Singularities in the Keller-Segel system

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Outline

1 - Introduction

- The Keller-Segel system:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^d. \qquad \textbf{(KS)}$$

- Modeling features:
  - Mathematical Biology: the *chemotaxis* phenomena, [Patlak '53], [Keller-Segel '70], [Nanjundiah '73], [Hillen-Painter '09]; Astrophysics: the gravitational interacting massive particles in a cloud, [Othmer-Stevens '90], [Stevens '00]), [Chavanis '08], [Hillen-Painter '08]; related to the miscroscopic description of the dynamics of particle systems in kinetic theory, [Chalub-Markowich-Perthame-Schmeiser '04]; etc.
  - Competition between dispersion of cells (diffusion) and aggregation;
  - Rich model from mathematical point of view, [Horstman '03 & '04], [Calvez-Carrillo-Hoffmann '16], [Biler '20], etc. The cross diffusive terms like $\nabla \cdot (u \nabla \Phi_u)$ lead to substantial difficulties compared to the standard theory of parabolic systems.

Basis features

The Keller-Segel system: $\boxed{\partial_t u = \nabla \cdot \left(\nabla u - u\nabla\Phi_u\right)}$

$$\Phi_u = -\mathcal{K}_d * u, \quad \mathcal{K}_d(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{for } d = 2, \\ \frac{1}{(d-2)\sigma_d}|x|^{2-d} & \text{for } d \geq 3. \end{cases}$$

- mass conservation: $M = \int_{\mathbb{R}^d} u_0(x)dx = \int_{\mathbb{R}^d} u(x,t)dx$;
- scaling invariance: $u_\gamma(x,t) = \frac{1}{\gamma^2}u\left(\frac{x}{\gamma}, \frac{t}{\gamma^2}\right), \quad \Phi_{u_\gamma}(x,t) = \Phi_u\left(\frac{x}{\gamma}, \frac{t}{\gamma^2}\right)$;
- $L^{\frac{d}{2}}$-critical: $\int_{\mathbb{R}^d} u_\gamma^{\frac{d}{2}} = \int_{\mathbb{R}^d} u^{\frac{d}{2}}$; L^1-supercritical for $d \geq 3$: $\int_{\mathbb{R}^d} u_\gamma = \gamma^{d-2}\int_{\mathbb{R}^d} u$;
- variational structure: $\mathcal{F}(u) = \int_{\mathbb{R}^d} u\left(\ln u - \frac{1}{2}\Phi_u\right)$ (*entropy* or *free energy*):

$$\frac{d}{dt}\mathcal{F}(u) = -\int_{\mathbb{R}^d} u|\nabla \log u - \nabla\Phi_u|^2 \leq 0.$$

The 8π problem in the 2D case

- If $M < 8\pi$: global existence + spreading, [Blanchet-Dolbeault-Perthame '06]. The proof mainly relies on the free energy functional $\mathcal{F}(u)$ and the Log HLS inequality.
- If $M = 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 u < +\infty$: **blowup in infinite time**, [Blanchet-Carrillo-Masmoudi '08]. Constructive approaches by [Ghoul-Masmoudi '18] (radial), [Davila-del Pino-Dolbeault-Musso-Wei '20] (full nonradial):

$$\|u(t)\|_{L^\infty} \sim c_0 \log t \quad \text{as} \quad t \to +\infty.$$

- If $M > 8\pi$: **blowup in finite time**, [Childress-Percus '81], [Jager-Luckhaus '92], [Nagai-Senba '98], [Senba-Suzuki '03]:

$$\text{(virial identity)} \quad \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)dx = \frac{M}{2\pi}(8\pi - M).$$

Constructive approaches in the radial setting by [Herrero-Velázquez '96], [Raphaël-Schweyer '14]:

$$\|u(t)\|_{L^\infty} \sim C_0 \frac{e^{\sqrt{2|\log(T-t)|}}}{T-t} \quad \text{as} \quad t \to T.$$

A numerical simulation for $d = 2$

A numerical simulation of blowup for the 2D Keller-Segel system

$$\partial_t u = \Delta u - \nabla.(u \nabla \Phi_u), \quad -\Delta \Phi_u = u.$$

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$$\partial_t u = \Delta u - \nabla.(u\nabla\Phi_u), \quad -\Delta\Phi_u = u.$$

Underlying problem

Existence and Stability of blowup solutions.

$$\boxed{\partial_t u = \nabla \cdot \big(\nabla u - u\nabla\Phi_u\big), \quad -\Delta\Phi_u = u \quad \text{in } \mathbb{R}^d}$$

2 - The two dimensional case: Statement of the result

Finite time blowup for the 2DKS

$$\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)$$

- Type I does not exist, [Senba-Suzuki '11]: $\partial_t u = \Delta u - \nabla u.\nabla\Phi_u + u^2$.
- Type II: "Δ dominates ∂_t" $\rightsquigarrow$ profile, **unknown blowup rates**.

Theorem 1 ([Collot-Ghoul-Masmoudi-Ng., 2021).

- There exists a set $\mathcal{O} \subset L^1 \cap \mathcal{E}$, where $\mathcal{E} = \{u : \sum_{k=0}^{2} \|\langle x\rangle^k \nabla^k u\|_{L^2} < +\infty\}$, of initial data u_0 (not necessary radially symmetric) such that

$$u(x,t) = \frac{1}{\lambda^2(t)}\left[Q\left(\frac{x-a(t)}{\lambda(t)}\right) + \varepsilon(x,t)\right], \quad Q(x) = \frac{8}{(1+|x|^2)^2},$$

where $a(t) \to \bar{a} \in \mathbb{R}^2$ and $\sum_{k=0}^{1} \|\langle y\rangle^k \nabla^k \varepsilon(t)\|_{L^2} \to 0$ as $t \to T$, and λ is given by either

$$\lambda(t) \sim 2e^{-\frac{\gamma+2}{2}}\sqrt{T-t}\exp\left(-\frac{\sqrt{|\log(T-t)|}}{\sqrt{2}}\right), \qquad \textbf{(C1)}$$

or

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{2}}|\log(T-t)|^{-\frac{\ell}{2(\ell-1)}}, \quad \ell \geq 2 \text{ integer.} \qquad \textbf{(C2)}$$

- Case (**C1**) is stable and Case (**C2**) is $(\ell-1)$-codimension stable.

Comments

$$\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u), -\Delta\Phi_u = u \text{ in } \mathbb{R}^2$$

- Existing results: formal level (numerical observation, formal matching asymptotic expansions) and in the radial setting to remove the nonlocal structure difficulty, i.e. $u(x,t) = u(r,t)$,

$$m(r) = \int_0^r u(\zeta)\zeta d\zeta, \quad u(r) = \frac{\partial_r m(r)}{r}, \quad \partial_r \Phi_u(r) = -\frac{m(r)}{r}, \quad r = |x|,$$

$$\partial_t u = \frac{1}{r}\partial_r\Big(r\partial_r u - ru\partial_r\Phi_u\Big) \quad \Longrightarrow \quad \boxed{\partial_t m = \partial_r^2 m - \frac{\partial_r m}{r} + \frac{\partial_r m^2}{2r}}$$

Refs: [Herrero-Velazquez '96 & '97], [Velazquez '02], [Schweyer-Raphael '14], [Dyachenko-Lushnikov-Vladimirova '13], ...

- The new result: full nonradial setting, refined description of the stable blowup mechanism, new (unstable) blowup dynamics, a nature approach via spectral analysis/robust energy-type method, a step toward a classification of all possible blowup behaviors, ...

2 - The two dimensional case: Existing formal/rigorous analysis

Formal analysis

$$\partial_t u = \nabla \cdot \left(\nabla u - u \nabla \Phi_u\right), -\Delta \Phi_u = u \text{ in } \mathbb{R}^2$$

- Formal analysis via matched asymptotic expansions [Velazquez '02]: working with the *self-similar variables*

$$u(x,t) = \frac{1}{T-t} w(z,\tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t),$$

$$\partial_\tau w = \nabla \cdot \left(\nabla w - w \nabla \Phi_w\right) - \frac{1}{2} \nabla \cdot (zw)$$

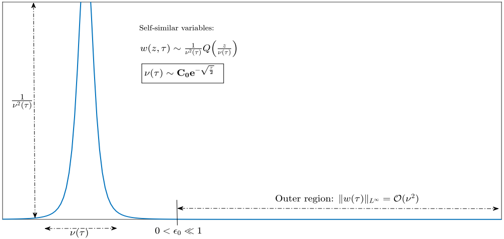

Fig 2: Understanding of the matched asymptotic expansions

Matched asymptotic expansions $\partial_\tau w = \nabla\cdot(\nabla w - w\nabla\Phi_w) - \frac{1}{2}\nabla\cdot(zw)$

- Inner approximate solution: $w^{\text{inn}}(z,\tau) = \frac{1}{\nu(\tau)^2} P\left(\frac{z}{\nu}, \tau\right),$

$$\nu^2\partial_\tau P = \nabla\cdot(\nabla P - P\nabla\Phi_P) + \sigma(\tau)\nabla\cdot(yP), \quad \sigma = \nu\nu_\tau - \frac{\nu^2}{2}.$$

- Expanding P: $P(y,\tau) = Q(y) + \sigma(\tau)T_1(y) + T_2(y,\tau)$, where

$$\mathscr{L}_0 T_1 = -\nabla\cdot(yQ), \quad \mathscr{L}_0 T_2 = \nu^2\sigma_\tau T_1 - \sigma^2(\tau)\nabla\cdot(yT_1) + lot.$$

$$\mathscr{L}_0 f = \nabla\cdot(\nabla f - f\nabla\Phi_Q - Q\nabla\Phi_f).$$

- Inner expansion: for $\nu \ll |z| < \epsilon_0$,

$$w^{inn}(z,\tau) = \underbrace{\frac{8\nu^2}{|z|^4}}_{Q} - \underbrace{\frac{4\sigma}{|z|^2}}_{\sigma T_1} \underbrace{-\sigma_\tau\left[\log|z| - \log\nu - \frac{5}{4}\right] + \frac{\sigma^2}{\nu^2}}_{T_2} + lot.$$

Matched asymptotic expansions $\partial_\tau w = \nabla \cdot (\nabla w - w\nabla\Phi_w) - \frac{1}{2}\nabla\cdot(zw)$

- Outer approximate solution: $w^{out} = \mathcal{O}(\nu^2)$, $\quad \frac{\partial \Phi_Q}{\partial |z|} \sim -\frac{4}{|z|}$ for $|z| \to 0$,

$$\partial_\tau w^{\text{out}} = \Delta w^{\text{out}} + \frac{4}{|z|}\frac{\partial w^{\text{out}}}{\partial |z|} - \frac{1}{2}\nabla\cdot\left(zw^{\text{out}}\right) := \mathscr{H} w^{out}.$$

- Expanding w^{out}: $\quad w^{out} = \nu^2 W_1 + \nu_\tau \nu W_2$, where $\mathscr{H} W_1 = 0$, $\quad \mathscr{H} W_2 = 2W_1$.
- Outer expansion: for $\nu \ll |z| < \epsilon_0$,

$$w^{out}(z,\tau) = \nu^2 \underbrace{\left[\frac{8}{|z|^4} + \frac{2}{|z|^2}\right]}_{W_1} + \nu_\tau \nu \underbrace{\left[-\frac{4}{|z|^2} + \log|z| - \frac{3}{4} - \frac{\log 4}{2} + \frac{\gamma}{2}\right]}_{W_2} + lot.$$

- Matching expansions yields the leading ODE:

$$\sigma_\tau \log \nu + \frac{5}{4}\sigma_\tau + \frac{\sigma^2}{\nu^2} = -\left(\frac{3}{4} + \frac{\log 4}{2} - \frac{\gamma}{2}\right)\nu\nu_\tau \quad \Longrightarrow \quad \boxed{\nu(\tau) = C_0 e^{-\sqrt{\frac{\tau}{2}}}}$$

- Analysis of the stability was formally done by [Velazquez '02] at the linear level.

Existing rigorous analysis $\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u), -\Delta\Phi_u = u$ in $\mathbb{R}^2$

- Rigorous analysis via modulation techniques [Schweyer-Raphael '14]: working with the *blowup variables*:

$$u(x,t) = \frac{1}{\lambda^2} v(y,s), \quad y = \frac{x}{\lambda}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad \Big(\lambda(t) > 0 \text{ unknown}\Big),$$

$$\boxed{\partial_s v = \Delta v - \nabla\cdot(v\nabla\Phi_v) - b\nabla\cdot(yv)} \quad b = -\lambda_t \lambda.$$

- Approximate solution:

$$v^{app}(y;b) = Q(|y|) + bT_1(|y|) + S_2(|y|;b),$$

$$\mathscr{L}_0 T_1 = \nabla.(yQ), \quad \mathscr{L}_0 S_2 = b^2\nabla.(yT_1) + b_s T_1 + lot.$$

Improving S_2 in the blowup zone $|y| \sim \frac{1}{\sqrt{b}}$ leads to the leading ODE ($m_{T_1} \sim c_1 \ln r$)

$$b_s = -\frac{2b^2}{|\log b|} \quad \Longrightarrow \quad \lambda(t) = \sqrt{T-t}\, e^{-\sqrt{\frac{|\log(T-t)|}{2}} + \mathcal{O}(1)}.$$

- Control of the remainder $\varepsilon = v - v^{app}$: based on the special structure

$$\mathscr{L}_0 \varepsilon = \nabla\cdot\Big(Q\nabla\mathscr{M}\varepsilon\Big), \quad \mathscr{M}\varepsilon = \frac{\varepsilon}{Q} - \Phi_\varepsilon.$$

Requirement: radial + L^1 smallness + a complicated treatment for $b\nabla.(y\varepsilon)$.

2 - The two dimensional case:
A new framework for blowup analysis

Strategy of the new constructive proof

$$\partial_t u = \nabla \cdot \left(\nabla u - u \nabla \Phi_u\right) \quad \text{in} \quad \mathbb{R}^2$$

- Self-similar variables:

$$u(x,t) = \frac{1}{T-t} w(z,\tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t),$$

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2}\nabla \cdot (zw)$$

- Linearized problem: $w(z,\tau) = Q_\nu(z) + \eta(z,\tau)$, where $Q_\nu(z) = \frac{1}{\nu^2} Q\left(\frac{z}{\nu}\right)$ and η solves

$$\partial_\tau \eta = \mathscr{L}^\nu \eta + \left(\frac{\nu_\tau}{\nu} - \frac{1}{2}\right) \nabla \cdot (zQ_\nu) - \nabla \cdot \Big(\eta \Phi_\eta\Big), \qquad \nu \to 0 \ \text{ unknown},$$

$$\mathscr{L}^\nu \eta = \underbrace{\nabla \cdot \left(\nabla \eta - \eta \nabla \Phi_{Q_\nu} - Q_\nu \nabla \Phi_\eta\right)}_{\equiv \mathscr{L}_0^\nu \eta} - \frac{1}{2}\nabla \cdot (z\eta)$$

- Structure of $\mathscr{L}_0^\nu$:

$$\mathscr{L}_0^\nu \eta = \nabla \cdot \left(Q_\nu \nabla \mathscr{M}^\nu \eta\right), \quad \mathscr{M}^\nu \eta = \frac{\eta}{Q_\nu} - \Phi_\eta.$$

($\mathscr{M}^\nu$ comes from the linearization of the energy functional $\mathcal{F}$ around Q_ν).

Key properties for the linear analysis: radial sector

- In the radial sector, the (nonlocal) operator $\mathscr{L}^\nu$ becomes a local operator through the partial mass setting, i.e. $\zeta = |z|$, $m_f(\zeta) = \int_0^\zeta f(r) r dr$,

$$\mathscr{L}^\nu f = \frac{1}{\zeta}\partial_\zeta\left(\mathscr{A}^\nu m_f\right), \quad \boxed{\mathscr{A}^\nu \phi = \zeta\partial_\zeta\left(\frac{\partial_\zeta \phi}{\zeta}\right) - \frac{\partial_\zeta(m_{Q_\nu}\phi)}{\zeta} - \frac{1}{2}\zeta\partial_\zeta\phi} \equiv \mathscr{A}_0^\nu\phi - \frac{1}{2}\zeta\partial_\zeta\phi.$$

- [Collot-Ghoul-Masmoudi-Ng., '21]: $\mathscr{A}^\nu$ is self-adjoint in $L^2_{\frac{\omega_\nu}{\zeta}}$, its eigenvalues are

$$\boxed{\operatorname{spec}(\mathscr{A}^\nu) = \left\{\alpha_{n,\nu} = 1 - n + \frac{1}{2\ln\nu} + \mathcal{O}\left(\frac{1}{|\ln\nu|^2}\right), \quad n \in \mathbb{N}\right\}} \quad \omega_\nu = \frac{e^{-\frac{\zeta^2}{4}}}{Q_\nu}.$$

The eigenfunction $\phi_{n,\nu}$ solving $\mathscr{A}^\nu\phi_{n,\nu} = \alpha_{n,\nu}\phi_{n,\nu}$ is defined by

$$\phi_{n,\nu}(\zeta) = \sum_{j=0}^{n} c_{n,j}\nu^{2j-2}T_j\left(\frac{\zeta}{\nu}\right) + \text{l.o.t}, \quad \mathscr{A}_0^\nu T_{j+1} = -T_j, \quad T_0 = \xi\partial_\xi m_Q.$$

Proof: Schrödinger type operator $\rightsquigarrow$ *discreteness*, Sturm comparison principle $\rightsquigarrow$ *uniqueness*, matching asymptotic expansions + implicit function theorem $\rightsquigarrow$ $(\alpha_{n,\nu}, \phi_{n,\nu})$.

Key properties of the linear analysis: nonradial sector

- First expression:

$$\mathscr{L}^{\nu} f = \mathscr{L}_0^{\nu} f - \frac{1}{2}\nabla.(zf) \quad \text{with} \quad \mathscr{L}_0^{\nu} f = \nabla \cdot (Q_\nu \nabla \mathscr{M}^{\nu} f) \text{ and } \mathscr{M}^{\nu} f = \frac{f}{Q_\nu} - \Phi_f,$$

The operator $\mathscr{L}_0^{\nu}$ is self-adjoint in L^2 with respect to the inner product

$$\langle f, g\rangle_{\mathscr{M}^\nu} = \int_{\mathbb{R}^2} f \mathscr{M}^{\nu} g \, dz, \quad \text{(positivity)} \quad \langle f, f\rangle_{\mathscr{M}^\nu} \sim \int_{\mathbb{R}^2} \frac{f^2}{Q_\nu} \, dz.$$

- Second expression:

$$\mathscr{L}^{\nu} f = \mathscr{H}^{\nu} f - \nabla Q_\nu \cdot \nabla \Phi_f \quad \text{with} \quad \mathscr{H}^{\nu} f = \frac{1}{\omega_\nu}\nabla \cdot \Big(\omega_\nu \nabla f\Big) + (2Q_\nu - 2)f.$$

The operator $\mathscr{H}^{\nu}$ is self-adjoint in $L^2_{\omega_\nu}$ with $\omega_\nu = \frac{e^{-\frac{|z|^2}{4}}}{Q_\nu}$.

- The well-adapted scalar product and coercivity [Collot-Ghoul-Masmoudi-Ng., '21]:

$$\boxed{\int_{\mathbb{R}^2} \mathscr{L}^{\nu}(f\sqrt{\rho})\mathscr{M}^{\nu}(f\sqrt{\rho}) \leq -c \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{Q_\nu} \rho dz} \quad \rho = e^{-\frac{|z|^2}{4}}.$$

Approximate solution

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2}\nabla \cdot (zw)$$

- The approximate solution: for $\ell \geq 1$ integer,

$$w^{app}(z,\tau) = Q_\nu(z) + \underbrace{a_\ell(\tau)\big[\varphi_{\ell,\nu}(|z|) - \varphi_{0,\nu}(|z|)\big]}_{\text{modification driving the law of blowup}} \quad \text{with} \quad \varphi_{n,\nu} = \frac{\partial_\zeta \phi_{n,\nu}}{\zeta}.$$

A suitable projection onto $\varphi_{\ell,\nu}$ and compatibility condition, we obtain the leading ODE

$$(\ell = 1, \text{ stable}) \quad \frac{\nu_\tau}{\nu} = \frac{1}{4\ln \nu} + \frac{e_2}{|\ln \nu|^2} \implies \boxed{\nu = C_0 e^{-\sqrt{\frac{\tau}{2}}}}$$

$$(\ell \geq 2, \text{ unstable}) \quad \frac{\nu_\tau}{\nu} = \frac{1-\ell}{2} + \frac{\ell}{4\ln \nu} \implies \boxed{\nu = C_\ell e^{\frac{(1-\ell)\tau}{2}} \tau^{\frac{\ell}{2(1-\ell)}}}$$

- The linearized equation: $\varepsilon = w - w^{app}$,

$$\partial_\tau \varepsilon = \mathscr{L}^\nu \varepsilon + Error + SmallLinear + Nonlinear.$$

Control of the remainder

$$\partial_\tau \varepsilon = \mathscr{L}^\nu \varepsilon + Error + \cdots$$

- Decomposition: $\varepsilon = \varepsilon^0 + \varepsilon^\perp$, $\varepsilon^0(\zeta) = \frac{\partial_\zeta m_\varepsilon}{\zeta}$,

$$\partial_\tau m_\varepsilon = \mathscr{A}^\nu m_\varepsilon + m_E + \cdots, \quad \partial_\tau \varepsilon^\perp = \mathscr{L}^\nu \varepsilon^\perp + \cdots .$$

- For the radial part, we use the spectral gap

$$\langle m_\varepsilon, \mathscr{A}^\nu m_\varepsilon \rangle_{L^2_{\frac{\omega_\nu}{\zeta}}} \leq \alpha_{N+1,\nu} \|m_\varepsilon\|^2_{L^2_{\frac{\omega_\nu}{\zeta}}} \quad \text{for} \quad m_\varepsilon \perp \phi_{n,\nu}, \quad n = 0, ..., N.$$

$$\Longrightarrow \quad \boxed{\frac{d}{d\tau}\|m_\varepsilon\|^2_{L^2_{\frac{\omega_\nu}{\zeta}}} \leq -\|m_\varepsilon\|^2_{L^2_{\frac{\omega_\nu}{\zeta}}} + C\frac{\nu^2}{|\ln \nu|^2}}$$

- For the nonradial part, we use the coerivity of $\mathscr{L}^\nu$ and the well-adapted norm

$$\|\varepsilon^\perp\|_0^2 = \int_{\mathbb{R}^2} \varepsilon^\perp \sqrt{\rho}\mathscr{M}^\nu(\varepsilon^\perp \sqrt{\rho})\, dz \sim \int_{\mathbb{R}^2} \frac{|\varepsilon^\perp|^2}{Q_\nu}\rho\, dz.$$

$$\Longrightarrow \quad \boxed{\frac{d}{d\tau}\|\varepsilon^\perp\|_0^2 \leq -c\|\varepsilon^\perp\|_0^2 + Ce^{-2\kappa\tau}} \quad 0 < \kappa \ll 1.$$

A significant issue in the nonlinear analysis

- The perturbation ε can be large near the origin due to the present of the resonance $\nabla.(zQ_\nu)$, and the sole $L^2_{\omega_\nu}$ orthogonality conditions do not allow for a dissipation type estimate here.
- The idea is to slightly modify the decomposition according to the new parameter $\tilde\nu \sim \nu$:

$$w = Q_{\tilde\nu} + a_\ell \Big[\varphi_{\ell,\tilde\nu} - \varphi_{0,\tilde\nu}\Big] + \tilde\varepsilon,$$

and impose the local orthogonality condition

$$\int_{\mathbb{R}^2} \tilde\varepsilon \nabla.(zQ_{\tilde\nu})\chi_M = 0.$$

- The spectral structure of the perturbation operator in the radial sector:

$$\bar{\mathscr{A}} = \frac{\mathscr{A}^\nu + \mathscr{A}^{\tilde\nu}}{2},$$

remains the same, and the spectral gap still holds true.

3 - Higher dimensional cases: Collapsing-ring/Traveling blowup solutions

Collapsing-ring/Traveling blowup solutions for $d \geq 3$

- Basis features: mass conservation, scaling symmetry $u_\gamma(x,t) = \gamma^2 u(\gamma x, \gamma^2 t)$, L^1-supercritical: $\int_{\mathbb{R}^d} u_\gamma = \gamma^{d-2} \int_{\mathbb{R}^d} u$.
- Radial setting: $r = |x|$, $m_u(r,t) = \int_0^r u(\zeta, t)\zeta^{d-1} d\zeta$,

$$\partial_t u = \partial_r^2 m_u + \frac{d-1}{r}\partial_r m_u + \frac{\partial_r(u m_u)}{r^{d-1}},$$

$$\boxed{\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}}$$

- Collapsing-ring/traveling solutions blow up in finite time:

$$m_u(r,t) = M_0 Q\left(\frac{r - R(t)}{\lambda(t)}\right), \quad 0 < \lambda(t) \ll R(t) \to 0 \ \text{as}\ t \to T,$$

- Blowup solutions with arbitrary mass $M_0 < +\infty$, different from the 2D case.

Traveling blowup solution in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

Theorem 2 ([Collot-Ghoul-Masmoudi-Ng., 2021]).

- There exists a set $\mathcal{O} \subset W^{1,\infty}(\mathbb{R}_+)$ of initial data $m_u(0)$ such that

$$m_u(r,t) = M(t)\left[Q\left(\frac{r-R(t)}{\lambda(t)}\right) + m_\varepsilon(r,t)\right], \quad Q(\xi) = \frac{e^{\frac{\xi}{2}}}{1+e^{\frac{\xi}{2}}},$$

where $\|m_\varepsilon(t)\|_{W^{1,\infty}(\mathbb{R}_+)} \to 0$ as $t \to T$,

$$\partial_t M \sim 0, \quad \lambda = \frac{R^{d-1}}{M}, \quad R(t) \sim \left[(d/2)M(T-t)\right]^{\frac{1}{d}}.$$

- The constructed solution is stable under small perturbation in $\mathcal{O}$.

3 - Higher dimensional cases: Formal derivation of the blowup law

Formal explanation of the blowup law

The equation in the radial setting:

$$\partial_t u = \partial_r^2 u + \frac{d-1}{r}\partial_r u + \frac{1}{r^{d-1}}\partial_r(u\, m_u).$$

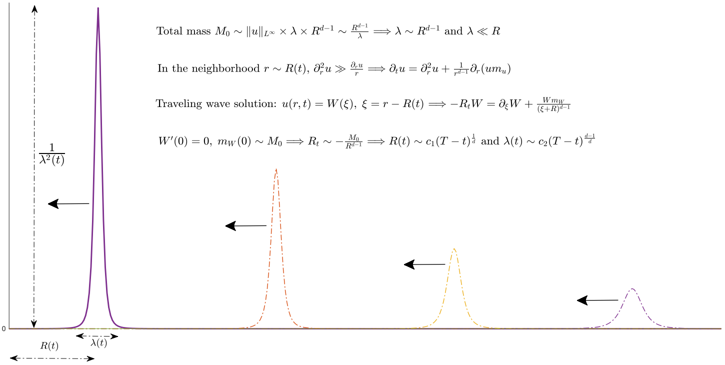

Fig 3: Collapsing-ring/traveling blowup solutions.

Traveling shock solutions in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

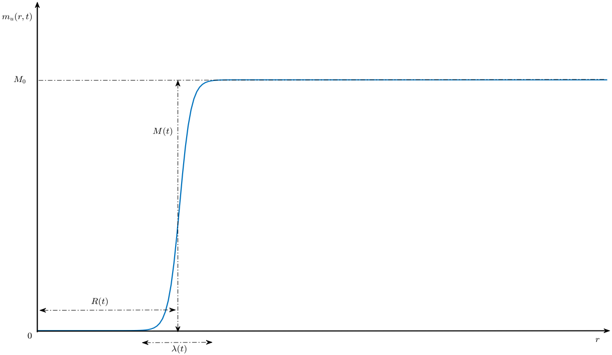

Fig 4: Illustration of a traveling shock solution to the partial mass equation.

A numerical simulation for $d = 3$

Fig 5: (horizontally zoomed solution) The initial data $m_u(r, 0) = MQ\left(\frac{r - M^{\frac{1}{3}}\epsilon}{M^{-\frac{1}{3}}\epsilon^2}\right)$*, where* $M = 27$ *and* $\epsilon = 0.7$*. With* $\epsilon = 0.7$*, the theoretical blowup time is* $T = \epsilon^3 \approx 0.343$*. Maple solver gives an approximation of the blowup time by saying "could not compute solution for* $t > 0.32$*: Newton iteration is not converging".*

3 - Higher dimensional cases: Ideas of the analysis

Renormalized variables

- Hyperbolic inviscid variables: $m_u(r,t) = M(t)m_w(\zeta,\tau), \quad \zeta = \frac{r}{R}, \quad \frac{d\tau}{dt} = \frac{M}{R^d}$

$$\partial_\tau m_w = \left(\frac{m_w}{\zeta^{d-1}} - \frac{1}{2}\zeta\right)\partial_\zeta m_w + \nu\left(\partial_\zeta^2 m_w - \frac{d-1}{\zeta}\partial_\zeta m_w\right) + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\zeta\partial_\zeta m_w - \frac{M_\tau}{M}m_w, \quad \nu = \frac{R^{d-2}}{M}.$$

- Blowup variables: $m_w(\zeta,\tau) = m_v(\xi,s), \quad \xi = \frac{\zeta-1}{\nu}, \quad \frac{ds}{d\tau} = \frac{1}{\nu}$

$$\partial_s m_v = \partial_\xi^2 m_v + m_v\partial_\xi m_v - \frac{1}{2}\partial_\xi m_v + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\partial_\xi m_v - \frac{M_s}{M}m_v + \left(\frac{1}{(1+\xi\nu)^{d-1}} - 1\right)m_v\partial_\xi m_v - \nu\frac{d-1}{1+\nu\xi}\partial_\xi m_v + \left(\frac{R_\tau}{R}\nu + \nu_\tau\right)\xi\partial_\xi m_v.$$

Linearized equation

- Linearized equation: $m_v = Q + m_q$, where

$$Q'' - \frac{1}{2}Q' + QQ' = 0, \quad \lim_{y\to-\infty} Q(\xi) = 0, \quad Q(\xi) = \frac{e^{\frac{\xi}{2}}}{1+e^{\frac{\xi}{2}}},$$

$$\partial_s m_q = \mathscr{L}_0 m_q + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\partial_\xi Q - \frac{M_s}{M}Q + L(m_q) + \frac{m_q \partial_\xi m_q}{(1+\nu\xi)^{d-1}} + \Psi,$$

where $\mathscr{L}_0$ is a self-adjoint operator in $L^2_{\omega_0}$ with $\omega_0 = Q^{-2}e^{\frac{\xi}{2}}$,

$$\mathscr{L}_0 = \partial_\xi^2 - \left(\frac{1}{2} - Q\right)\partial_\xi + Q', \quad \mathscr{L}_0 Q' = 0,$$

- Orthogonality conditions:

$$\int_{-1/\nu}^{\infty} m_q Q' \chi_A \omega_0 d\xi = 0 \implies \langle m_q, \mathscr{L}_0 m_q\rangle_{L^2_{\omega_0}} \leq -\delta_0 \|m_q\|^2_{H^1_{\omega_0}},$$

and

$$\int_0^\infty m_q \chi_{|\log\nu|}\Big(\xi - 4|\log\nu|\Big) d\xi = 0 \implies \exists \xi^* \in (|\log\nu|, 6|\log\nu|) \ \ s.t \ \ m_q(\xi^*) = 0.$$

Modulation equations and Bootstrap regime

- Solution decomposition:

$$m_q^{\text{inn}} = m_q \chi_{2|\log \nu|}, \quad m_q^{\text{out}} = m_q \left(1 - \chi_{|\log \nu|}\right).$$

- Modulation equations:

$$\left| \frac{R_\tau}{R} + \frac{1}{2} \right| \lesssim \|m_q^{\text{inn}}\|_{L^2_{\omega_0}} + \nu + \left| \frac{M_s}{M} \right| \quad \text{and} \quad \left| \frac{M_s}{M} \right| \lesssim \|\partial_\xi m_q^{\text{out}}\|_{L^\infty} + \nu^2.$$

- Bootstrap estimates: for $0 < K^{-1}, \kappa \ll 1$,

$$\|m_q\|_{\text{inn}} \leq K e^{-\kappa\tau}, \quad \|\partial_\xi m_q^{\text{out}}\|_{L^\infty} \leq \sqrt{K} \nu e^{-\kappa\tau},$$

where

$$\|m_q\|_{\text{inn}}^2 = -\langle m_q^{\text{inn}}, \mathscr{L}_0 m_q^{\text{inn}} \rangle_{L^2_{\omega_0}} \sim \|m_q^{\text{inn}}\|_{H^1_{\omega_0}}^2.$$

- Improve estimates:

$$\frac{d}{ds} \|m_q\|_{\text{inn}}^2 \leq -c_0 \|m_q\|_{\text{inn}}^2 + C\nu^{-2} \|\partial_\xi m_q^{\text{out}}\|_{L^\infty} + C\nu^2,$$

and $m_\varepsilon^{\text{out}}(\zeta, \tau) = m_q^{\text{out}}(\xi, s)$,

$$\frac{d}{d\tau} \|\partial_\zeta m_\varepsilon^{\text{out}}\|_{L^\infty} \leq -\frac{1}{2} \|\partial_\zeta m_\varepsilon^{\text{out}}\|_{L^\infty} + \ldots$$

4. Conclusion & Perspectives

Conclusion and Perspectives

- Existence/Stability of blowup solutions via constructive approaches.
- Adaptability and Flexibility for studying singularity formation in other nonlinear problems, especially for wave-type equations.
- Interesting problems:
  - multiple-collapse phenomena/ interaction-collision of multi-solitons;
  - *classification* of blowup dynamics (rates & profiles);
  - Numerical methods for blowup problems (detection, rates & profiles).