psi}\Delta_k^{\mathrm{h}}\partial_y\varphi_\Phi\|_{L^2(t_0,t;L^2_+)}^2\Big).\end{aligned}\tag{3.47}$$

$$\begin{aligned}&\int_{t_0}^{t}\hbar(t')\big|\big(e^{\Psi}\Delta_k^{\mathrm{h}}[(u+u^s+\varepsilon f(t')\chi(y))\partial_x\varphi]_\Phi\mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\big|\,dt'\\&\qquad\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}^2.\end{aligned}\tag{3.48}$$

$$\int_{t_0}^{t}\hbar(t')\big|\big(e^{\Psi}\int_y^{\infty}\Delta_k^{\mathrm{h}}[\partial_y u\partial_x\varphi]_\Phi\,dy'\mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\big|\,dt'\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}^2.\tag{3.49}$$

$$\begin{aligned}&\int_{t_0}^{t}\hbar(t')\big(e^{\Psi}\Delta_k^{\mathrm{h}}(\partial_t\varphi_\Phi-\partial_{yy}\varphi_\Phi)\ |\ e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\,dt'\\&\geq\frac{1}{2}\Big(\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t)\|_{L^2_+}^2-\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t_0)\|_{L^2_+}^2\\&\qquad-\int_{t_0}^{t}\hbar'(t')\|e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t')\|_{L^2_+}^2\,dt'+\|e^{\Psi}\Delta_k^{\mathrm{h}}\partial_y\varphi_\Phi\|_{L^2(t_0,t;L^2_+)}^2\Big).\end{aligned}\tag{3.47}$$

$$\begin{aligned}&\int_{t_0}^{t}\hbar(t')\big|\big(e^{\Psi}\Delta_k^{\mathrm{h}}[(u+u^s+\varepsilon f(t')\chi(y))\partial_x\varphi]_\Phi\ |\ e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\big|\,dt'\\&\qquad\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}^2.\end{aligned}\tag{3.48}$$

$$\int_{t_0}^{t}\hbar(t')\big|\big(e^{\Psi}\int_y^\infty\Delta_k^{\mathrm{h}}[\partial_y u\partial_x\varphi]_\Phi\,dy'\ |\ e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\big|\,dt'\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}^2.\tag{3.49}$$

$$\lambda\dot\theta(t)\big(e^{\Psi}|D_h|\Delta_k^{\mathrm{h}}\varphi_\Phi\ |\ e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\big)_{L^2_+}\geq c\lambda\dot\theta(t)2^k\|e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi(t)\|_{L^2_+}^2.\tag{3.50}$$

$$
\begin{aligned}
&\int_{t_0}^{t} \hbar(t')\Big|\Big(e^{\Psi}\int_y^{\infty}\Delta_k^{\mathrm{h}}\big[\partial_y(u^s+\varepsilon f(t')\chi(y'))\partial_x\varphi\big]_{\Phi}\,dy' \mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}\Big)_{L^2_+}\Big|\,dt' \\
&\lesssim 2^k\int_0^t \dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}(t')\|_{L^2_+}^2\,dt' \lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_{\Phi}\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}.
\end{aligned}
\tag{3.51}
$$

$$\begin{aligned}&\int_{t_0}^{t} \hbar(t')\Big|\Big(e^{\Psi}\int_{y}^{\infty}\Delta_k^{\rm h}\big[\partial_y(u^s+\varepsilon f(t')\chi(y'))\partial_x\varphi\big]_{\Phi}\,dy' \mid e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}\Big)_{L^2_+}\Big|\,dt'\\ &\lesssim 2^k\int_0^t \dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t')\|_{L^2_+}^2\,dt' \lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_{\Phi}\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}.\end{aligned}\tag{3.51}$$

$$\begin{aligned}&\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}\|^2_{L^\infty(t_0,t;L^2_+)}+2c\lambda 2^k\int_{t_0}^{t}\dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t')\|^2_{L^2_+}\,dt'\\ &\quad+\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\partial_y\varphi_{\Phi}\|^2_{L^2(t_0,t;L^2_+)}\leq\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t_0)\|^2_{L^2_+}\\ &+\int_{t_0}^{t}\hbar'(t')\|e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t')\|^2_{L^2_+}\,dt'+Cd_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_{\Phi}\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})},\end{aligned}\tag{3.52}$$

Taking $t_0 = 0$ and $\hbar(t) = \langle t \rangle^{\frac{1}{2}}$ in (3.52), we obtain

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi} \Delta_k^{\rm h} \varphi_\Phi\|^2_{L^\infty_t(L^2_+)} + 2c\lambda 2^k \int_0^t \dot\theta(t') \|\langle t'\rangle^{\frac{1}{4}} \Delta_k^{\rm h} \varphi_\Phi(t')\|^2_{L^2_+} \, dt'$$

$$\leq \left\| e^{\frac{y^2}{8}} e^{\delta|D_x|} \Delta_k^{\rm h} \varphi_0 \right\|^2_{L^2_+} + C d_k^2 2^{-k} \|\langle t\rangle^{\frac{1}{4}} e^{\Psi} \varphi_\Phi\|^2_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{1,0})}.$$

Taking $t_0 = 0$ and $\hbar(t) = \langle t \rangle^{\frac{1}{2}}$ in (3.52), we obtain

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi} \Delta_k^{\mathrm{h}} \varphi_\Phi\|_{L^\infty_t(L^2_+)}^2 + 2c\lambda 2^k \int_0^t \dot\theta(t') \|\langle t'\rangle^{\frac{1}{4}} \Delta_k^{\mathrm{h}} \varphi_\Phi(t')\|_{L^2_+}^2 \, dt' \leq \|e^{\frac{y^2}{8}} e^{\delta|D_x|} \Delta_k^{\mathrm{h}} \varphi_0\|_{L^2_+}^2 + C d_k^2 2^{-k} \|\langle t\rangle^{\frac{1}{4}} e^{\Psi} \varphi_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{1,0})}^2.$$

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi} \varphi_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{1}{2},0})} + \sqrt{2c\lambda} \|\langle t'\rangle^{\frac{1}{4}} e^{\Psi} \varphi_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{1,0})} \leq \|e^{\frac{y^2}{8}} e^{\delta|D_x|} \varphi_0\|_{\mathcal{B}^{\frac{1}{2},0}} + \sqrt{C} \|\langle t'\rangle^{\frac{1}{4}} e^{\Psi} \varphi_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{1,0})}. \tag{3.53}$$

Taking $t_0 = 0$ and $\hbar(t) = \langle t \rangle^{\frac{1}{2}}$ in (3.52), we obtain

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_\Phi\|^2_{L^\infty_t(L^2_+)} + 2c\lambda 2^k \int_0^t \dot{\theta}(t')\|\langle t'\rangle^{\frac{1}{4}}\Delta_k^{\mathrm{h}}\varphi_\Phi(t')\|^2_{L^2_+}\,dt' \leq \|e^{\frac{y^2}{8}} e^{\delta|D_x|}\Delta_k^{\mathrm{h}}\varphi_0\|^2_{L^2_+} + Cd_k^2 2^{-k}\|\langle t\rangle^{\frac{1}{4}} e^{\Psi}\varphi_\Phi\|^2_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{1,0})}.$$

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{1}{2},0})} + \sqrt{2c\lambda}\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{1,0})} \leq \|e^{\frac{y^2}{8}} e^{\delta|D_x|}\varphi_0\|_{\mathcal{B}^{\frac{1}{2},0}} + \sqrt{C}\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{1,0})}. \tag{3.53}$$

$$\|\langle t'\rangle^{\frac{1}{4}} e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{1}{2},0})} \leq C\|e^{\frac{y^2}{8}} e^{\delta|D_x|}\varphi_0\|_{\mathcal{B}^{\frac{1}{2},0}}. \tag{3.54}$$

For any nonnegative and non-decreasing function $\hbar \in C^1(\mathbb{R}_+)$,

$$\int_{t_0}^{t} \hbar(t') \big| \big(e^{\Psi} \Delta_k^{\mathrm{h}} [u \partial_x \varphi]_\Phi \mid e^{\Psi} \Delta_k^{\mathrm{h}} \varphi_\Phi \big)_{L^2_+} \big| \, dt' \lesssim d_k^2 2^{-k} \| \hbar^{\frac{1}{2}} e^{\Psi} \varphi_\Phi \|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}. \tag{3.55}$$

For any nonnegative and non-decreasing function $\hbar \in C^1(\mathbb{R}_+)$,

$$\int_{t_0}^{t} \hbar(t') \big| \big(e^{\Psi} \Delta_k^{\mathrm{h}} [u\partial_x \varphi]_\Phi \mid e^{\Psi} \Delta_k^{\mathrm{h}} \varphi_\Phi \big)_{L^2_+} \big| \, dt' \lesssim d_k^2 2^{-k} \| \hbar^{\frac{1}{2}} e^{\Psi} \varphi_\Phi \|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}. \tag{3.55}$$

By applying Bony's decomposition in the horizontal variable to $u\partial_x\varphi$, we write

$$u\partial_x\varphi = T^{\mathrm{h}}_u \partial_x \varphi + T^{\mathrm{h}}_{\partial_x\varphi} u + R^{\mathrm{h}}(u, \partial_x \varphi).$$

For any nonnegative and non-decreasing function $\hbar \in C^1(\mathbb{R}_+)$,

$$\int_{t_0}^{t} \hbar(t') \big| \big(e^{\Psi}\Delta_k^{\rm h}[u\partial_x \varphi]_\Phi \mid e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi\big)_{L^2_+} \big| \, dt' \lesssim d_k^2 2^{-k} \|\hbar^{\frac{1}{2}} e^{\Psi}\varphi_\Phi\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}. \tag{3.55}$$

By applying Bony's decomposition in the horizontal variable to $u\partial_x\varphi$, we write

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$$\begin{aligned}
&\int_{t_0}^{t} \hbar(t') \big| \big(e^{\Psi}\Delta_k^{\rm h}\big[T^{\rm h}_u \partial_x\varphi\big]_\Phi \mid e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi\big)_{L^2_+} \big| \, dt' \\
&\qquad \lesssim \sum_{|k'-k|\leq 4} \int_{t_0}^{t} \|S^{\rm h}_{k'-1} u_\Phi(t')\|_{L^\infty_+} \|\hbar^{\frac{1}{2}} e^{\Psi}\Delta^{\rm h}_{k'}\partial_x\varphi_\Phi(t')\|_{L^2_+} \|\hbar^{\frac{1}{2}} e^{\Psi}\Delta^{\rm h}_{k}\varphi_\Phi(t')\|_{L^2_+} \, dt'.
\end{aligned}$$

For any nonnegative and non-decreasing function $\hbar \in C^1(\mathbb{R}_+)$,

$$\int_{t_0}^{t} \hbar(t') \left| \left(e^{\Psi} \Delta_k^{\rm h} [u \partial_x \varphi]_\Phi \mid e^{\Psi} \Delta_k^{\rm h} \varphi_\Phi \right)_{L^2_+} \right| dt' \lesssim d_k^2 2^{-k} \| \hbar^{\frac{1}{2}} e^{\Psi} \varphi_\Phi \|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}. \tag{3.55}$$

By applying Bony's decomposition in the horizontal variable to $u\partial_x\varphi$, we write

$$u\partial_x\varphi = T^{\rm h}_u \partial_x \varphi + T^{\rm h}_{\partial_x\varphi} u + R^{\rm h}(u, \partial_x \varphi).$$

$$\begin{aligned}
&\int_{t_0}^{t} \hbar(t') \left| \left(e^{\Psi} \Delta_k^{\rm h} \left[T^{\rm h}_u \partial_x \varphi \right]_\Phi \mid e^{\Psi} \Delta_k^{\rm h} \varphi_\Phi \right)_{L^2_+} \right| dt' \\
&\qquad \lesssim \sum_{|k'-k|\leq 4} \int_{t_0}^{t} \| S^{\rm h}_{k'-1} u_\Phi(t') \|_{L^\infty_+} \| \hbar^{\frac{1}{2}} e^{\Psi} \Delta^{\rm h}_{k'} \partial_x \varphi_\Phi(t') \|_{L^2_+} \| \hbar^{\frac{1}{2}} e^{\Psi} \Delta^{\rm h}_k \varphi_\Phi(t') \|_{L^2_+} \, dt'.
\end{aligned}$$

$$\begin{aligned}
\| e^{\frac{3}{4}\Psi} \Delta_k^{\rm h} u_\Phi(t') \|_{L^\infty_{\rm v}(L^2_{\rm h})} \lesssim & \| e^{\frac{3}{4}\Psi} \int_y^\infty \Delta_k^{\rm h} \partial_y u_\Phi(t') \, dy' \|_{L^\infty_{\rm v}(L^2_{\rm h})} \\
\lesssim & \| e^{\frac{3}{4}\Psi} \left(\int_y^\infty e^{-\frac{7}{4}\Psi} \, dy \right)^{\frac{1}{2}} \|_{L^\infty_{\rm v}} \| e^{\frac{7}{8}\Psi} \Delta_k^{\rm h} \partial_y u_\Phi(t') \|_{L^2_+} \\
\lesssim & d_k(t') 2^{-\frac{k}{2}} \langle t' \rangle^{\frac{1}{4}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{\mathcal{B}^{\frac{1}{2},0}},
\end{aligned} \tag{3.56}$$

$$
\begin{aligned}
\|S^{\mathrm{h}}_{k-1}u_\Phi(t')\|_{L^\infty_+} &\lesssim \sum_{k'\leq k-2} 2^{\frac{k}{2}}\|\Delta^{\mathrm{h}}_k u_\Phi(t')\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}\\
&\lesssim \langle t'\rangle^{\frac14}\|e^{\Psi}\partial_y G_\Phi(t')\|_{\mathcal{B}^{\frac12,0}} \lesssim \dot{\theta}(t').
\end{aligned}
$$

$$\|S^{\mathrm{h}}_{k-1}u_\Phi(t')\|_{L^\infty_+}\lesssim\sum_{k'\le k-2}2^{\frac{k}{2}}\|\Delta^{\mathrm{h}}_k u_\Phi(t')\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}$$
$$\lesssim\langle t'\rangle^{\frac14}\|e^{\Psi}\partial_y G_\Phi(t')\|_{\mathcal{B}^{\frac12,0}}\lesssim\dot{\theta}(t').$$

$$\int_{t_0}^t\hbar(t')\big|\big(e^{\Psi}\Delta^{\mathrm{h}}_k\big[T^{\mathrm{h}}_u\partial_x\varphi\big]_\Phi\ |\ e^{\Psi}\Delta^{\mathrm{h}}_k\varphi_\Phi\big)_{L^2_+}\big|\,dt'$$
$$\lesssim\sum_{|k'-k|\le4}2^{k'}\Big(\int_{t_0}^t\dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta^{\mathrm{h}}_{k'}\varphi_\Phi(t')\|^2_{L^2_+}\,dt'\Big)^{\frac12}\Big(\int_{t_0}^t\dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta^{\mathrm{h}}_{k}\varphi_\Phi(t')\|^2_{L^2_+}\,dt'\Big)^{\frac12}$$
$$\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}. \tag{3.57}$$

$$
\begin{aligned}
&\int_{t_0}^{t} \hbar(t')\Big|\big(e^{\Psi}\Delta_k^{\mathrm{h}}\big[T^{\mathrm{h}}_{\partial_x\varphi}u\big]_{\Phi}\;\big|\;e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}\big)_{L^2_+}\Big|\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}\int_{t_0}^{t}\|\hbar^{\frac12}e^{\Psi}S^{\mathrm{h}}_{k'-1}\partial_x\varphi_{\Phi}(t')\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}\|\Delta^{\mathrm{h}}_{k'}u_{\Phi}(t')\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}(t')\|_{L^2_+}\,dt',
\end{aligned}
$$

$$\int_{t_0}^{t} \hbar(t')\left|\left(e^{\Psi}\Delta_k^{\mathrm{h}}\left[T^{\mathrm{h}}_{\partial_x\varphi}u\right]_{\Phi} \mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}\right)_{L^2_+}\right| dt'$$

$$\lesssim \sum_{|k'-k|\le 4}\int_{t_0}^{t}\|\hbar^{\frac12}e^{\Psi}S^{\mathrm{h}}_{k'-1}\partial_x\varphi_{\Phi}(t')\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}\|\Delta^{\mathrm{h}}_{k'}u_{\Phi}(t')\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}(t')\|_{L^2_+}\,dt',$$

$$\int_{t_0}^{t} \hbar(t')\left|\left(e^{\Psi}\Delta_k^{\mathrm{h}}\left[T^{\mathrm{h}}_{\partial_x\varphi}u\right]_{\Phi} \mid e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}\right)_{L^2_+}\right| dt'$$

$$\lesssim \sum_{|k'-k|\le 4}2^{-\frac{k'}{2}}\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}S^{\mathrm{h}}_{k'-1}\partial_x\varphi_{\Phi}(t')\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}(t')\|_{L^2_+}\,dt'$$

$$\lesssim \sum_{|k'-k|\le 4}2^{-\frac{k'}{2}}\left(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}S^{\mathrm{h}}_{k'-1}\partial_x\varphi_{\Phi}(t')\|^2_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}\,dt'\right)^{\frac12}$$

$$\times\left(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\mathrm{h}}\varphi_{\Phi}(t')\|^2_{L^2_+}\,dt'\right)^{\frac12}.$$

$$\Big(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}S^{\rm h}_{k'-1}\partial_x\varphi_\Phi(t')\|^2_{L^2_{\rm v}(L^\infty_{\rm h})}\,dt'\Big)^{\frac12}$$

$$\lesssim\sum_{j\le k'-2}2^{\frac{3j}{2}}\Big(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}\Delta^{\rm h}_j\varphi_\Phi(t')\|^2_{L^2}\,dt'\Big)^{\frac12} \tag{3.58}$$

$$\lesssim d_{k'}2^{\frac{k'}{2}}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|_{\widetilde{L}^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}.$$

$$\Big(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}S^{\rm h}_{k'-1}\partial_x\varphi_\Phi(t')\|^2_{L^2_{\rm v}(L^\infty_{\rm h})}\,dt'\Big)^{\frac12}$$

$$\lesssim\sum_{j\le k'-2}2^{\frac{3j}{2}}\Big(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}\Delta^{\rm h}_j\varphi_\Phi(t')\|^2_{L^2}\,dt'\Big)^{\frac12} \tag{3.58}$$

$$\lesssim d_{k'}2^{\frac{k'}{2}}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|_{\widetilde L^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}.$$

$$\int_{t_0}^{t}\hbar(t')\big|\big(e^{\Psi}\Delta^{\rm h}_k\big[T^{\rm h}_{\partial_x\varphi}u\big]_\Phi\;|\;e^{\Psi}\Delta^{\rm h}_k\varphi_\Phi\big)_{L^2_+}\big|\,dt'\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|^2_{\widetilde L^2_{t_0,t;\dot\theta(t)}(\mathcal{B}^{1,0})}. \tag{3.59}$$

$$\int_{t_0}^{t} \hbar(t')\big|\big(e^{\Psi}\Delta_k^{\rm h}\big[R^{\rm h}(u,\partial_x\varphi)\big]_{\Phi} \mid e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}\big)_{L^2_+}\big|\,dt'$$

$$\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_{t_0}^{t}\|\widetilde{\Delta}_{k'}^{\rm h}u_{\Phi}(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\hbar^{\frac12}e^{\Psi}\Delta_{k'}^{\rm h}\partial_x\varphi_{\Phi}(t')\|_{L^2_+}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t')\|_{L^2_+}\,dt'$$

$$\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}\Delta_{k'}^{\rm h}\varphi_{\Phi}(t')\|_{L^2_+}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t')\|_{L^2_+}\,dt'$$

$$\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\Big(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}\Delta_{k'}^{\rm h}\varphi_{\Phi}(t')\|_{L^2_+}^2\,dt'\Big)^{\frac12}\Big(\int_{t_0}^{t}\dot\theta(t')\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_{\Phi}(t')\|_{L^2_+}^2\,dt'\Big)^{\frac12},$$

$$
\begin{aligned}
&\int_{t_0}^{t} \hbar(t')\big|\big(e^{\Psi}\Delta_k^{\rm h}[R^{\rm h}(u,\partial_x\varphi)]_{\Phi} \mid e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi\big)_{L^2_+}\big|\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_{t_0}^{t}\|\widetilde{\Delta}_{k'}^{\rm h}u_\Phi(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\hbar^{\frac12}e^{\Psi}\Delta_{k'}^{\rm h}\partial_x\varphi_\Phi(t')\|_{L^2_+}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi(t')\|_{L^2_+}\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\int_{t_0}^{t}\dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta_{k'}^{\rm h}\varphi_\Phi(t')\|_{L^2_+}\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi(t')\|_{L^2_+}\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\Big(\int_{t_0}^{t}\dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta_{k'}^{\rm h}\varphi_\Phi(t')\|_{L^2_+}^2\,dt'\Big)^{\frac12}\Big(\int_{t_0}^{t}\dot{\theta}(t')\|\hbar^{\frac12}e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi(t')\|_{L^2_+}^2\,dt'\Big)^{\frac12},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{t_0}^{t} \hbar(t')\big|\big(e^{\Psi}\Delta_k^{\rm h}[R^{\rm h}(u,\partial_x\varphi)]_{\Phi} \mid e^{\Psi}\Delta_k^{\rm h}\varphi_\Phi\big)_{L^2_+}\big|\,dt'\\
&\lesssim d_k 2^{-\frac{k}{2}}\Big(\sum_{k'\geq k-3}d_{k'}2^{-\frac{k'}{2}}\Big)\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}\\
&\lesssim d_k^2 2^{-k}\|\hbar^{\frac12}e^{\Psi}\varphi_\Phi\|^2_{\widetilde{L}^2_{t_0,t;\dot{\theta}(t)}(\mathcal{B}^{1,0})}.
\end{aligned}
\tag{3.60}
$$

$$
\begin{aligned}
&\hbar(t)\big(e^{\Psi}\Delta_k^{\mathrm{h}}(\partial_t u_\Phi-\partial_{yy}u_\Phi)\mid e^{\Psi}\Delta_k^{\mathrm{h}}u_\Phi\big)_{L^2_+}+\lambda\dot{\theta}(t)\hbar(t)\big(e^{\Psi}|D_h|\Delta_k^{\mathrm{h}}u_\Phi\mid e^{\Psi}\Delta_k^{\mathrm{h}}u_\Phi\big)_{L^2_+}\\
&\quad+\hbar(t)\big(e^{\Psi}\Delta_k^{\mathrm{h}}[(u+u^s+\varepsilon f(t)\chi(y))\partial_x u]_\Phi\mid e^{\Psi}\Delta_k^{\mathrm{h}}u_\Phi\big)_{L^2_+}\\
&\quad+\hbar(t)\big(e^{\Psi}\Delta_k^{\mathrm{h}}\big[v\partial_y(u+u^s+\varepsilon f(t)\chi(y))\big]_\Phi\mid e^{\Psi}\Delta_k^{\mathrm{h}}u_\Phi\big)_{L^2_+}=0.
\end{aligned}
\tag{3.61}
$$

$$\begin{aligned}
&\hbar(t)\big(e^{\Psi}\Delta_k^{\rm h}(\partial_t u_\Phi-\partial_{yy}u_\Phi)\mid e^{\Psi}\Delta_k^{\rm h}u_\Phi\big)_{L^2_+}+\lambda\dot\theta(t)\hbar(t)\big(e^{\Psi}|D_h|\Delta_k^{\rm h}u_\Phi\mid e^{\Psi}\Delta_k^{\rm h}u_\Phi\big)_{L^2_+}\\
&\quad+\hbar(t)\big(e^{\Psi}\Delta_k^{\rm h}[(u+u^s+\varepsilon f(t)\chi(y))\partial_x u]_\Phi\mid e^{\Psi}\Delta_k^{\rm h}u_\Phi\big)_{L^2_+}\\
&\quad+\hbar(t)\big(e^{\Psi}\Delta_k^{\rm h}\big[v\partial_y(u+u^s+\varepsilon f(t)\chi(y))\big]_\Phi\mid e^{\Psi}\Delta_k^{\rm h}u_\Phi\big)_{L^2_+}=0.
\end{aligned}\tag{3.61}$$

$$\begin{aligned}
&\hbar(t)\big(e^{\Psi}\Delta_k^{\rm h}\big(\partial_t G_\Phi-\partial_{yy}G_\Phi+\langle t\rangle^{-1}G_\Phi\big)\mid e^{\Psi}\Delta_k^{\rm h}G_\Phi\big)_{L^2_+}\\
&+\lambda\dot\theta(t)\hbar(t)\big(e^{\Psi}|D_h|\Delta_k^{\rm h}G_\Phi\mid e^{\Psi}\Delta_k^{\rm h}G_\Phi\big)_{L^2_+}+\hbar(t)\big(e^{\Psi}|D_h|\Delta_k^{\rm h}[v\partial_y G]_\Phi\mid e^{\Psi}\Delta_k^{\rm h}G_\Phi\big)_{L^2_+}\\
&+\hbar(t)\big(e^{\Psi}\Delta_k^{\rm h}\big[(u+u^s+\varepsilon f(t)\chi(y))\partial_x G\big]_\Phi\mid e^{\Psi}\Delta_k^{\rm h}G_\Phi\big)_{L^2_+}\\
&+\hbar(t)\big(e^{\Psi}\Delta_k^{\rm h}\big[\partial_y(u^s+\varepsilon f(t)\chi(y))v-\frac{1}{2}\langle t\rangle^{-1}v\partial_y(y\varphi)\big]_\Phi\mid e^{\Psi}\Delta_k^{\rm h}G_\Phi\big)_{L^2_+}\\
&+\langle t\rangle^{-1}\hbar(t)\big(e^{\Psi}y\int_y^\infty\Delta_k^{\rm h}\big[\partial_y\big(u+u^s+\varepsilon f(t)\chi(y')\big)\partial_x\varphi\big]_\Phi\,dy'\mid e^{\Psi}\Delta_k^{\rm h}G_\Phi\big)_{L^2_+}=0.
\end{aligned}\tag{3.62}$$

Global small analytic solutions of MHD boundary layer equations

Two-dimensional MHD boundary layer equations in the upper space $\mathbb{R}^2_+ \overset{\text{def}}{=} \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}_+\}$,

$$\begin{cases} \partial_t u_1 - \partial_y^2 u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 + \partial_x p = b_1 \partial_x b_1 + b_2 \partial_y b_1, \\ \partial_t b_1 - \kappa \partial_y^2 b_1 + u_1 \partial_x b_1 + u_2 \partial_y b_1 = b_1 \partial_x u_1 + b_2 \partial_y u_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \ \ \partial_x b_1 + \partial_y b_2 = 0, \\ u_1|_{y=0} = u_2|_{y=0} = 0, \ \ \partial_y b_1|_{y=0} = b_2|_{y=0} = 0, \\ \lim_{y \to +\infty} u_1 = U_1, \ \ \lim_{y \to +\infty} b_1 = B_1, \\ u_1|_{t=0} = u_{1,0}, \ \ b_1|_{t=0} = b_{1,0}, \end{cases} \tag{4.63}$$

Theorem (Ning Liu and Ping Zhang, *JDE*, 281 (2021), 199–257)

Let $\kappa \in]0,2[$, $\bar{B}_\kappa = \begin{cases} 1 & \text{if } \kappa = 1, \\ 0 & \text{otherwise}, \end{cases}$ *and* $\varepsilon, \delta > 0$. *We assume that the far field states* (U, B) *satisfy*

$$\|\langle t\rangle^{\frac{9}{4}} e^{\delta|D_x|}(U,B)\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}_h^{\frac{3}{2}})} + \|\langle t\rangle^{\frac{7}{4}} e^{\delta|D_x|}(\partial_t U, \partial_t B, U, B)\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}_h^{\frac{1}{2}})} \le \varepsilon, \tag{4.64a}$$

$$\int_0^\infty \langle t\rangle^{\frac{5}{4}} \|e^{\delta|D_x|}(U,B)\|_{\mathcal{B}_h^{\frac{1}{2}}}\, dt \le \varepsilon. \tag{4.64b}$$

Let the initial data $(u_0, b_0, \varphi_0, \psi_0)$ *satisfy the compatibility condition:* $u_0|_{y=0} = \partial_y b_0|_{y=0} = 0$, $\int_0^\infty u_0\, dy = \int_0^\infty b_0\, dy = 0$, *and*

$$\|e^{\frac{y^2}{8}} e^{\delta|D_x|}(u_0, b_0, \varphi_0, \psi_0)\|_{\mathcal{B}^{\frac{1}{2},0}} < \infty \quad \text{and} \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(G_0, H_0)\|_{\mathcal{B}^{\frac{1}{2},0}} \le \sqrt{\varepsilon}, \tag{4.65}$$

where $G_0 \stackrel{def}{=} u_0 + \frac{y}{2\langle t\rangle}\varphi_0$ *and* $H_0 \stackrel{def}{=} b_0 + \frac{y}{2\kappa\langle t\rangle}\psi_0$. *Then there exist positive constants* $\lambda, \mathfrak{c}$ *and* $\varepsilon_0(\lambda, \mathfrak{c}, \kappa, \delta)$ *so that for* $\varepsilon \le \varepsilon_0$ *and* $l_\kappa \stackrel{def}{=} \frac{\kappa(2-\kappa)}{4} \in]0, 1/4]$, *the system (4.63) has a unique global solution* (u, b) *which satisfies* $\sup_{t\in[0,\infty[} \theta(t) \le \frac{\delta}{2\lambda}$,

Theorem

$$\|e^{\Psi}(u,b)_{\Phi}\|_{\widetilde{L}^{\infty}_t(\mathcal{B}^{\frac{1}{2},0})}+\sqrt{l_{\kappa}}\|e^{\Psi}\partial_y(u,b)_{\Phi}\|_{\widetilde{L}^{2}_t(\mathcal{B}^{\frac{1}{2},0})}\leq\|e^{\frac{y^2}{8}}e^{\delta|D_x|}(u_0,b_0)\|_{\mathcal{B}^{\frac{1}{2},0}}+C\sqrt{\varepsilon}. \quad (4.66)$$

Furthermore, for any $t>0$ and $\gamma\in]0,1[$, there hold

$$\|\langle\tau\rangle^{\frac{1}{2}+l_{\kappa}-\mathfrak{c}\varepsilon}e^{\Psi}(u,b)_{\Phi}\|_{\widetilde{L}^{\infty}_t(\mathcal{B}^{\frac{1}{2},0})}+\|\langle\tau\rangle^{\frac{1}{2}+l_{\kappa}-\mathfrak{c}\varepsilon}e^{\Psi}\partial_y(u,b)_{\Phi}\|_{\widetilde{L}^{2}_{[\frac{t}{2},t]}(\mathcal{B}^{\frac{1}{2},0})} \quad (4.67a)$$

$$\leq C\big(\|e^{\frac{y^2}{8}}e^{\delta|D_x|}(u_0,b_0,\varphi_0,\psi_0)\|_{\mathcal{B}^{\frac{1}{2},0}}+\sqrt{\varepsilon}\big),$$

$$\|\langle\tau\rangle^{1+l_{\kappa}-\mathfrak{c}\varepsilon}e^{\Psi}(G,H)_{\Phi}\|_{\widetilde{L}^{\infty}_t(\mathcal{B}^{\frac{1}{2},0})}+\|\langle\tau\rangle^{1+l_{\kappa}-\mathfrak{c}\varepsilon}e^{\Psi}\partial_y(G,H)_{\Phi}\|_{\widetilde{L}^{2}_{[\frac{t}{2},t]}(\mathcal{B}^{\frac{1}{2},0})} \quad (4.67b)$$

$$\leq C\sqrt{\varepsilon}\big(1+\|e^{\frac{y^2}{8}}e^{\delta|D_x|}(u_0,b_0)\|_{\mathcal{B}^{\frac{1}{2},0}}\big),$$

$$\|\langle\tau\rangle^{1+l_{\kappa}-\mathfrak{c}\varepsilon}e^{\gamma\Psi}(u,b)_{\Phi}\|_{\widetilde{L}^{\infty}_t(\mathcal{B}^{\frac{1}{2},0})}+\|\langle\tau\rangle^{1+l_{\kappa}-\mathfrak{c}\varepsilon}e^{\gamma\Psi}\partial_y(u,b)_{\Phi}\|_{\widetilde{L}^{2}_{[\frac{t}{2},t]}(\mathcal{B}^{\frac{1}{2},0})} \quad (4.67c)$$

$$\leq C\sqrt{\varepsilon}\big(1+\|e^{\frac{y^2}{8}}e^{\delta|D_x|}(u_0,b_0)\|_{\mathcal{B}^{\frac{1}{2},0}}\big).$$

The global well-posedness in the optimal Gevrey class

For a clear presentation, we shall take $U^{\mathrm{E}} = 0$ and $P^{\mathrm{E}} = C$ in the rest of this paper, so that (u, v) solves

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\ \partial_x u + \partial_y v = 0, \\ (u, v)|_{y=0} = 0, \quad \lim_{y \to +\infty} u = 0, \\ u|_{t=0} = u_0(x, y). \end{cases} \tag{4.68}$$

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$$\begin{cases} \partial_t u+u\partial_x u+v\partial_y u-\partial_y^2 u=0, \quad (t,x,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+,\\ \partial_x u+\partial_y v=0,\\ (u,v)|_{y=0}=0, \quad \lim_{y\to+\infty} u=0,\\ u|_{t=0}=u_0(x,y). \end{cases} \tag{4.68}$$

Let $\langle\xi\rangle \stackrel{\text{def}}{=} \left(1+\xi^2\right)^{\frac{1}{2}}$, we denote

$$e^{\Phi(t,D_x)} \stackrel{\text{def}}{=} e^{\delta(t)|D|^{\frac{1}{2}}} \quad \text{with} \quad \delta(t)=\delta-\lambda\theta(t) \quad \text{and} \quad f_\Phi \stackrel{\text{def}}{=} e^{\Phi(t,D_x)}f, \tag{4.69}$$

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$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \quad (t,x,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\ \partial_x u + \partial_y v = 0, \\ (u,v)|_{y=0} = 0, \quad \lim\limits_{y\to+\infty} u = 0, \\ u|_{t=0} = u_0(x,y). \end{cases} \tag{4.68}$$

Let $\langle\xi\rangle \stackrel{\text{def}}{=} \left(1+\xi^2\right)^{\frac{1}{2}}$, we denote

$$e^{\Phi(t,D_x)} \stackrel{\text{def}}{=} e^{\delta(t)|D|^{\frac{1}{2}}} \quad \text{with} \quad \delta(t) = \delta - \lambda\theta(t) \quad \text{and} \quad f_\Phi \stackrel{\text{def}}{=} e^{\Phi(t,D_x)} f, \tag{4.69}$$

and the weighted anisotropic Sobolev norm

$$\|f\|_{H^{s,k}_\Psi} \stackrel{\text{def}}{=} \sum_{0\le\ell\le k} \left(\int_0^\infty e^{2\Psi(t,y)} \|\partial_y^\ell f(\cdot,y)\|^2_{H^s_{\mathrm{h}}}\, dy\right)^{\frac{1}{2}} \quad \text{with} \quad \Psi(t,y) \stackrel{\text{def}}{=} \frac{y^2}{8\langle t\rangle}. \tag{4.70}$$

Theorem (Chao Wang, Yuxi Wang and Ping Zhang)

Let $u_0 = \partial_y \varphi_0$ *satisfy* $u_0(x,0) = 0$ *and* $\int_0^\infty u_0\, dy = 0$. *Let* $G_0 \stackrel{def}{=} u_0 + \frac{y}{2}\varphi_0$. *For some sufficiently small but fixed* $\eta \in \left(0, \frac{1}{6}\right)$, *we denote*

$$\begin{aligned}\mathrm{E}(t) \stackrel{def}{=} & \|\langle t\rangle^{\frac{1-\eta}{4}} u_\Phi\|^2_{H_\Psi^{\frac{11}{2},0}} + \|\langle t\rangle^{\frac{3-\eta}{4}} \partial_y u_\Phi\|^2_{H_\Psi^{5,0}} + \|\langle t\rangle^{\frac{5-\eta}{4}} G_\Phi\|^2_{H_\Psi^{4,0}} \\ & + \|\langle t\rangle^{\frac{7-\eta}{4}} \partial_y G_\Phi\|^2_{H_\Psi^{3,0}} + \|\langle t\rangle^{\frac{9-\eta}{4}} \partial_y^2 G\|^2_{H_\Psi^{3,0}} + \|\langle t\rangle^{\frac{11-\eta}{4}} \partial_y^3 G\|^2_{H_\Psi^{2,0}},\end{aligned} \tag{4.71}$$

and

$$\begin{aligned}\mathrm{D}(t) \stackrel{def}{=} & \|\langle t\rangle^{\frac{1-\eta}{4}} \partial_y u_\Phi\|^2_{H_\Psi^{\frac{11}{2},0}} + \|\langle t\rangle^{\frac{3-\eta}{4}} \partial_y^2 u_\Phi\|^2_{H_\Psi^{5,0}} + \|\langle t\rangle^{\frac{5-\eta}{4}} \partial_y G_\Phi\|^2_{H_\Psi^{4,0}} \\ & + \|\langle t\rangle^{\frac{7-\eta}{4}} \partial_y^2 G_\Phi\|^2_{H_\Psi^{3,0}} + \|\langle t\rangle^{\frac{9-\eta}{4}} \partial_y^3 G\|^2_{H_\Psi^{3,0}} + \|\langle t\rangle^{\frac{11-\eta}{4}} \partial_y^4 G\|^2_{H_\Psi^{2,0}}.\end{aligned} \tag{4.72}$$

Theorem (Chao Wang, Yuxi Wang and Ping Zhang)

We assume moreover that

$$\|u_\Phi(0)\|^2_{H_\Psi^{\frac{25}{4},0}} + \mathrm{E}(0) \leq \varepsilon^2. \tag{4.73}$$

Then there exists $\varepsilon_0, \lambda_0 > 0$ *so that for* $\varepsilon \leq \varepsilon_0$ *and* $\lambda \geq \lambda_0$, $\theta(t)$ *in* (4.69) *satisfies* $\sup_{t\in[0,\infty)} \theta(t) \leq \frac{\delta}{4\lambda}$, *and the system (4.68) has a unique global solution u which satisfies*

$$\mathrm{E}(t) + c\eta \int_0^t \mathrm{D}(t')\,dt' \leq C\varepsilon^2 \quad \forall\, t \in \mathbb{R}. \tag{4.74}$$

Remark

(1) *As in [Dietert and Gérard-Varet 19], there is a loss on the Gevrey radius of the solution* u. *Moreover, we need* $u_\Phi(0) \in H_\Psi^{\frac{25}{4},0}(\mathbb{R}^2_+)$ *in order to guarantee that* $u_\Phi(t) \in H_\Psi^{\frac{11}{2},0}(\mathbb{R}^2_+)$. *These loss of regularities is due to the instabilities.*

(2) *Compared with [Paicu-Zhang19] with analytic data, here our smallness condition can not be presumed only on the initial data of the good quantity,* G. *Moreover, the decay of Gevrey solutions has a loss of* $\frac{\eta}{4}$ *in* (4.74). *We do not know if we could get rid of this loss or not.*

(3) *The main idea to prove* (4.74) *is to combine the method in [Dietert and Gérard-Varet 19], which is based on both a tricky change of unknown and appropriate choice of test function, and the time weighted energy estimate method in [Paicu-Zhang19]. Furthermore, as in [PZ19], we shall use the faster decay estimate of the good quantity,* G, *to control the Gevrey radius of the solutions, which will be one of the crucial step to construct the global solutions of* (4.68) *with optimal Gevrey regular data.*

Step 1. New formulation of the problem
By applying Bony's decomposition and $e^{\Phi(t,D_x)}$

$$\begin{aligned}(u\partial_x u)_\Phi &= T^{\mathrm{h}}_u\partial_x u_\Phi + \frac{\delta(t)}{2}T^{\mathrm{h}}_{D_x u}\Lambda(D_x)\partial_x u_\Phi + \big(T^{\mathrm{h}}_{\partial_x u}u + R^{\mathrm{h}}(u,\partial_x u)\big)_\Phi + f_1,\\ (v\partial_y u)_\Phi &= T^{\mathrm{h}}_{\partial_y u}v_\Phi + T^{\mathrm{h}}_v\partial_y u_\Phi + \frac{\delta(t)}{2}T^{\mathrm{h}}_{\partial_y D_x u}\Lambda(D_x)v_\Phi + R^{\mathrm{h}}(v,\partial_y u)_\Phi + f_2,\end{aligned} \tag{4.75}$$

where the remainder terms f_1, f_2 are defined by

$$\begin{aligned}f_1 &\stackrel{\text{def}}{=} (T^{\mathrm{h}}_u\partial_x u)_\Phi - T^{\mathrm{h}}_u\partial_x u_\Phi - \frac{\delta(t)}{2}T^{\mathrm{h}}_{D_x u}\Lambda(D_x)\partial_x u_\Phi,\\ f_2 &\stackrel{\text{def}}{=} (T^{\mathrm{h}}_{\partial_y u}v)_\Phi - T^{\mathrm{h}}_{\partial_y u}v_\Phi - \frac{\delta(t)}{2}T^{\mathrm{h}}_{\partial_y D_x u}\Lambda(D_x)v_\Phi + (T^{\mathrm{h}}_v\partial_y u)_\Phi - T^{\mathrm{h}}_v\partial_y u_\Phi.\end{aligned} \tag{4.76}$$

Here and in the sequel, we always denote $D_x \stackrel{\text{def}}{=} \frac{1}{i}\partial_x$.

Lemma

Let $\Phi(t,\xi) \stackrel{def}{=} \delta(t)\langle\xi\rangle^{\frac{1}{2}}$. *Let* $s \in \mathbb{R}$ *and* $\sigma > \frac{3}{2}$. *Then one has*

$$\|(T^{\mathrm{h}}_a \partial_x f)_\Phi - T^{\mathrm{h}}_a \partial_x f_\Phi\|_{H^s_{\mathrm{h}}} \leq C\delta(t)\|a_\Phi\|_{H^\sigma_{\mathrm{h}}}\|f_\Phi\|_{H^{s+\frac{1}{2}}_{\mathrm{h}}}. \tag{4.77}$$

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Lemma

Let $s \in \mathbb{R}$, $\sigma > \frac{5}{2}$ *and* $D_x \stackrel{def}{=} \frac{1}{i}\partial_x$. *Let* $\Phi(t,\xi) \stackrel{def}{=} \delta(t)\langle\xi\rangle^{\frac{1}{2}}$, *and* $\Lambda(\xi) \stackrel{def}{=} \xi(1+\xi^2)^{-\frac{3}{4}}$. *Then if* $0 < \delta(t) \leq L$, *one has*

$$\|(T^{\rm h}_a \partial_x f)_\Phi - T^{\rm h}_a \partial_x f_\Phi - \frac{\delta(t)}{2} T^{\rm h}_{D_x a}\Lambda(D)\partial_x f_\Phi\|_{H^s_{\rm h}} \leq C_L\|a_\Phi\|_{H^\sigma_{\rm h}}\|f_\Phi\|_{H^s_{\rm h}}. \tag{4.78}$$

$$
\begin{cases}
\partial_t u_\Phi + \lambda\dot{\theta}\langle D_x\rangle^{\frac{1}{2}} u_\Phi + T^{\rm h}_u \partial_x u_\Phi + T^{\rm h}_v \partial_y u_\Phi + T^{\rm h}_{\partial_y u} v_\Phi \\
\qquad + \frac{\delta(t)}{2}\big(T^{\rm h}_{D_x u}\Lambda(D_x)\partial_x u_\Phi + T^{\rm h}_{\partial_y D_x u}\Lambda(D_x) v_\Phi\big) - \partial_y^2 u_\Phi = f, \\
\partial_x u_\Phi + \partial_y v_\Phi = 0, \\
(u_\Phi, v_\Phi)|_{y=0} = 0, \quad \lim_{y\to+\infty} u_\Phi = 0, \\
u_\Phi|_{t=0} = e^{\delta\langle D_x\rangle^{\frac{1}{2}}} u_0,
\end{cases}
\tag{4.79}
$$

where the source term f is given by

$$
f = -f_1 - f_2 - f_3 \quad \text{with} \quad f_3 \overset{\text{def}}{=} \big(T^{\rm h}_{\partial_x u} u + R^{\rm h}(u, \partial_x u) + R^{\rm h}(v, \partial_y u)\big)_\Phi. \tag{4.80}
$$

To construct the global small solution of (4.68) with Gevery 2 regularity in x variable, we need first to modify the definitions of the auxiliary functions H, ϕ, which are first introduced by Dietert and Gérard-Varet in (*Ann. PDE*, 2019) In order to do it, let $\Psi(t,y) = \frac{y^2}{8\langle t\rangle}$ and $\hbar(t)$ be a positive and non-decreasing smooth function, we define the operator

$$\mathcal{L} \stackrel{\text{def}}{=} \partial_t + \lambda\dot{\theta}(t)\langle D_x\rangle^{\frac{1}{2}} + T^{\rm h}_u\partial_x + T^{\rm h}_v\partial_y + \frac{\delta(t)}{2} T^{\rm h}_{D_x u}\Lambda(D)\partial_x - \partial_y^2, \tag{4.81}$$

and its adjoint operator in $L^2(\hbar(t)e^{2\Psi})$ is given by

$$\begin{aligned}\mathcal{L}^* \stackrel{\text{def}}{=} & -\partial_t - \frac{\hbar'}{\hbar} + \frac{y^2}{4\langle t\rangle^2} + \lambda\dot{\theta}(t)\langle D_x\rangle^{\frac{1}{2}} - \partial_x(T^{\rm h}_u)^* \\ & - (\partial_y + \frac{y}{2\langle t\rangle})(T^{\rm h}_v)^* - \frac{\delta(t)}{2}\Lambda(D)\partial_x(T^{\rm h}_{D_x u})^* - (\partial_y + \frac{y}{2\langle t\rangle})^2.\end{aligned} \tag{4.82}$$

We now introduce the function H via

$$\begin{cases} \mathcal{L}\int_y^\infty H dz = \dot{\theta}(t)\int_y^\infty u_\Phi dz, \\ \partial_y H|_{y=0} = 0, \quad H|_{y\to+\infty} = 0, \\ H|_{t=0} = 0, \end{cases} \tag{4.83}$$

and ϕ by

$$\begin{cases} \mathcal{L}^*\phi = \dot{\theta}(t)H, \\ \phi|_{y=0} = 0, \quad \phi|_{y\to+\infty} = 0, \\ \phi|_{t=T} = 0. \end{cases} \tag{4.84}$$

We have the following remark concerning the definitions of H and ϕ :

(1) The reason why we need the function $\hbar(t)$ in the adjoint operator $\mathcal{L}^*$ given by (4.82) is for the purpose of deriving the decay estimates of the solutions H, ϕ and u.

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(2) In (4.83) and (4.84), we define H and ϕ via $\int_y^\infty H\,dz$ and $\int_y^\infty \phi\,dz$ (instead of $\int_0^y H\,dz$ and $\int_0^y \phi\,dz$ as in [DG19]) is to make the solutions H and ϕ decay faster as y approaching to ∞.

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(3) We have the additional $\dot\theta(t)$ before $\int_y^\infty u_\Phi\,dz$ in (4.83) and H in (4.84) is for the purpose of controlling the Gevery radius of the solutions H and ϕ simultaneously with their norms. This idea goes back to [Ch04] where Chemin introduced a tool to make analytical type estimates and controlling the size of the analytic radius simultaneously to classical Navier-Stokes system.

Applying ∂_y to the first equation of (4.83) yields

$$\dot{\theta}(t)u_\Phi = \mathcal{L}H - T^{\rm h}_{\partial_y u}\partial_x\int_y^\infty H\,dz + T^{\rm h}_{\partial_y v}H - \frac{\delta(t)}{2}T^{\rm h}_{\partial_y D_x u}\Lambda(D_x)\partial_x\int_y^\infty H\,dz, \tag{4.85}$$

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Whereas by taking ∂_x to the first equation of (4.83) and using the fact $v = \int_y^\infty \partial_x u dz$, we find

$$\begin{aligned} \dot{\theta}(t)v_\Phi =& \mathcal{L}\partial_x \int_y^\infty H\, dz + T^{\rm h}_{\partial_x u}\partial_x \int_y^\infty H\, dz \\ & - T^{\rm h}_{\partial_x v}H + \frac{\delta(t)}{2} T^{\rm h}_{\partial_x D_x u}\Lambda(D_x)\partial_x \int_y^\infty H\, dz. \end{aligned} \tag{4.86}$$

In the process of the estimates, we shall use the formulas (4.85) and (4.86), and thus the time derivative of $\dot{\theta}(t)$ gets involved in. This makes the evolution equations for $\theta(t)$ as in all the previous references like in [PZ5], which is defined via

$$\begin{cases} \dot{\theta}(t) = \langle t \rangle^{\frac{1}{4}} \|e^{\Psi} \partial_y G_{\Phi}(t)\|_{B^{\frac{1}{2},0}}, \\ \theta|_{t=0} = 0. \end{cases} \tag{4.87}$$

to be impossible here.

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to be impossible here.

Fortunately, we observe that what we really used in [PZ5] is the time decay estimate of $\| e^{\Psi} \partial_y G_{\Phi}(t) \|_{B^{\frac{1}{2},0}}$. Motivated by this observation, for some $\beta > 1$, we define

$$\begin{cases} \dot{\theta}(t) = \varepsilon^{\frac{1}{2}} \langle t \rangle^{-\beta}, \\ \theta(0) = 0. \end{cases} \tag{4.88}$$

Thank you for your attention!