

Soliton resolution for equivariant wave maps

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Energy-critical wave maps equation

"Definition" An application $\Psi: \mathbb{R}^{1+2} \rightarrow S^2 \subset \mathbb{R}^3$ is a wave map if it is a critical point of the Lagrangian

$$\mathcal{L}(\Psi) := \frac{1}{2} \iint (|\partial_t \Psi|^2 - |\partial_x \Psi|^2) dx dt.$$

Natural analogs of linear waves in a nonlinear, geometric setting.

Euler-Lagrange equation:

$$\partial_t^2 \Psi(t, x) - \Delta \Psi(t, x) = -(|\partial_t \Psi(t, x)|^2 - |\nabla \Psi(t, x)|^2) \Psi(t, x).$$

Local well-posedness and global well-posedness for small data:

Klainerman, Selberg, Sterbenz, Tataru, Tao, ... (1993-2002)

Equivariant wave maps

We study the dynamics (long time behavior) of large solutions, but only in a special case:

$$\Psi(t, r\cos\theta, r\sin\theta) = (\sin\psi(t, r)\cos(k\theta), \sin\psi(t, r)\sin(k\theta), \cos\psi(t, r)).$$

Here, $k \in \{1, 2, \dots\}$ is the equivariance degree, $t \in \mathbb{R}$, $r \in (0, \infty)$

Equation for ψ :

$$(WM_k) \quad \partial_t^2 \psi(t, r) - \partial_r^2 \psi(t, r) - \frac{1}{r} \partial_r \psi(t, r) + \frac{k^2}{2r^2} \sin(2\psi(t, r)) = 0.$$

Lagrangian: $\mathcal{L} := \pi \iint \left((\partial_t \psi)^2 - (\partial_r \psi)^2 - \frac{k^2 \sin^2 \psi}{r^2} \right) r dr dt$

Energy: $E(\psi_0, \dot{\psi}_0) := \pi \int_0^\infty \left(\underbrace{\dot{\psi}_0^2}_{\text{kinetic}} + \underbrace{(\partial_r \psi_0)^2 + \frac{k^2 \sin^2 \psi_0}{r^2}}_{\text{potential}} \right) r dr$

Scaling invariance and criticality

If ψ solves (WM_k) and $\lambda > 0$, then

$\psi_\lambda(t, r) := \psi\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right)$ solves (WM_k) as well.

Moreover, $E(\psi_\lambda, \dot{\psi}_\lambda) = E(\psi, \dot{\psi}) \rightsquigarrow$ energy-critical problem

Note: if $\lambda \ll 1$, then ψ_λ is concentrated and evolves fast.

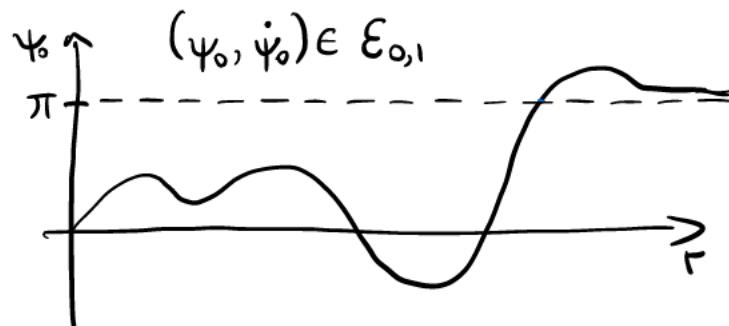
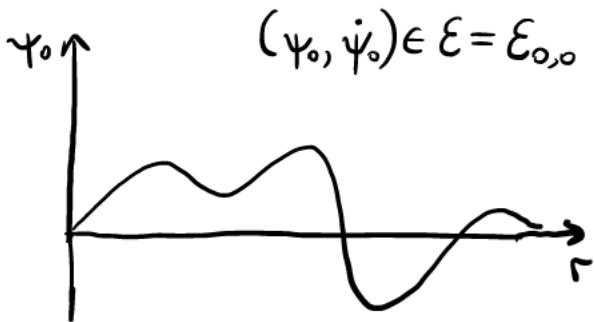
Local theory, small data theory

Energy norm: $\|(\psi_0, \dot{\psi}_0)\|_{\mathcal{E}}^2 := \|\dot{\psi}_0\|_{L^2}^2 + \|\psi_0\|_{H^1}^2$,

$$\|\dot{\psi}_0\|_{L^2}^2 := \int_0^\infty \dot{\psi}_0^2 r dr, \quad \|\psi_0\|_{H^1}^2 := \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2}{r^2} \psi_0^2 \right) r dr.$$

$$\|\psi_0\|_{L^\infty} \text{ small} \Rightarrow \|(\psi_0, \dot{\psi}_0)\|_{\mathcal{E}}^2 \approx E(\psi_0, \dot{\psi}_0).$$

Finite energy sectors : $\mathcal{E}_{m,n} := \left\{ (\psi_0, \dot{\psi}_0) : E(\psi_0, \dot{\psi}_0) < \infty, \lim_{r \rightarrow 0} \psi_0(r) = m\pi, \lim_{r \rightarrow \infty} \psi_0(r) = n\pi \right\}.$



Theorem (Shatah-Struwe 1994, $k=1$
Planchon-Stalker-Tahvildar Zadeh 2003, $k \geq 2$)

Equation (WM_k) is locally well-posed in each finite energy sector, in the sense of strong solutions.

Linearisation around $\psi=0$: $\partial_t^2 \psi_L - \partial_r^2 \psi_L - \frac{1}{r} \partial_r \psi_L + \frac{k^2}{r^2} \psi_L = 0$.

If ψ is small, the nonlinear effects become negligible for large times:

Theorem If $E(\psi_0, \dot{\psi}_0)$ is small enough, then ψ exists globally and

$$\lim_{t \rightarrow \pm\infty} \|(\psi(t), \partial_t \psi(t)) - (m\pi + \psi_L^\pm(t), \partial_t \psi_L^\pm(t))\|_{\mathcal{E}} = 0 \quad (\text{scattering})$$

These results are consequences of Strichartz estimates.

Stationary solutions ("solitons")

Minimisers of E :

* on $\mathcal{E}_{m,m}$ \rightsquigarrow constant functions

* on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow (m\pi + 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$

* on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow (m\pi - 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$

* on other sectors $\rightsquigarrow \emptyset$

We denote $Q(r) := 2\arctan(r^k)$, $Q_\lambda := Q(r/\lambda)$ for $\lambda > 0$.

- Key role in the description of the dynamics of large solutions.

Sequential soliton resolution

Theorem (Côte - Kenig - Lawrie - Schlag 12, Côte 13, Jia-Kenig 15)

Let ψ a finite energy wave map for $k \in \{1, 2\}$.

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$, $t_n \rightarrow T_+$,

$0 < \lambda_{1,n} \ll \lambda_{2,n} \ll \dots \ll \lambda_{N,n} \ll T_+ - t_n$, $z_1, z_2, \dots, z_N \in \{-l, l\}$

and $(\psi_0^*, \dot{\psi}_0^*) \in \mathcal{E}_{0,l}$ such that

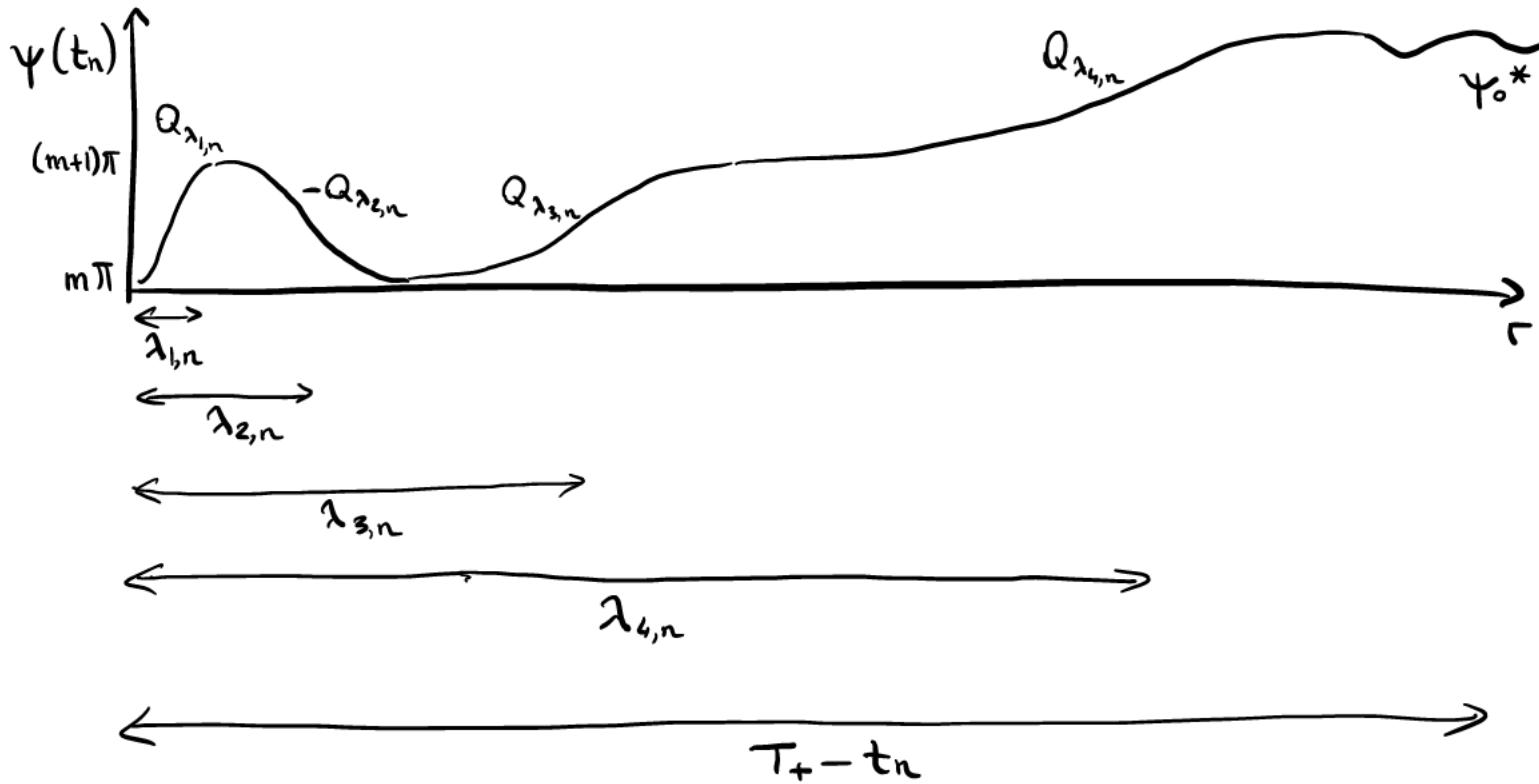
$$\lim_{n \rightarrow \infty} \|(\psi(t_n), \dot{\psi}(t_n)) - (m\pi + \sum_{j=1}^N z_j Q_{\lambda_{j,n}} + \psi_0^*, \dot{\psi}_0^*)\|_{\mathcal{E}} = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$, $t_n \rightarrow \infty$,

$0 < \lambda_{1,n} \ll \lambda_{2,n} \ll \dots \ll \lambda_{N,n} \ll t_n$, $z_1, z_2, \dots, z_N \in \{-l, l\}$

and ψ_L solution of the linear wave equation such that

$$\lim_{n \rightarrow \infty} \|(\psi(t_n), \dot{\psi}(t_n)) - (m\pi + \sum_{j=1}^N z_j Q_{\lambda_{j,n}} + \psi_L(t_n), \dot{\psi}_L(t_n))\|_{\mathcal{E}} = 0.$$

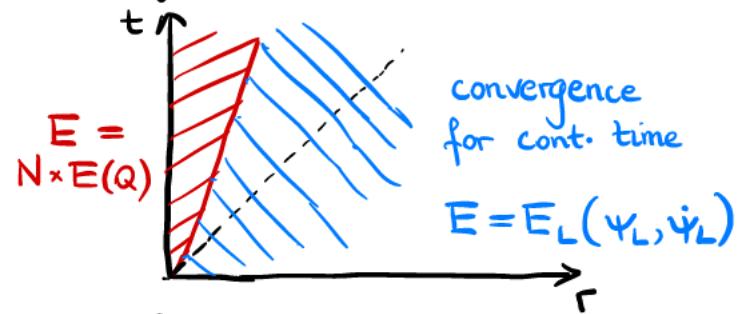
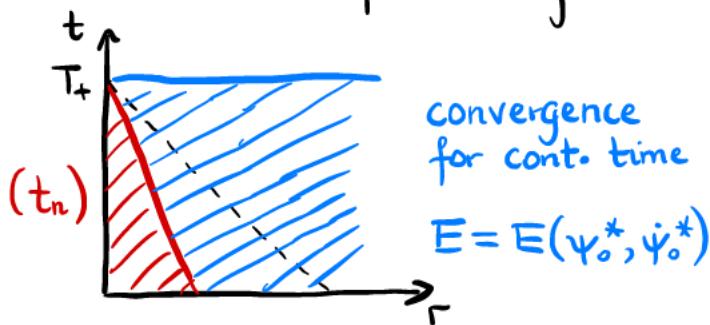


Corollary (CKLS)

If $(\psi, \dot{\psi}) \in E$ and $E(\psi, \dot{\psi}) < 8k\pi$, then ψ scatters.

Remarks

- * In the blow-up case, fundamental earlier results of Christodoulou, Shatah-Tahvildar Zadeh and Struwe
- * Outside of the light cone, convergence in continuous time



- * Generalizes to any $k \in \{1, 2, \dots\}$.

Proposition (Côte)

If, in the above result, $N=1$, or $\gamma_1 = \gamma_2 = \dots = \gamma_N$,
then the decomposition holds for continuous time.

Do these decompositions hold in continuous time unconditionally?

Soliton Resolution Conjecture

Theorem (Duyckaerts - Kenig - Martel - Merle, March 2021)

Let ψ a finite energy wave map for $k=1$.

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$,

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll T_+ - t, \quad z_1, z_2, \dots, z_N \in \{-l, l\}$$

and $(\psi_0^*, \dot{\psi}_0^*) \in E_{0,l}$ such that

$$\lim_{t \rightarrow T_+} \|(\psi(t), \dot{\psi}(t)) - (m\pi + \sum_{j=1}^N z_j Q_{\lambda_j(t)} + \psi_0^*, \dot{\psi}_0^*)\|_\varepsilon = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$,

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll t, \quad z_1, z_2, \dots, z_N \in \{-l, l\}$$

and ψ_L solution of the linear wave equation such that

$$\lim_{t \rightarrow \infty} \|(\psi(t), \dot{\psi}(t)) - (m\pi + \sum_{j=1}^N z_j Q_{\lambda_j(t)} + \psi_L(t), \dot{\psi}_L(t))\|_\varepsilon = 0.$$

Remarks

- * The first result of this type for a model which is not completely integrable was obtained by Duyckaerts, Kenig and Merle in 2012 for the energy-critical focusing wave equation with the power nonlinearity, introducing the method of energy channels.
- * Generalized to all odd dimension in 2019 by the same authors
- * The proof of the last theorem : energy channels

Our contribution is to prove, using rather different ideas, that the last result holds for any $k \in \{1, 2, 3, \dots\}$

Soliton Resolution Conjecture

Theorem (J.- Lawrie, June 2021)

Let ψ a finite energy wave map for $k \in \{1, 2, 3, \dots\}$

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$,

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll T_+ - t, \quad z_1, z_2, \dots, z_N \in \{-1, 1\}$$

and $(\psi_0^*, \dot{\psi}_0^*) \in \mathcal{E}_{0,l}$ such that

$$\lim_{t \rightarrow T_+} \|(\psi(t), \dot{\psi}(t)) - (m\pi + \sum_{j=1}^N z_j Q_{\lambda_j(t)} + \psi_0^*, \dot{\psi}_0^*)\|_\varepsilon = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$,

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll t, \quad z_1, z_2, \dots, z_N \in \{-1, 1\}$$

and ψ_L solution of the linear wave equation such that

$$\lim_{t \rightarrow \infty} \|(\psi(t), \dot{\psi}(t)) - (m\pi + \sum_{j=1}^N z_j Q_{\lambda_j(t)} + \psi_L(t), \dot{\psi}_L(t))\|_\varepsilon = 0.$$

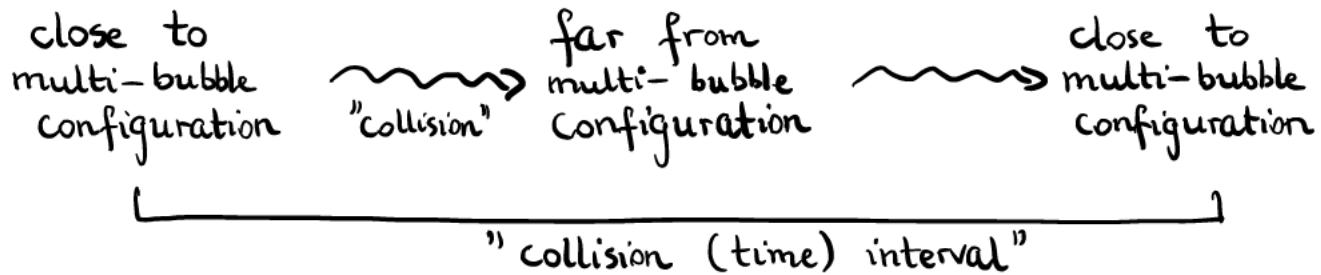
Remarks

- * The radial critical Yang-Mills equation is closely related to the 2-equivariant wave maps equation and could also be treated by our method.
- * No solutions with $N > 1$ have been constructed for $k=1$.
For $k \geq 2$, J. and J.-Lawrie constructed and studied the "two-bubble solutions".
Prior constructions include Krieger-Schlag-Tataru for $k \in \{1, 2\}$ and Raphaël-Rodnianski, Rodnianski-Sterbenz ($N=1$)
- * The soliton resolution problem is inspired by the theory of completely integrable systems and numerical simulations.

Main ideas of the proof

The desired decomposition holds for a time sequence.

We need to prevent the following scenario:



We need a no-return lemma.

- * Inspired by Duyckaerts-Merle, Nakanishi-Schlag, Krieger-Nakanishi-Schlag for single soliton which is linearly unstable.
- * Here, inter-soliton interactions play a similar role as linear instability in those works (cf. J - Lawrie '17)

Virial identity

If ψ is a smooth wave map, then

$$\operatorname{div}_{t,r} \left(\partial_t \phi \, r^2 \partial_r \phi, -\frac{1}{2} r^2 (\partial_t \phi)^2 - \frac{1}{2} r^2 (\partial_r \phi)^2 + \frac{k^2}{2} \sin^2 \phi \right) = -r (\partial_r \phi)^2$$

We estimate the boundary terms from above and the space-time integral of $r (\partial_r \phi)^2$ from below.

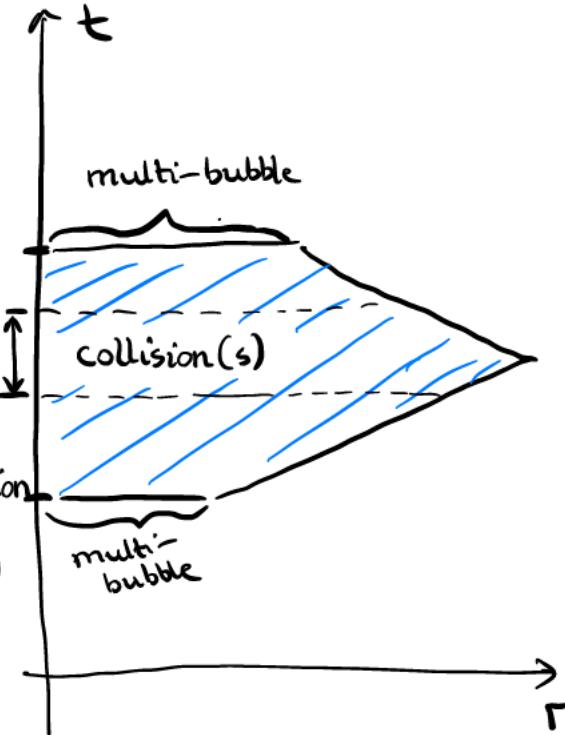
"horizontal" boundary : \ll spatial scale

$\iint (\partial_r \phi)^2 r dr dt$: \gtrsim collision duration

("Compactness Lemma")

remaining boundary : ignore for now

Collision duration \gtrsim spatial scale ?



Interior and exterior bubbles

Let $K \in \{1, 2, \dots, N\}$, $[a, b] \subset [T_0, T_f)$, $0 < \varepsilon < \eta$.

We say $[a, b]$ is a collision interval with $N-K$ exterior bubbles if:

- * $d(a) \leq \varepsilon$, $d(b) \leq \varepsilon$, $\exists c \in [a, b] : d(c) \geq \eta$

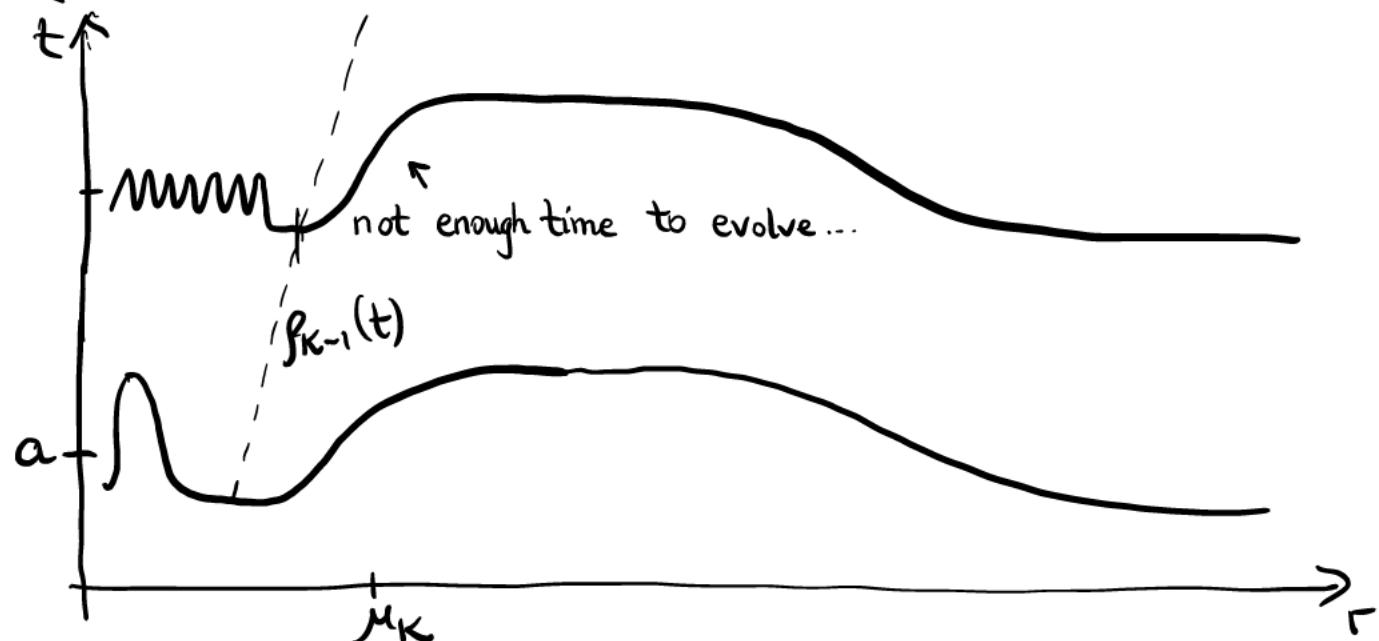
where $d(t)$ is the distance of $\gamma(t)$ to (multi-bubble + radiation)

- * \exists curve $r = g_K(t)$ such that in the region $r \geq g_K(t)$
 $\gamma(t, r)$ is ε -close to $[(N-K)\text{-bubble} + \text{radiation}]$.

We now set K to be the smallest number such that
there exist $\eta > 0$, a sequence $\varepsilon_n \rightarrow 0$
and a sequence of collision intervals $[a_n, b_n]$
corresponding to these parameters ε_n, η and K .

Lemma $\exists C = C(\gamma) > 0$ and $\varepsilon > 0$ such that
 if $[a, b]$ collision interval with parameters (ε, η, K) ,
 then $b - a \geq C \min(\mu_K(a), \mu_K(b))$,
 where μ_K is the scale of the K -th bubble.

Proof If not, K would not be smallest possible.



Compactness Lemma

Let $\rho_n > 0$ and ψ_n a sequence of wave maps of bounded energy, ψ_n defined on the time interval $[0, \rho_n]$.

Suppose $\exists R_n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} \int_0^{\rho_n} \int_0^{R_n} (\partial_t \psi_n(t, r))^2 r dr dt = 0.$$

Then, up to passing to a subsequence, $\exists t_n \in [0, \rho_n]$ and $1 \ll r_n \leq R_n$ such that $\psi_n(t_n)$ converges to a multi-bubble for $0 \leq r \leq r_n \rho_n$.

Proof Quite related to the existing proofs of sequential soliton resolution (Côte, Jia-Kenig), based on the Bahouri-Gérard profile decomposition.

Modulation

- * Near a_n and b_n , ψ is close to a multi-bubble and the analysis above does not apply.
- * In this case, the main dynamical information are the scales of the bubbles
- * We obtain differential inequalities on these scales.
Informally: $\lambda_j'' \approx -r_j r_{j+1} \omega^2 \frac{\lambda_j^{k-1}}{\lambda_{j+1}^k} + r_j r_{j-1} \omega^2 \frac{\lambda_{j-1}^k}{\lambda_j^{k+1}}$.
 $\lambda_{j-1} \ll \lambda_j \ll \lambda_{j+1}$
- * Error bounded by the energy of attractive interactions.
- * The influence of the exterior bubbles and radiation can essentially be neglected by enlarging \mathcal{E}_n .
- * Refined modulation parameters: Raphael-Szeftel '11, J.-Lauwne '17.

"Lemma" If d starts growing at t_0 , then $\forall t_* \geq t_0$

$$\int_{t_0}^{t_*} d(t) dt \leq C_0 d(t_*)^{2k} \mu_k(t_0) \quad \text{if } k \geq 2$$

$$\int_{t_0}^{t_*} d(t) dt \leq C_0 d(t_*)^2 \sqrt{-\log d(t_*)} \mu_k(t_0) \quad \text{if } k=1$$

The final step is to partition $[a_n, b_n]$ into

