On the asymptotic stability of shear flows and vortices

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We consider the incompressible Euler equation in 2D

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \qquad \mathrm{div}\, u = 0.$$

In the vorticity formulation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \qquad u = \nabla^\perp = (-\partial_y, \partial_x)\psi,$$

where $\omega : \mathcal{D} \to \mathbb{R}$ is the vorticity and $\psi = \Delta^{-1}\omega$ is the stream function.

Smooth solutions remain smooth and exist globally in time (Wolibner, Yudovich). The vorticity is transported, i.e. the evolution conserves all L^p norms of the vorticity.

The long time behavior of smooth solutions is however hard to understand, due to the lack of a global relaxation mechanism.

A more realistic goal is to study the global nonlinear dynamics of solutions that are close to steady states of the 2D Euler equation. Coherent structures, such as shear flows and vortices, are particularly important in the study of the 2D Euler equation, since precise numerical simulations and physical experiments show that they tend to form dynamically and become the dominant feature of the solution for a long time.

The study of stability property of these steady states is a classical subject and a fundamental problem in hydrodynamics. Early investigations were started by Rayleigh and Kelvin, with a focus on mode stability.

Rayleigh: "On the stability or instability of certain fluid motions" (1880)
Kelvin: "Stability of fluid motion: rectilinear motion of viscous fluid between two plates" (1887).

Shear flows: an example

The velocity field $U : \mathbb{T} \times I \to \mathbb{R}^2$ is given by $(b(y), 0)) + u(x, y)$, where I is an interval and $b : I \to \mathbb{R}$ is a smooth function. The vorticity given is by $-b'(y) + \omega$ and the Euler equation becomes

$$\begin{cases} \partial_t \omega + b(y)\partial_x \omega - b''(y)\partial_x \psi + u \cdot \nabla \omega = 0, \\ u = (u^x, u^y) = (-\partial_y \psi, \partial_x \psi), \qquad \Delta \psi = \omega, \end{cases}$$

with suitable boundary condtitions for ψ.

In the simplest case $b(y) = y$ (the Couette flow) and $I = \mathbb{R}$, the linearized equation becomes

$$\partial_t \omega + y \partial_x \omega = 0.$$

One can then solve this equation explicitly and calculate

$$\omega(t, x, y) = \omega_0(x - yt, y).$$

Shear flows: an example

Therefore

$$\omega_k(t,y) = e^{-ikty}\omega_{0,k}(y),$$

so the 0 mode is constant during the linear evolution, while all the other modes converge weakly (but not strongly) to 0 as $t \to \infty$.

The equation for the stream function gives

$$\widetilde{\psi}(t,k,\xi) = -\frac{\widetilde{\omega}(t,k,\xi)}{k^2+|\xi|^2} = -\frac{\widetilde{\omega_{0,k}}(\xi+kt)}{k^2+|\xi|^2}.$$

If ω_0 is smooth then we can view ξ as $\xi = -kt + O(1)$. Therefore ψ_k decays (qualitatively) like $|k|^{-2}\langle t\rangle^{-2}$ for each $k \neq 0$, so u^x decays like $|k|^{-1}\langle t\rangle^{-1}$ and u^y decays like $|k|^{-1}\langle t\rangle^{-2}$ for all $k \neq 0$. Hence, the velocity field converges to another shear flow $(u_\infty(y), 0)$.

Shear flows

This is the mechanism of *inviscid damping*: the nonzero modes of the vorticity oscillate faster and faster as $t \to \infty$, which makes the corresponding modes of the velocity fields go to 0 (at least qualitatively). This could lead to convergence both in the linearized and in the nonlinear problem.

Major issues to understand for asymptotic stability:

- the presence of non-decaying solutions for other shear flows $b(y)$;
- the effect of the boundary in the case of finite channels, which disrupts the mechanism described above;
- in the nonlinear problem, the final state of the evolution is a different shear flow $(u_\infty(y), 0)$, which has to be identified dynamically by the flow.

Linear stability

There are many results on various forms of linear stability of coherent states of the Euler and Navier-Stokes equations:

Rayleigh, Kelvin, Orr, Taylor, Faddeev, McWilliams,

...
Bedrossian-Coti Zelati-Vicol, Zillinger,
Grenier–Nguyen–Rousset–Soffer, Wei–Zhang–Zhao.

Orbital stability: Arnold proved a general stability result for the nonlinear equation, using the energy Casimir method, but this method does not give asymptotic information on the global dynamics.

Nonlinear asymptotic stability

There are only 3 asymptotic stability nonlinear results for the Euler equation, all of them in 2D.

- The Couette flow: Bedrossian-Masmoudi (2015) in the infinite cylinder domain $\mathbb{T} \times \mathbb{R}$, extended to the finite channel $\mathbb{T} \times [0, 1]$ by I-Jia to allow for finite energy solutions;

- Point vortices I-Jia;

- General monotonic shear flows satisfying a suitable spectral assumption in the finite channel $\mathbb{T} \times [0, 1]$: I-Jia and Masmoudi-Zhao.

The main theorem

Assume that the initial data ω_0 has compact support in $\mathbb{T} \times [2\vartheta_0, 1 - 2\vartheta_0]$, and satisfies

$$\|\omega_0\|_{\mathcal{G}^{\beta_0,1/2}(\mathbb{T}\times\mathbb{R})} = \epsilon \leq \overline{\epsilon}.$$

Let $\omega : [0, \infty) \times \mathbb{T} \times [0, 1] \to \mathbb{R}$ denote the global smooth solution to the Euler equation

$$\begin{cases} \partial_t\omega + b(y)\partial_x\omega - b''(y)\partial_x\psi + u \cdot \nabla\omega = 0, \\ u = (-\partial_y\psi, \partial_x\psi), \quad \Delta\psi = \omega, \quad \psi(t, x, 0) = \psi(t, x, 1) = 0, \end{cases}$$

where $b : [0, 1] \to \mathbb{R}$ satisfies condition (B) below.
Then $\operatorname{supp}\omega(t) \subseteq \mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$ for any $t \geq 0$ and

$$\|\omega(t, x + tb(y) + \Phi(t, y), y) - F_\infty(x, y)\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])} \lesssim \frac{\epsilon}{\langle t\rangle},$$

for some $\beta_1 > 0$, where

$$\Phi(t, y) := \int_0^t \langle u^x\rangle(\tau, y)\, d\tau.$$

The main theorem

We define the smooth functions $u_\infty : [0,1] \to \mathbb{R}$ by

$$\partial_y^2 \psi_\infty = \langle F_\infty \rangle, \qquad \psi_\infty(0) = \psi_\infty(1) = 1, \qquad u_\infty(y) := -\partial_y \psi_\infty.$$

Then the velocity field $u = (u^x, u^y)$ satisfies

$$\|\langle u^x \rangle(t,y) - u_\infty(y)\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])} \lesssim \epsilon \langle t \rangle^{-2},$$

$$\|u^x(t,x,y) - \langle u^x \rangle(t,y)\|_{L^\infty(\mathbb{T}\times[0,1])} \lesssim \epsilon \langle t \rangle^{-1},$$

$$\|u^y(t,x,y)\|_{L^\infty(\mathbb{T}\times[0,1])} \lesssim \epsilon \langle t \rangle^{-2}.$$

Remarks

- The spaces $\mathcal{G}^{\lambda,s}$, $s \in (0,1]$, $\lambda > 0$, are Gevrey spaces defined by the norms

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})} := \left\| e^{\lambda\langle k,\xi\rangle^s} \widetilde{f}(k,\xi) \right\|_{L^2_{k,\xi}}.$$

Gevrey regularity is needed in all nonlinear asymptotic stability results (Deng-Masmoudi).

- The function $b \in \mathcal{G}^{\beta_0,1/2}$ satisfies the (necessary) conditions:

(i) For some $\vartheta_0 \in (0, 1/10]$

$$\vartheta_0 \le b'(y) \le 1/\vartheta_0 \text{ for } y \in [0,1],$$
$$b''(y) \equiv 0 \text{ for } y \notin [2\vartheta_0, 1-2\vartheta_0].$$

(ii) The associated linearized operator
$L_k : L^2(0,1) \to L^2(0,1), k \in \mathbb{Z}\backslash\{0\}$, given by

$$L_k f = b(y)f - b''(y)\varphi_k, \qquad \partial_y^2\varphi_k - k^2\varphi_k = f, \;\; \varphi_k(0) = \varphi_k(1) = 0,$$

has no discrete eigenvalues.

Numerical simulations and physical experiments show that vortices emerge naturally in two dimensional fluid flows, in high Reynolds number viscous flows and in perfect fluids.

Video of two dimensional turbulence at high Reynolds number 50,000

A first step is to understand the stability of vortices. At the linearized level this was done recently by Bedrossian-Coti Zelati-Vicol and I.-Jia.

Point vortices

In this talk, we consider the simplest class of two dimensional vortices, the point vortices, which model the physically relevant situation when the vorticity concentrates sharply in small regions.

More precisely, we consider solutions of the form

$$\text{vorticity} = \kappa\,\delta(P(t)) + \omega,$$
$$\text{velocity field} = \nabla^{\perp}\Delta^{-1}\delta(P(t)) + u,$$

where $\kappa \in \mathbb{R}\backslash\{0\}$ is the strength of the point vortex, $\delta(P(t))$ is the Dirac mass centered at $P(t) = (P_1(t), P_2(t)) \in \mathbb{R}^2$.

The main theorem

The perturbation ω, u satisfy the equation

$$\partial_t \omega + U \cdot \nabla \omega + u \cdot \nabla \omega = 0, \qquad \text{for } (x, y, t) \in \mathbb{R}^2 \times [0, \infty),$$

where

$$U = \nabla^\perp \Delta^{-1} \delta(P(t)) = \frac{\kappa}{2\pi} \nabla^\perp \log |(x, y) - P(t)|,$$

and the velocity field u and the stream function ψ are determined through

$$u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi), \qquad \Delta \psi = \omega, \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

In addition, the center $P(t)$ satisfies the ODE

$$P'(t) = \nabla^\perp \psi(t, P(t)).$$

The main theorem

A rough version of our main result can be stated as follows.

Theorem (I.-Jia)

Assume smooth $\omega_0 \in C_0^\infty(\mathbf{R}^2)$ *is compactly supported away from the origin and is small in Gevrey-2 spaces, i.e.*

$$\int_{\mathbf{R}^2} e^{\lambda\langle\xi,\eta\rangle^{1/2}} \left|\widetilde{\omega_0}(\xi,\eta)\right|^2 d\xi d\eta \leq \epsilon^2.$$

Then $\omega(t)$ *is global, supported away from* $P(t)$ *for all times, and* $P(t)$ *stabilizes rapidly to a final position* P_∞*.*

Moreover, $\omega(t)$ *converges weakly to a smooth* ω_∞ *which is radial with respect to* P_∞*, as* $t \to \infty$*.*

In the above P_∞ can be determined by the initial data through conservation laws.

Assume $\Omega : \mathbb{R}^2 \to \mathbb{R}$ is a given radial and radially-decreasing function (the vortex we would like to perturb around) and write the solution ω in the form

$$\omega(t, x, y) = \Omega((x, y) - P(t)) + f(t, x, y),$$

where $P(t) \in \mathbb{R}^2$ denotes the moving center of the vortex.
Our analysis in reveals a new phenomenon: even at the linearized level, the flow has no "uniformly smooth profile" even after taking off oscillatory factors. Instead, we should decompose the modes g_k of the solution

$$g_k(t, r) = f_{k1}(t, r)e^{-ik(U(r)/r)t} + f_{k2}(t, r),$$

where f_{k1} and f_{k2} are both uniformly smooth in r over $t \in (0, \infty)$. This decomposition has natural physical meaning, since the first term comes from the interior of the fluid itself, while the second term is generated by the "boundary" corresponding to $r = 0$.

This structure presents a new and possibly significant difficulty for the nonlinear problem, since the method used in all the nonlinear inviscid damping results relies crucially on proving smoothness of a well defined profile for the vorticity. We propose a different strategy: instead of studying the profile of the vorticity function, which is difficult to even define we focus on what we call *the spectral density function*.

In new coordinates $r = e^v$, $v \in \mathbf{R}$, the spectral density function $\Theta_k(v,w) : \mathbb{R}^2 \to \mathbb{R}$ is defined in such a way that

$$\phi_k(t,v) = -\frac{1}{2\pi}\int_{\mathbf{R}} e^{-ikB(w)t}\Theta_k(v-w,w)B'(w)\,dw,$$
$$h_k(t,v) = -e^{-2v}(k^2-\partial_v^2)\phi_k(t,v),$$

where $\phi_k(t,v) = \psi_k(t,e^v)$, $h_k(t,v) = g_k(t,e^v)$, and $B(v) = e^{-v}U(e^v)$. The main point is to prove suitable smoothness of the spectral density function Θ, and the estimates on the stream function and velocity fields can then be obtained easily.

We consider now the generalized surface quasi-geostrophic equations (gSQG)

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, & (t,x)\in[0,T)\times\mathcal{D}, \\ u = -\nabla^{\perp}(-\Delta)^{-1+\alpha/2}\theta, \end{cases}$$

where $\alpha\in[0,2]$ and $\mathcal{D}$ is a domain in $\mathbb{R}^2$. The case $\alpha=1$ corresponds to the surface quasi-geostrophic (SQG) equation. Notice that the case $\alpha=0$ corresponds to the 2D incompressible Euler equations and the case $\alpha=2$ produces stationary solutions.

The construction of nontrivial global solutions for the gSQG equations is a very challenging open problem for all parameters $\alpha\in(0,2)$, both in the smooth and in the patch case. The only known non-stationary global solutions of finite energy, both in the smooth and the patch setting, are special rotating solutions, periodic in time (construction of Castro–Córdoba–Gómez-Serrano, extended recently to quasi-periodic solutions by Hassainia–Hmidi–Masmoudi).

It is tempting to try to use the mechanism of inviscid damping to construct families of nontrivial global solutions of the gSQG equations, at least for some parameters $\alpha \in (0, 2)$, by perturbing around stationary solutions. The easiest would be to perturb around shear flows on the finite channel domain $\mathcal{D} = \mathbb{T} \times [0, 1]$, in particular around the Couette flow corresponding to $\theta(t, x, y) = -1$.

The fractional Laplacian $(-\Delta)^{-1+\alpha/2}$ on this domain can be defined using explicit spectral theory. The vorticity deviation $\omega = \theta + 1 : [0, T] \times \mathbb{T} \times [0, 1] \to \mathbb{R}$ satisfies the system

$$\partial_t\omega + \partial_y a(y)\partial_x\omega - \partial_y\psi\partial_x\omega + \partial_x\psi\partial_y\omega = 0,$$
$$\psi = -(-\Delta)^{-1+\alpha/2}\omega, \qquad \psi(t, x, 1) = \psi(t, x, 0) = 0,$$

where $a = a(y)$ is given by $(-\partial_y^2)^{1-\alpha/2}a(y) = -1$, $a(0) = a(1) = 0$.

At first glance it seems plausible to adapt the ideas used in the analysis of the Euler equation to prove global regularity of this system, at least for some $\alpha > 0$ small. One can still perform a nonlinear change of variables and derive a system of equations for a profile F, as before A simplified version of this system is the closed equation

$$\partial_t F - \partial_v P_{\neq 0}\Phi\, \partial_z F = 0,$$

$$\widetilde{P_{\neq 0}\Phi}(t,k,\xi) = \frac{\widetilde{F}(t,k,\xi)}{[k^2 + (\xi - tk)^2]^{1-\alpha/2}} \mathbf{1}_{\mathbb{Z}^*}(k)$$

for the smooth function $F : [0,T] \times \mathbb{T} \times \mathbf{R} \to \mathbf{R}$.

Surprisingly, our analysis (in collaboration with Javier Gómez-Serrano) reveals that this system is completely unstable, for any $\alpha > 0$.