Long time behavior of MHD waves

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Contents

The 2D incompressible MHD equations in a channel $\mathbb{T} \times [-1, 1]$:

$$(MHD) \begin{cases} \partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} - \mathbf{B} \cdot \nabla \mathbf{B} + \nabla \mathbf{P} = 0, \\ \partial_t \mathbf{B} + \mathbf{V} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{V} = 0, \\ \nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{B} = 0, \\ \mathbf{V}_2(t,x,y)\big|_{y=-1,1} = 0, \quad \mathbf{B}_2(t,x,y)\big|_{y=-1,1} = 0, \end{cases}$$

Equilibrium (Background velocity and magnetic field):

$$\mathbf{V}_s = \big(u(y), 0\big), \ \mathbf{B}_s = \big(b(y), 0\big), \ \mathbf{P}_s = constant.$$

Notations:

- $u(y) = 0$, non-flowing plasma,
    - $b(y) \equiv 1$ homogeneous case,
    - general $b(y)$ inhomogeneous case;
- $u(y) \neq 0$, flowing plasma.

Question

We would like to study the long time behavior of the solutions near the equilibrium (shear flow and sheared magnetic field) at both linear and nonlinear level.

Let $V = (V_1, V_2) = \mathbf{V} - (u(y), 0)$ and $H = (H_1, H_2) = \mathbf{B} - (b(y), 0)$ be the perturbed velocity and magnetic field.

2D linearized MHD equation

The linearized MHD equation around $(u(y), 0)$ and $(b(y), 0)$.

$$\begin{cases} \partial_t V_1 + u\partial_x V_1 + \partial_x p + u' V_2 - b\partial_x H_1 - b' H_2 = 0, \\ \partial_t V_2 + u\partial_x V_2 + \partial_y p - b\partial_x H_2 = 0, \\ \partial_t H_1 + u\partial_x H_1 + b' V_2 - b\partial_x V_1 - u' H_2 = 0, \\ \partial_t H_2 + u\partial_x H_2 - b\partial_x V_2 = 0, \\ \nabla \cdot V = 0, \quad \nabla \cdot H = 0. \end{cases} \tag{1}$$

We introduce the vorticity $w = \partial_x V_2 - \partial_y V_1$ and the current density $j = \partial_x H_2 - \partial_y H_1$ which satisfy

$$\begin{cases} \partial_t w + u\partial_x w - b\partial_x j = u'' V_2 - b'' H_2, \\ \partial_t j + u\partial_x j - b\partial_x w = b'' V_2 - u'' H_2 + 2u' \partial_x H_1 + 2b' \partial_y V_2, \end{cases}$$

Let $w = -\Delta\psi$ and $j = -\Delta\phi$, where ψ and ϕ are stream function and the magnetic potential function.

Historical Comments: non-flowing plasma, homogeneous

Non-flowing plasma, homogeneous case: $(\partial_{tt} - \partial_{xx})(j, w) = 0$, no decay (1-D wave).

Global nonlinear stability: C. Bardos, C. Sulem, and P. L. Sulem, 1988, Tran. Amer. Math. Soci. Note $x \in \mathbb{R}$ the decay in space gives the decay in time.

Key idea(observation):

- Elsässer valuables: $z^+ = \mathbf{V} + \mathbf{B}$ and $z^- = \mathbf{V} - \mathbf{B}$;
- The fluctuations $z^\pm$ propagate along the background magnetic field in opposite directions. [1]
- The strong magnetic field reduce the nonlinear interactions and inhibit formation of strong gradients. [from nonlinear PDE to quasilinear PDE]

[1] $\mathbf{B} \sim (1, 0)$ and $\mathbf{V} \sim 0$.

Historical comments: inhomogeneous case

Inhomogeneous case:
Grossmann and Tataronis predicted decay rate $\frac{1}{t}$ in 1973.

Z. Physik 261, 203–216 (1973)

Decay of MHD Waves by Phase Mixing

I. The Sheet-Pinch in Plane Geometry

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Max-Planck-Institut für Plasmaphysik, Euratom Association**,
Garching bei München, Germany

Received March 13, 1973

The mechanism leading to the damping is the phase mixing phenomenon.

Historical comments: flowing plasma

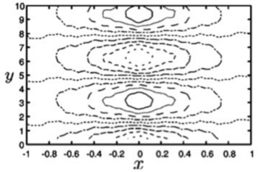


FIG. 4. Contour plot of $\tilde{b}_x(x,t)e^{ik_y y}$ at $t=40$ with the parameters $\alpha=0.5$, $\beta=1$, $k_y=1$, and $[-L_1,L_2]=[-1,1]$. This is a typical numerical solution of the initial value problem (16).

algebraic growth of $\tilde{b}_\parallel(x_r,t)\propto t$ in addition to $\tilde{v}_\parallel(x_r,t)\propto t$. The algebraic instability we have shown can be more important than that of Lau and Liu,[17,18] for the growth of the latter instability is transient and saturated in a finite time.

Asymptotic behavior of the master variables, $\tilde{v}_x(x,t)$ and $\tilde{b}_x(x,t)$, changes discontinuously when the flow shear (α) exceeds the magnetic shear (β). It is interesting to note that for large B_z(=const) and small k_z, the master variables are, respectively, similar to the stream function and the flux function in the reduced MHD theory (see Appendix C). For weak flow shear ($|\alpha|<|\beta|$), the asymptotic behavior given by (67) and (68) represents the generation of a magnetic island as shown in Fig. 4. On the other hand, for strong flow shear ($|\alpha|>|\beta|$), the asymptotic behavior makes the transition to (73) and (74), where the stretching effect of shear flow de-

Flowing plasma: Hirota, Tatsuno and Yoshida studied the spectral of the linear operator and formally proved the existence of "magnetic island".

$$(67).\ \widehat{V_2}(t) \to \frac{\alpha}{\beta}\hat{H}_2^{in}(\alpha, y),$$

$$(68).\ \widehat{H_2}(t) \to \hat{H}_2^{in}(\alpha, y).$$

Figure: Physics of Plasmas 12, 012107 (2005)

Magnetic island [H. Zohm, Magnetohydrodynamic Stability of Tokamaks]

Reconnection of field lines is the process by which the topology of a flux surface structure in a plasma can change. It occurs in situation in which magnetic field lines of opposing direction occur close to each other, indicating the presence of a current sheet. While this current sheet persists infinitely long in ideal MHD, it decays on the resistive timescale in resistive MHD. By forming topologically new objects, the so-called magnetic islands, the free energy of the system can be reduced.This instability, which tears and reconnects field lines, is called a tearing mode.

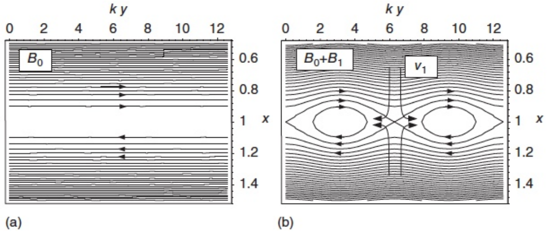


Figure 8.1 (a and b) Reconnection of field lines in an area of opposing field. During the process, magnetic islands are formed. Also indicated is the flow pattern of plasma into the island during formation.

Magnetic islands and magnetic reconnection

Magnetic islands are the regions enclosed by magnetic field lines and separated by reconnection points.

Magnetic reconnection is a ubiquitous plasma process which changes the magnetic field topology. The process starts when two oppositely directed magnetic field lines bend towards each other and touch at a reconnection point. After that, field lines break, pair and reconnect. This generates closed regions called magnetic islands

Main results: Linear damping

Theorem (Damping, Ren, Z, SIAM Jour. Math. Anal. 2017)

Let $b(y) \in C^4([-1,1])$ *be a strictly monotone and positive function with* $b(y) \geq c_0$, $b'(y) \geq c_0$ *for a fixed constant* $c_0 > 0$. *Assume that* $\int_{\mathbb{T}} \omega_0(x,y)dx = 0$, $\int_{\mathbb{T}} j_0(x,y)dx = 0$. *Then it holds that,*

1. *if* $\omega_0, j_0 \in H_x^{k-1}L_y^2$, *for* $k \in \mathbb{N}$, *then*

$$\|\partial_t^k H_2\|_{L^2_{x,y}} + \|\partial_t^k V_2\|_{L^2_{x,y}} \lesssim \langle t\rangle^{-1}\big(\|\omega_0\|_{H_x^{k-1}L_y^2} + \|j_0\|_{H_x^{k-1}L_y^2}\big),$$

2. *if* $\omega, j_0 \in H_x^{l-\frac{3}{2}}H_y^1$, *for* $l = 0, 1$, *then*

$$\|\nabla^l_{x,y} P\|_{L^2_{x,y}} \leq \frac{C}{\langle t\rangle(1+\ln\langle t\rangle)^2}\big(\|\omega_0\|_{H_x^{l-\frac{3}{2}}H_y^1} + \|j_0\|_{H_x^{l-\frac{3}{2}}H_y^1}\big),$$

3. *if* $u_1, u_2, b_1, b_2 \in L^2_{x,y}$, *then there exists* $\mathcal{A}^{\pm}_{\infty}(x,y) \in L^2_{x,y}$, *such that,*

$$\big\|(H_1 \pm V_1)(t, x \mp tb(y), y) - \mathcal{A}^{\pm}_{\infty}(x,y)\big\|_{L^2_{x,y}} \longrightarrow 0 \quad as \quad t \to +\infty.$$

Main results: Magnetic island

Theorem (Magnetic island, Zhai, Zhang, Z. 2018)

Assume that $u(y), b(y) \in C^5([-1,1])$ *satisfy* $u(0) = b(0) = 0$ *and* $b'(y) > |u'(y)|$ *and let* $\big(\psi(t,x,y), \phi(t,x,y)\big)$ *be the solution of linearized MHD with initial data* $(\psi_0, \phi_0) \in H^3(-1,1) \times H^4(-1,1)$. *There holds that,*

1. for $y = 0$, *as* $t \to +\infty$ *and* $\alpha \neq 0$

$$\widehat{\psi}(t,\alpha,0) \to \frac{u'(0)}{b'(0)} \widehat{\phi}_0(\alpha,0), \quad \widehat{\phi}(t,\alpha,0) \equiv \widehat{\phi}_0(\alpha,0);$$

2. as $t \to +\infty$, *there exists* $\Gamma(\alpha,y)$ *such that*

$$\widehat{\psi}(t,\alpha,y) \to -\frac{u(y)}{b(y)} \big(b(y)\Gamma(\alpha,y)\big) \widehat{\phi}_0(\alpha,0),$$

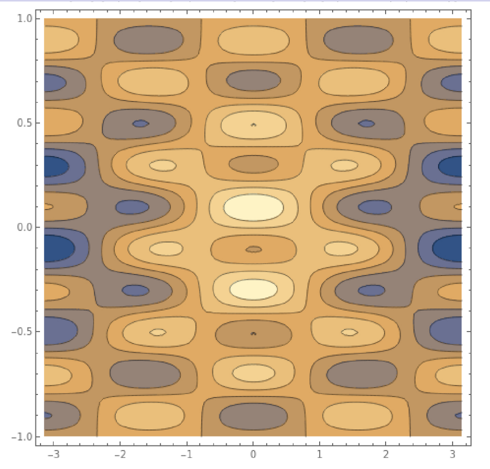
$$\widehat{\phi}(t,\alpha,y) \to -\big(b(y)\Gamma(\alpha,y)\big) \widehat{\phi}_0(\alpha,0),$$

Note that $\widehat{V}_2 = i\alpha\widehat{\psi}$ and $\widehat{H}_2 = i\alpha\widehat{\phi}$.

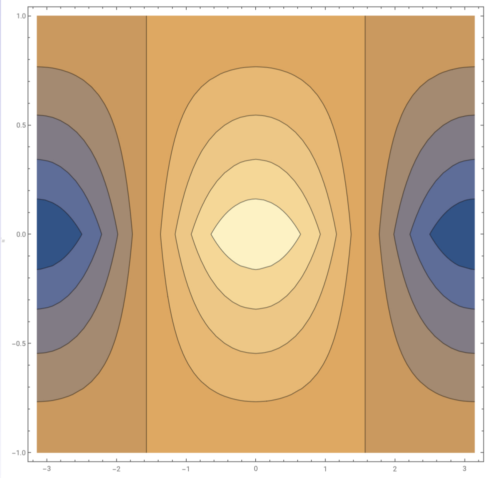
Remarks

- We have representation formula of the final state:
$$\Gamma(\alpha,y)=\begin{cases}\Gamma^{+}(\alpha,y), & y\geq 0,\\ \Gamma^{-}(\alpha,y), & y\leq 0.\end{cases}$$ with
$$\Gamma^{\pm}(\alpha,y)=\frac{1}{b'(0)}\int_{\pm 1}^{y}\frac{\varphi_{\pm}(\alpha,y)(u'(0)^2-b'(0)^2)}{(u(y')^2-b(y')^2)\varphi_{\pm}(\alpha,y')^2}dy',$$
where $\varphi_{\pm}$ solves $\partial_y\left((u^2-b^2)\partial_y\varphi_{\pm}\right)-\alpha^2(u^2-b^2)\varphi_{\pm}=0$ with boundary conditions $\varphi_{\pm}(\alpha,0)=1$ and $\partial_y\varphi_{\pm}(\alpha,0)=0$.
- If $u(y)=ky$, $b(y)=k_0 y$ for some constant $k_0>|k|\geq 0$, then $b(y)\Gamma^{\pm}(\alpha,y)$ are harmonic functions on $\mathbb{T}\times[0,\pm 1]$ with boundary condition $b(0)\Gamma^{\pm}(\alpha,0)=-1$ and $b(\pm 1)\Gamma^{\pm}(\alpha,\pm 1)=0$.
- If $-5u'(0)u''(0)+u''(0)b'(0)-u'(0)b''(0)+5b'(0)b''(0)\neq 0$, then there exists a positive constant C such that
$$\left|\partial_y\left(b(y)\Gamma^{\pm}(\alpha,y)\right)\right|\geq C^{-1}\left(1+\left|\ln|y|\right|\right),$$
which implies $\widehat{V}_1(t=\infty),\widehat{H}_1(t=\infty)\notin L^{\infty}$.
- Even if $u=0$, there exists island structure at infinite time.

pictures



$$\phi_0(x, y) = \frac{\cos(\pi y) + 1}{2} \cos x + \sin(5\pi y) \cos(2x).$$



$$\phi(t,x,y) \to \phi_\infty(x,y) = \frac{\sinh(1-|y|)}{\sinh 1}\cos x$$

If $u = 0$, the sheared magnetic field changes the direction, i.e., $b(y)$ changes the sign, then there exists magnetic islands structures at infinite time.

Historical comments: fluid driven damping

In 2020, Ren, Wei and Zhang considered the linear sheared velocity and magnetic field $b(y) = k_1 y$ and $u(y) = k_2 y$ with $k_1 < k_2$ and proved linear damping for the vertical components of velocity and magnetic field.

- The stern stability condition $|b(y)| \geq |u(y)|$ fails.
- The strong shear flow destructs the magnetic island structure.

Main results: Linear Damping and depletion

Theorem (Vertical damping, Liu, Masmoudi, Zhai, Z, JMPA 2021)

Let $u, b \in C^3(\mathbb{T})$ *be such that* $b > |u| \geq 0$ *and the critical points of* $(u \pm b)$ *are non-degenerate. Let* $\alpha \neq 0$ *be a fixed wave number and let* $\left(\widehat{\psi}, \widehat{\phi}\right)$ *solve LMHD with initial data* $\left(\widehat{\psi}_0, \widehat{\phi}_0\right) \in (H^3 \times H^3)$*. Then the following space-time estimate holds:*

$$\left\| \left(\widehat{\psi}, \widehat{\phi}\right) \right\|_{H^1_t L^2_y} \leq C_\alpha \left\| \left(\widehat{\psi}_0, \widehat{\phi}_0\right) \right\|_{H^3_y}. \tag{2}$$

In particular, $\lim_{t\to\infty} \left\| \left(\widehat{V}_2,\ \widehat{H}_2\right) \right\|_{L^2_y} = 0.$

Theorem (Horizontal depletion, Liu, Masmoudi, Zhai, Z, JMPA 2021)

Let y_0 *be a critical point of* $(u + b)$ *or* $(u - b)$*. Let* $\left(\widehat{V}_1, \widehat{H}_1\right)$ *correspond to the solution to LMHD. Then it holds that*

$$\lim_{t\to\infty} \left| \left(\widehat{V}_1,\ \widehat{H}_1\right)(t, \alpha, y_0) \right| = 0.$$

Summary

To highlight the differences among the long time behaviors of the solutions to the linearized MHD equations in various cases, we show the following table:

Conditions				Results	References
Monotonicity	Uniform direction $b>0$	Stern stability $\lvert u\rvert \leq \lvert b\rvert$	Other conditions		
Yes	Yes	Yes	$u\equiv 0$	Damping	Ren, Z, 2017
Yes	No	Yes	$u(0)=b(0)=0$	Magnetic Island	Zhai, Zhang, Z, 2018
Yes	No	No	$u=k_1y$, $b=k_2y$ $k_1>k_2\geq 0$	Damping	Ren, Wei, Zhang, 2020
No	Yes	Yes $\lvert u\rvert < b$	Non-degenerate critical points	Damping & Depletion	Liu, Masmoudi, Zhai, Z, 2021

Why damping?

Mixing leads to damping.

To better illustrate the mixing mechanism, let us recall the system in terms of (V_1, H_1):

$$\begin{cases} \partial_t V_1 + u\partial_x V_1 - b\partial_x H_1 = L_1, \\ \partial_t H_1 + u\partial_x H_1 - b\partial_x V_1 = L_2, \end{cases} \tag{3}$$

where $(L_1, L_2) :=$
$\big(b'H_2 - u'V_2 - 2\partial_x\Delta^{-1}(b'\partial_x H_2 - u'\partial_x V_2), u'H_2 - b'V_2\big)$ can be seen as nonlocal forcing terms depending on V_2 and H_2.
By the incompressibility condition, we can check that

$$\|(\partial_x V_1, \partial_x H_1)\|_{L^2_x H^{-1}_y} \sim \|(V_2, H_2)\|_{L^2_{x,y}}.$$

Then the mixing of $(V_1 \pm H_1)$ would lead to the linear damping of (V_2, H_2).

Toy model

Let us consider a toy model, obtained by neglecting the nonlocal forcing terms (L_1, L_2) in the linearized system (3), i.e.,

$$\begin{cases} \partial_t V_1 + u\partial_x V_1 - b\partial_x H_1 = 0, \\ \partial_t H_1 + u\partial_x H_1 - b\partial_x V_1 = 0. \end{cases} \tag{4}$$

Lemma

Let $u, b \in C^3(\mathbb{T})$ be such that $u \pm b$ have only non-degenerate critical points. Then the solution of (4) with initial data $(V_{1,in}, H_{1,in})$ satisfies

$$\|(\partial_x V_1, \partial_x H_1)\|_{L^2_x H^{-1}_y} \lesssim \frac{1}{\langle t \rangle^{\frac{1}{2}}} \|(V_{1,in}, H_{1,in})\|_{H_x^{\frac{1}{2}} H_y^1}. \tag{5}$$

Moreover, if the initial data $(U_{1,in}, H_{1,in})$ vanish at all the critical points of $(u \pm b)$, then it holds that

$$\|(\partial_x V_1, \partial_x H_1)\|_{L^2_x H^{-1}_y} \lesssim \frac{1}{\langle t \rangle} \|(V_{1,in}, H_{1,in})\|_{H_x^{-1} H_y^2}. \tag{6}$$

Depletion

The space-time estimate fails to hold for the toy model (4). Exploring the mechanisms behind the enhanced damping for the complete system (3), we found a new dynamical phenomenon apart from velocity mixing: the **depletion** of horizontal velocity and magnetic field (V_1, H_1) at the critical points of $u \pm b$.

Deduction of the problem

Step 1 Rewrite the equation $\partial_t \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(t,\alpha,y) = -i\alpha M_\alpha \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(t,\alpha,y)$.

Step 2 Representation formula of the solution:

$$\begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(t,\alpha,y) = \frac{1}{2\pi i}\int_{\partial\Omega} e^{-i\alpha tc}(cI - M_\alpha)^{-1}\begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(0,\alpha,y)dc.$$

Step 3 Solve $(cI - M_\alpha)^{-1}\begin{pmatrix} \widehat{\psi}_0 \\ \widehat{\phi}_0 \end{pmatrix}(\alpha,y) = \begin{pmatrix} \Psi_1 \\ \Phi_1 \end{pmatrix}(\alpha,y,c)$, which is

$$\partial_y\Big[\Big((u-c)^2 - b^2\Big)\partial_y\Phi\Big] - \alpha^2\Big((u-c)^2 - b^2\Big)\Phi = F,$$

where $\Phi_1(\alpha,y,c) = b(y)\Phi(\alpha,y,c)$ then
$\Psi_1(\alpha,y,c) = (u(y)-c)\Phi(\alpha,y,c) + \widehat{\phi}_0(\alpha,y)/b(y)$.
This is the so-called Sturmian type equation.

Step 4 Estimate and take the limit. $c \to Ran(u+b) \cap Ran(u-b)$

Uniform estimate

$$\partial_y\left[\left((u-c)^2-b^2\right)\partial_y\Phi\right]-\alpha^2\left((u-c)^2-b^2\right)\Phi=F, \qquad (7)$$

Proposition

There exists $\epsilon_0>0$ such that for $c\in(\Omega_{\epsilon_0}\setminus(\mathrm{Ran}\,Z_+\cup\mathrm{Ran}\,Z_-))$, the solution to (7) satisfies the following bound, uniform with respect to c

$$\|\Phi(\alpha,\cdot,c)\|_{L^2}+\|(Z_--c)(Z_+-c)\partial_y\Phi(\alpha,\cdot,c)\|_{H^1}\le C\|F(\alpha,\cdot,c)\|_{H^1}.$$

Proposition

For $c\in(\mathrm{Ran}\,Z_+\cup\mathrm{Ran}\,Z_-)$, there exist $\Phi^\pm(\alpha,\cdot,c)\in L^2$ such that as $\epsilon\to0^+$, $\Phi(\alpha,\cdot,c\pm i\epsilon)\to\Phi^\pm(\alpha,\cdot,c)$ in L^r with $r\in(1,2)$ and

$$\left\|\Phi^\pm(\alpha,\cdot,c)\right\|_{L^2}\le C\|F(\alpha,\cdot,c)\|_{H^1}.$$

Here $Z_\pm=u\pm b$.

Recalling that $\Psi_1 = (u-c)\Phi + \frac{\widehat{\phi}_0}{b}$ and $\Phi_1 = b\Phi$ we have, by the representation formula that

$$\begin{pmatrix}\widehat{\psi}\\ \widehat{\phi}\end{pmatrix}(t,\alpha,y) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{\partial\Omega_\epsilon} e^{-i\alpha t c}\begin{pmatrix}\Psi_1\\ \Phi_1\end{pmatrix}(\alpha,y,c)\,\mathrm{d}c$$

$$= \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\Bigg(\int_{\text{Ran }Z_+\cup\text{Ran }Z_-} e^{-i\alpha t(c-i\epsilon)}\begin{pmatrix}u-(c-i\epsilon)\\ b\end{pmatrix}\Phi(\alpha,y,c-i\epsilon)\,\mathrm{d}c$$

$$-\int_{\text{Ran }Z_+\cup\text{Ran }Z_-} e^{-i\alpha t(c+i\epsilon)}\begin{pmatrix}u-(c+i\epsilon)\\ b\end{pmatrix}\Phi(\alpha,y,c+i\epsilon)\,\mathrm{d}c\Bigg).$$

For $c \in (\text{Ran}\,(u+b)\cup\text{Ran}\,(u-b))$, we denote $c_\epsilon := c + i\epsilon$ with $-\epsilon_0 < \epsilon < \epsilon_0$. We recall that $\Phi(\alpha,y,c_\epsilon)$ solves

$$\partial_y\left((Z_+-c_\epsilon)(Z_--c_\epsilon)\partial_y\Phi\right) - \alpha^2(Z_+-c_\epsilon)(Z_--c_\epsilon)\Phi = F.$$

Let $\widetilde{\Phi}(\alpha,y,c) := \Phi^-(\alpha,y,c) - \Phi^+(\alpha,y,c)$.

Differentiating in t yields

$$\partial_t \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} (t,\alpha,y) = \frac{1}{2\pi i} \int_{\text{Ran } Z_+ \cup \text{Ran } Z_-} i\alpha c e^{-i\alpha t c} \begin{pmatrix} (c-u)\widetilde{\Phi} \\ -b\widetilde{\Phi} \end{pmatrix} (\alpha,y,c)\, \mathrm{d}c.$$

By Plancherel's theorem, we have the following estimates

$$\left\| \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} \right\|^2_{L^2_t L^2_y} + \left\| \partial_t \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} \right\|^2_{L^2_t L^2_y} = \int_{\mathbb{T}} \int_{-\infty}^{\infty} \left(\left| \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} \right|^2 + \left| \partial_t \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} \right|^2 \right) \mathrm{d}t\, \mathrm{d}y$$

$$= \int_{\mathbb{T}} \int_{\text{Ran } (u+b) \cup \text{Ran } (u-b)} (1+(\alpha c)^2) \left| \begin{pmatrix} (u-c)\widetilde{\Phi} \\ b\widetilde{\Phi} \end{pmatrix} (\alpha,y,c) \right|^2 \mathrm{d}c\, \mathrm{d}y.$$

Invoking Proposition and the boundedness of b, we have

$$\left\| \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} \right\|^2_{L^2_t L^2_y} + \left\| \partial_t \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix} \right\|^2_{L^2_t L^2_y} \leq C_\alpha \int_{\text{Ran } (u+b) \cup \text{Ran } (u-b)} \|\widetilde{\Phi}(\alpha,\cdot,c)\|^2_{L^2_y}\, \mathrm{d}c$$

$$\leq C_\alpha \|F\|^2_{H^1_y} \lesssim \left\| \begin{pmatrix} \widehat{\psi}_0 \\ \widehat{\phi}_0 \end{pmatrix} \right\|^2_{H^3_y}.$$

Estimate at critical points

Lemma

Let $y_0 \in \left((Z_+)^{-1}(c) \cup (Z_-)^{-1}(c)\right)$ *be a critical point, i.e.,* $Z'_+(y_0) = 0$ *or* $Z'_-(y_0) = 0$. *Then it holds that*

$$|\Phi(y_0)| \leq C|(Z_-(y_0) - c)(Z_+(y_0) - c)|^{-\frac{1}{4}}, \quad (8)$$

$$|\partial_y \Phi(y_0)| \leq C|(Z_-(y_0) - c)(Z_+(y_0) - c)|^{-\frac{3}{4}}. \quad (9)$$

The lemma doesn't hold for the toy model, for which it holds that

$$|\partial_y \Phi(y_0)| \leq C|(Z_-(y_0) - c)(Z_+(y_0) - c)|^{-1} \notin L^1_c$$

The above lemma implies uniform bounds on both
$\Phi(\alpha, y_0, \cdot \pm i\epsilon_0)$ in L^ρ_c, $\rho \in [1, 4)$
and $\partial_y \Phi(\alpha, y_0, \cdot \pm i\epsilon)$ in L^p_c, $p \in [1, \frac{4}{3})$.
Thus, there exists a subsequence $\epsilon_n \to 0^+$ as well as $\Lambda^\pm \in L^\rho_c$ and $\Theta^\pm \in L^p_c$ such that as $\epsilon_n \to 0^+$,

$$\Phi(\alpha, y_0, \cdot \pm i\epsilon_n) \rightharpoonup \Lambda^\pm(\alpha, y_0, \cdot),$$
$$\partial_y \Phi(\alpha, y_0, \cdot \pm i\epsilon_n) \rightharpoonup \Theta^\pm(\alpha, y_0, \cdot).$$

$$\begin{pmatrix}\widehat{V}_1\\ \widehat{H}_1\end{pmatrix}(t,\alpha,y_0)=\partial_y\begin{pmatrix}\widehat{\psi}\\ \widehat{\phi}\end{pmatrix}(t,\alpha,y_0)$$

$$=\lim_{\epsilon_n\to 0^+}\frac{1}{2\pi i}\int_{\partial\Omega_{\epsilon_n}}e^{-i\alpha tc}\partial_y\begin{pmatrix}\Psi_1\\ \Phi_1\end{pmatrix}(\alpha,y_0,c)\,\mathrm{d}c$$

$$=\lim_{\epsilon_n\to 0^+}\frac{1}{2\pi i}\int_{\partial\Omega_{\epsilon_n}}e^{-i\alpha tc}\begin{pmatrix}(u-c)\partial_y\Phi+u'\Phi\\ b\partial_y\Phi+b'\Phi\end{pmatrix}(\alpha,y_0,c)\,\mathrm{d}c$$

$$=\frac{1}{2\pi i}\int_{\mathrm{Ran}\,(u+b)\cup\mathrm{Ran}\,(u-b)}e^{-i\alpha tc}\begin{pmatrix}(u-c)(\Theta^--\Theta^+)\\ b(\Theta^--\Theta^+)\end{pmatrix}(\alpha,y_0,c)\,\mathrm{d}c$$

$$+\frac{1}{2\pi i}\int_{\mathrm{Ran}\,(u+b)\cup\mathrm{Ran}\,(u-b)}e^{-i\alpha tc}\begin{pmatrix}u'(\Lambda^--\Lambda^+)\\ b'(\Lambda^--\Lambda^+)\end{pmatrix}(\alpha,y_0,c)\,\mathrm{d}c.$$

The desired conclusion follows from Riemann-Lebesgue lemma, as $(\Theta^- - \Theta^+)(\alpha,y_0,\cdot)\in L^1_c$ and $(\Lambda^- - \Lambda^+)(\alpha,y_0,\cdot)\in L^1_c$.

Proof of uniform estimate

- A contradiction argument.
- Separate the monotonic region and critical region.
- Energy estimate and representation formula.
- From weak convergence to strong convergence.

On-going project

- Rate; (to be continued)
- Nonlinear (in)stability

Theorem (preprint, Liu, Masmoudi, Zhai, Z)

Under the Stern stability condition $|u| \leq |b|$ with some non-degeneracy assumption, it holds that:

- *if $Ran(u+b) \cap Ran(u-b) = \emptyset$, then damping and depletion at critical points;*
  [Ren-Z 2017, Liu-Masmoudi-Zhai-Z 2021]
- *if $Ran(u+b) \cap Ran(u-b) = \{0\}$, then damping except for the zeros;*
- *if $Ran(u+b) \cap Ran(u-b) \setminus \{0\} \neq \emptyset$, then there exists island structure.*
  [Zhai-Zhang-Z 2018]

Thank you for your attention!