General Decay in Viscoelasticity: Overview and recent development

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Outline

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The elastic solid

- Has a definite shape.
- Deformed by external forces into a new equilibrium shape.
- Reverts exactly to its original form on removal of external forces.
- Stores all the energy obtained from the work done by external forces during deformation.
- This energy remains available to restore the body to its original shape when these forces are removed.
- Responds only to the total stress level at every instant of time.

The elastic solid

Example 1 (Wave Equation)

$$u_{tt} - \Delta u = 0 \ \text{ in } \Omega,$$

under $u \equiv 0$ on $\partial\Omega$, the total energy

$$E(t) = \frac{1}{2}\int_{\Omega}\left(u_t^2 + |\nabla u|^2\right)dx$$

satisfies $E'(t) \equiv 0$. Hence $E(t) = E(0)$, $\forall t \geq 0$.

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satisfies $E'(t) \equiv 0$. Hence $E(t) = E(0)$, $\forall t \geq 0$. Energy conserved

The elastic solid

Example 2 (Damped Wave Equation)

$$u_{tt} - \Delta u + g(u_t) = 0 \text{ in } \Omega,$$

under $u \equiv 0$ on $\partial\Omega$, $g(0) = 0$ and g is increasing, we have

$$E'(t) = -\int_\Omega g(u_t)u_t dx \leq 0.$$

So $E(t)$ is decreasing.

Many stability results were obtained: Kopackova, Haraux, Zuazua, Lasiecka, Guesmia, Soufyane, Messaoudi, Benaissa, Tataru,...

The elastic solid

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Newtonian viscous fluid

- ☞ Responds to a suddenly applied state of uniform shear stress by a steady flow process.
- ☞ Has no definite shape and flows irreversibly under the action of external forces.

Viscoelastic Materials

- Other materials have properties which are intermediate between those of an elastic solid and a viscous liquid.
- The most interesting examples are polymers.
- A polymer can show all the features of a glassy, brittle solid, an elastic rubber or a viscous liquid depending on the temperature and time scale of measurement.
- Polymers are usually described as viscoelastic materials.

Viscoelastic Materials

Picture2.jpg

Viscoelastic Materials

- This type of material possesses a characteristic which can be referred to as a memory effect.
- That is, the material response is not only determined by the current state of stress, but is also determined by all past states of stress.
- To understand this phenomenon, several early models were introduced by Maxwell, Kelvin-Voight, Boltzmann (1874), and Volterra (1909).

Linear Kelvin-Voight Viscoelastic Model

$$u_{tt} = \operatorname{div} S, \;\; S = a\nabla u + b\nabla u_t, \;\; a, b > 0.$$

As a result:

$$u_{tt} - a\Delta u - b\Delta u_t = 0.$$

(Viscoelastic or strongly damped wave equation)

For Dirichlet boundary condition

$$E'(t) = -b\int_\Omega |\nabla u_t|^2 dx.$$

In fact:

$$E(t) \le ce^{-\lambda t}, \;\; c, \lambda > 0.$$

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Boltzmann Model

- In the isothermal viscoelasticity, the stress-strain relation is given by

$$\sigma(t) = \sigma_0\varepsilon(t) - \int_0^t g(t-\tau)\varepsilon(\tau)d\tau$$

- g characterizes the mechanical properties of the material and is referred to as **relaxation functions** and ε is the strain.
- It can be considered to be the formulation of Boltzmann's superposition principle.

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$$u_{tt} - \sigma_0\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0.$$

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(Viscoelastic equation)

Literature Review

✓ Dafermos (1970) discussed a certain one-dimensional viscoelastic problem

- established some existence results
- proved that solutions go to zero as t goes to infinity (smooth monotone decreasing relaxation functions)
- no rate of decay has been specified.

✓ Hrusa (1985) considered

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)\left(\psi(u_x(x,s))\right)_x ds = f(x,t)$$

- proved several global existence results for large data
- proved exponential decay, for strong solutions, when $m(s) = e^{-s}$ and ψ satisfies certain conditions.

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- ✓ Rivera (1994) considered equations for linear isotropic homogeneous viscoelastic solids of integral type
  - ☞ For bounded domains: proved an exponential decay result for exponentially decaying relaxation functions
  - ☞ For $\mathbb{R}^n$: showed that only the polynomial decay can be obtained even if the kernel is of exponential decay.

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Literature Review

The viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0 \\ u(x,t) = 0, \quad x \in \partial\Omega\ , t \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

in $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ bounded with $\partial\Omega$ regular, $\gamma > 0$, $g \geq 0$ discussed by many mathematicians.

✓ Cavalcanti *et al.* (EJDE 2002) proved an exponential decay under

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

$||g||_{L^1((0,+\infty))}$ is small enough and $a : \Omega \to \mathbb{R}^+$ such that

$$a(x) \geq a_0 > 0 \quad on \quad \emptyset \neq \omega \subset \Omega,$$

with ω satisfying some geometry restrictions.

✓ Berrimi and Messaoudi (EJDE 2004) obtained the same result under weaker conditions on g and ω. In particular, they allowed $\omega = \emptyset$ and the mechanical damping to be nonlinear.

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$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = 0, \quad \rho > 0.$$

solution-dependent density

☞ global existence result for $\gamma \geq 0$,

☞ exponential decay for $\gamma > 0$ were established.

✓ This last result has been extended to

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = b|u|^{p-1}u,$$

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$$g'(t) \leq 0, \quad \int_0^{+\infty} g(t)e^{\alpha t}dt < +\infty, \quad \alpha > 0.$$

✓ Many other results have been established by Munoz Rivera, Cavalcanti, Tatar, Alabau-Boussouira and Cannarsa, Messaoudi, Mustafa, Kafini, Soufyane, Guesmia, Said-Houari, Martinez, Park, Xiaosen and Mingxing ...

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General Decay

All results dealt mainly with either exponential decay

$$g'(t) \leq -\alpha g(t),$$

or polynomial decay

$$g'(t) \leq -\alpha g^{\rho}(t), \quad 1 < \rho < 3/2$$

General Decay

Question

How about other rates of decay?

To answer this question, Messaoudi (2008) investigated the situation when

$$g'(t) \leq -\xi(t)g(t), \tag{3.1}$$

where ξ is a positive function.
Consider

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0 \\ u(x,t) = 0, \quad x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{3.2}$$

in a bounded domain Ω and $t > 0$.

General Decay

Hypotheses

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function and

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(\tau)d\tau = l > 0.$$

(G2) There exists a differentiable function ξ such that

$$\begin{cases} \xi(t) > 0, \quad \xi'(t) \leq 0, \qquad \forall t > 0. \\ g'(t) \leq -\xi(t)g(t), \ \forall t \geq 0. \end{cases}$$

General Decay

Remark (1)

There are many functions satisfying **(G1)** and **(G2)**. Examples of such functions are

$$\begin{aligned} g(t) &= \frac{a}{(1+t)^{\nu}}, \quad \nu > 1 \\ g(t) &= ae^{-b(t+1)^p}, \quad 0 < p \leq 1. \end{aligned}$$

for a and b to be chosen properly.

Theorem 3 (Cavalcanti *et al.* 2001)

Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies (G1). Then problem (3.2) has a unique global solution

$$u \in \mathcal{C}(\mathbb{R}_+; H_0^1(\Omega)), \quad u_t \in \mathcal{C}(\mathbb{R}_+; L^2(\Omega))$$

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General Decay

The "modified" energy functional

$$E(t) := \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t), \qquad (3.3)$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_2^2 d\tau. \qquad (3.4)$$

Theorem 4 (Messaoudi 2008)

*Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g and ξ satisfy **(G1)** and **(G2)**. Then, for each $t_0 > 0$, there exist strictly positive constants K and λ such that the solution of (3.2) satisfies*

$$E(t) \le K e^{-\lambda \int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0.$$

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$$\boxed{E(t) \le K e^{-\lambda \int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0.}$$

Idea of proof

Let

$$F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{3.5}$$

where ε_1 and ε_2 are positive constants and

$$\Psi(t) \ : \ = \int_\Omega u u_t dx$$

$$\chi(t) \ : \ = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau dx.$$

General Decay

Lemma 5

If u is a solution of (3.2), then the energy satisfies

$$E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \tag{3.6}$$

Lemma 6

For ε_1 and ε_2 small enough, we have

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{3.7}$$

holds for two positive constants α_1 and α_2.

General Decay

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Lemma 7

Under the assumptions ***(G1)*** *and* ***(G2)****, the functional*

$$\Psi(t) := \int_\Omega u u_t dx$$

satisfies, along the solution of (3.2),

$$\Psi'(t) \leq \|u_t\|_2^2 - \frac{l}{2}\|\nabla u(t)\|_2^2 + C(g \circ \nabla u)(t). \tag{3.8}$$

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Lemma 8

Under the assumptions ***(G1)*** *and* ***(G2)***, *the functional*

$$\chi(t) := -\int_{\Omega} u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$$

satisfies, along the solution of (3.2), *for any* $\delta > 0$

$$\begin{aligned}\chi'(t) \le & -\left[\int_0^t g(\tau)d\tau - \delta\right]\|u_t\|_2^2 - \frac{C}{\delta}(g' \circ \nabla u)(t) \\ & + \delta\|\nabla u\|_2^2 + \frac{C}{\delta}(g \circ \nabla u)(t).\end{aligned} \tag{3.9}$$

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Lemma 8

*Under the assumptions **(G1)** and **(G2)**, the functional*

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$$\chi'(t) \leq -\left[\int_0^t g(\tau)d\tau - \delta\right] \|u_t\|_2^2 - \frac{C}{\delta}(g' \circ \nabla u)(t) + \delta\|\nabla u\|_2^2 + \frac{C}{\delta}(g \circ \nabla u)(t). \tag{3.9}$$

Idea of proof

Proof.

Since g is positive and $g(0) > 0$ then for any $t_0 > 0$ we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(\tau)d\tau = g_0 > 0, \ \forall t \geq t_0.$$

By using (3.5), (3.6), (3.8), (3.9), with suitable choice of constants we obtain for $t \geq t_0$,

$$F'(t) \leq -\beta_1 E(t) + \beta_2 (g \circ \nabla u)(t), \qquad \forall t \geq t_0. \tag{3.10}$$

Multiply (3.10) by $\xi(t)$ and recall Lemma 5

$$\begin{aligned} \xi(t)F'(t) &\leq -\beta_1 \xi(t) E(t) + \beta_2 (\xi g \circ \nabla u)(t) \\ &\leq -\beta_1 \xi(t) E(t) - \beta_3 (g' \circ \nabla u)(t) \\ &\leq -\beta_1 \xi(t) E(t) - K E'(t) \end{aligned}$$

Then

$$KE'(t) + \xi(t)F'(t) \leq -\beta_1 \xi(t) E(t)$$

Idea of proof

Note

$$\begin{aligned}(KE(t)+\xi(t)F(t))' &\leq KE'(t)+\xi(t)F'(t)\\ &\leq -\beta_1\xi(t)E(t)\end{aligned}$$

Use

$$L(t) = KE(t)+\xi(t)F(t) \backsim E(t) \tag{3.11}$$

to arrive at

$$\boxed{L'(t) \leq -\lambda\xi(t)L(t), \qquad \forall t \geq t_0}$$

A simple integration leads to

$$L(t) \leq L(t_0)e^{-\lambda\int_{t_0}^{t}\xi(\tau)d\tau}, \qquad \forall t \geq t_0.$$

Thus (3.11) yield

$$E(t) \leq Ce^{-\lambda\int_{t_0}^{t}\xi(\tau)d\tau}, \qquad \forall t \geq t_0. \tag{3.12}$$

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General Decay

Remark (2)

The estimate (3.12) is also true for $t \in [0, t_0]$ by virtue of continuity and boundedness of $E(t)$ and $\xi(t)$.

General Decay

Example 9

Let

$$g(t) = ae^{-(1+t)^{\nu}}, \;\; 0 < \nu \leq 1,$$

where $0 < a < 1$ is chosen so that $\int_0^{+\infty} g(t)dt < 1$. Then

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^{\nu}} = -\xi(t)g(t)$$

where $\xi(t) = \nu(1+t)^{\nu-1}$ which is nonincreasing and $\xi(0) > 0$. Therefore Theorem 4 gives

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General Decay

Example 10

Let

$$g(t) = \frac{a}{(1+t)^\nu}, \quad \nu > 2,$$

where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)dt < 1$.

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -\frac{\nu}{1+t}g(t) = -\xi(t)g(t), \tag{3.13}$$

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$$E(t) \leq \frac{c}{(1+t)^{\lambda\nu}}.$$

General Decay

Example 10

Let

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Alabau-Boussouira and Cannarsa (C. R. Acad. Sci. Paris (2009)) considered Problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0 \\ u(x,t) = 0, \quad x \in \partial\Omega \,, t \geq 0 \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{3.2}$$

in a bounded domain Ω and $t > 0$, with

$$g'(t) \leq -H\left(g(t)\right), \qquad \forall \; a.e. \; t \geq 0$$

- ☞ H is nonnegative measurable function on some interval $[0, k_0]$
- ☞ strictly increasing and of class C^1 on $[0, k_1]$, for $k_1 \leq k_0$
- ☞ $H(0) = H'(0) = 0$
- ☞ $H(s) \geq H_0 > 0, \quad \forall s \in [k_1, k_0]$
- ☞ $\int_0^{k_0} \frac{dx}{H(x)} = +\infty, \qquad \int_0^{k_0} \frac{xdx}{H(x)} < 1.$

Under the above hypotheses and an extra condition of the form

$$\lim_{s \to 0^+} \inf \frac{H(s)/s}{H'(s)} > \frac{1}{2},$$

they announced a decay result for the energy of (3.2), with an explicit rate of decay.
They also asked the question: how about

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General Decay

Mustafa and Messaoudi (2012) considered (3.2) under:

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$, with $H(0) = 0$, and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$ for some $r < 1$, such that

$$g'(t) \leq -H(g(t)), \quad \forall t \geq 0.$$

General Decay

Theorem 11

Let $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be given. Assume that* **(A1)** - **(A2)** *hold. Then there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the solution of* (3.2) *satisfies*

$$E(t) \leq k_3 H_1^{-1}(k_1 t + k_2) \qquad \forall t \geq 0, \tag{3.15}$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)} ds \qquad \text{and} \qquad H_0(t) = H(D(t))$$

provided that D *is a positive* C^1 *function, with* $D(0) = 0$, *for which* H_0 *is strictly increasing and strictly convex* C^2 *function on* $(0, r]$ *and*

$$\int_0^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))} ds < +\infty. \tag{3.16}$$

Moreover, if $\int_0^1 H_1(t)dt < +\infty$ *for some choice of* D, *then we have the improved estimate*

$$E(t) \leq k_3 G^{-1}(k_1 t + k_2) \qquad \text{where} \qquad G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)} ds.$$

General Decay

✓ Lasiecka, Messaoudi and Mustafa (2013) used iteration calculation to extend the range of the optimality in case of the polynomial decay.

✓ Cavalcanti et al (2016) characterized the energy decay by the solution of a corresponding ODE and obtained the optimality for the maximal range.

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Messaoudi and Al-Khulaifi (2017) considered

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0 \\ u(x,t) = 0, \quad x \in \partial\Omega\ , t \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{3.2}$$

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under:

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

(A2) There exists a differentiable function ξ such that

$$\begin{cases} \xi(t) > 0, \quad \xi'(t) \leq 0, & \forall t > 0. \\ g'(t) \leq -\xi(t)g^p(t), & 1 \leq p < \frac{3}{2}, \ \forall t \geq 0. \end{cases}$$

General Decay

Theorem 12

Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* ***(G1)*** *and* ***(G2)****. Then for each* $t_0 > 0$*, there exist strictly positive constants* K *and* λ *such that the solution of (3.2) satisfies, for all* $t \geq t_0$*,*

$$E(t) \leq K e^{-\lambda \int_{t_0}^{t} \xi(\tau) d\tau}, \qquad p = 1, \tag{3.17}$$

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(\tau) d\tau} \right]^{\frac{1}{2p-2}}, \qquad p > 1. \tag{3.18}$$

General Decay

Theorem 12

Moreover, if

$$\int_0^{+\infty} \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(\tau) d\tau} \right]^{\frac{1}{2p-2}} dt < +\infty, \qquad 1 < p < \frac{3}{2}, \qquad (3.19)$$

then

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{p}(\tau) d\tau} \right]^{\frac{1}{p-1}}, \qquad p > 1. \qquad (3.20)$$

General Decay

Example 13 (Revisited)

Let

$$g(t) = \frac{a}{(1+t)^{\nu}}, \quad \nu > 2,$$

where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)dt < 1$.

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^{\nu}}\right)^{\frac{\nu+1}{\nu}} = -bg^{p}(t), \qquad (3.21)$$

where $p = \frac{\nu+1}{\nu} < \frac{3}{2}$, $b > 0$.

General Decay

Example 13 (Revisited)

Therefore the condition (3.19), with $\xi(t) = b$, yields

$$\int_0^{+\infty} \left(\frac{1}{b^{2p-1}t + 1} \right)^{\frac{1}{2p-2}} dt < +\infty.$$

and hence by estimate (3.20) we get

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p-1}}} = \frac{C}{(1+t)^{\nu}},$$

which is the "optimal" decay rate.

General Decay

Mustafa (2017) considered (3.2) under

Hypotheses

(C1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function and

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(\tau)d\tau = l > 0.$$

(C2) There exist a differentiable function ξ and a C^2-function H which is ether linear or strictly increasing and strictly convex on $[0, r]$ with $H(0) = H'(0) = 0$ such that

$$\begin{cases} \xi(t) > 0, \quad \xi'(t) \leq 0, \qquad \forall t > 0. \\ g'(t) \leq -\xi(t)H(g(t)), \ \forall t \geq 0. \end{cases}$$

General Decay

Theorem 14

Let $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be given. Assume that* ***(C1)*** *-* ***(C2)*** *hold. Then there exist two positive constants* $k_1 \leq 1$ *and* k_2 *such that the energy functional of* (3.2) *satisfies*

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_{g^{-1}(r)}^{t} \xi(s) ds \right),$$

where

$$H_1(t) = \int_t^r \frac{ds}{sH'(s)} ds \qquad r \leq g(0).$$

Proof: Very technical and combines some new ideas with others from the proof of Theorem 11.

General Decay

Corollary 15

Under the conditions of Theorem 14, with

$$g'(t) \le -\xi(t) g^p(t), \qquad 1 \le p < 2,$$

the energy functional of (3.2) *satisfies*

$$E(t) \le k e^{-k_1 \int_0^t \xi(s) ds}, \qquad p = 1$$

$$E(t) \le k \left(1 + \int_0^t \xi(s) ds \right)^{\frac{-1}{p-1}}, \qquad 1 < p < 2.$$

Remark 3: This latter result of Mustafa extended the range of p from $[1, \frac{3}{2})$ to $[1, 2)$. So, the result of Al-Khulaifi and Messaoudi (2017) is only special case.

Summary

$$g'(t) \leq -\xi(t)H(g(t)), \;\; t \geq 0$$

✓ $\xi \equiv a > 0$, $H(s) = s^p$, $1 \leq p < \frac{3}{2} \implies g'(t) \leq -a g^p(t), \quad \forall t \geq 0.$
(Most of the work before 2008.)

✓ ξ is a function and $H(s) = s \implies g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0.$
General decay (Messaoudi 2008).

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Open Questions

✓ Case of "super" exponential

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Comments & Questions

THANK YOU FOR YOUR ATTENTION