Singularities in the Keller-Segel Equation

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Outline

1 - Introduction

The Keller-Segel equation

The Keller-Segel equation [Patlak '53], [Keller-Segel '70], [Nanjundiah '73]:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^d. \qquad \textbf{(KS)}$$

Modeling features:

- Describing the *chemotaxis* in biology, [Hillen-Painter '09]; interacting stochastic many-particles system, [Othmer-Stevens '90], [Stevens '00]), [Chavanis '08], [Hillen-Painter '08]; as a diffusion limit of a kinetic model [Chalub-Markowich-Perthame-Schmeiser '04], model of stellar dynamics under friction and fluctuations [Wolansky '92];
- Competition between diffusion of cells and aggregation;
- Rich model from mathematical point of view, [Horstman '03 & '04];

Basis features

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

The Keller-Segel equation:

$$\partial_t u = \nabla \cdot \big(u \nabla (\ln u - \Phi_u)\big),$$

$$\Phi_u = W * u, \quad W(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{for} \quad d = 2, \\ \frac{\Gamma(d/2-1)}{4\pi^{d/2}} |x|^{2-d} & \text{for} \quad d \geq 3. \end{cases}$$

- mass conservation: $M = \int_{\mathbb{R}^2} u_0(x)dx = \int_{\mathbb{R}^2} u(x,t)dx$;
- $L^{\frac{d}{2}}$-scaling invariance: $\forall \gamma > 0, \;\; u_\gamma(x,t) = \frac{1}{\gamma^2} u\left(\frac{x}{\gamma}, \frac{t}{\gamma^2}\right), \; \|u_\gamma\|_{L^{\frac{d}{2}}} = \|u\|_{L^{\frac{d}{2}}}$;
- free energy functional: $\mathcal{F}(u) = \int_{\mathbb{R}^d} u\left(\ln u - \frac{1}{2}\Phi_u\right), \quad \frac{d}{dt}\mathcal{F}(u) \leq 0$;
- stationary solution for $d = 2$: $Q_{\gamma,a}(x) = \frac{1}{\gamma^2} Q\left(\frac{x-a}{\gamma}\right)$, where

$$Q(x) = \frac{8}{(1+|x|^2)^2}, \quad \int_{\mathbb{R}^2} Q = 8\pi.$$

Diffusion vs. Aggregation in 2D

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

- If $M < 8\pi$: global existence + spreading, [Blanchet-Dolbeault-Perthame '06]. The proof mainly relies on the free energy functional $\mathcal{F}(u)$ and the Log HLS inequality.
- If $M = 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 u < +\infty$: **blowup in infinite time**, [Blanchet-Carrillo-Masmoudi '08]. Constructive approaches by [Ghoul-Masmoudi '18] (radial), [Davila-del Pino-Dolbeault-Musso-Wei '20] (full nonradial):

$$\|u(t)\|_{L^\infty} \sim c_0 \log t \quad \text{as} \quad t \to +\infty.$$

- If $M > 8\pi$: **blowup in finite time**, [Childress-Percus '81], [Jager-Luckhaus '92], [Nagai-Senba '98], [Senba-Suzuki '03]:

$$\text{(virial identity)} \quad \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)dx = \frac{M}{2\pi}(8\pi - M).$$

  Constructive approaches in the radial setting by [Herrero-Velázquez '96], [Raphaël-Schweyer '14]:

$$\|u(t)\|_{L^\infty} \sim C_0 \frac{e^{\sqrt{2|\log(T-t)|}}}{T-t} \quad \text{as} \quad t \to T.$$

A numerical simulation of finite time singularity

A numerical simulation of blowup for the 2D Keller-Segel system

$$\partial_t u = \Delta u - \nabla.(u \nabla \Phi_u), \quad -\Delta \Phi_u = u.$$

Diffusion vs. Aggregation in high dimensions $d \geq 3$

- A critical threshold in $L^{\frac{d}{2}}$ for global existence [Calvez-Corrias-Ebde '12]

$$\|u(0)\|_{L^{\frac{d}{2}}} < \frac{8}{d} C_{GN}^{-2(1+2/d)}(d/2, d) \quad \Longrightarrow \quad \text{global existence.}$$

- Existence of self-similar (type I) blowup solutions by [Herrero-Medina-Velázquez '98]:

$$u(x,t) = \frac{1}{T-t}\varphi\left(\frac{x}{\sqrt{T-t}}\right), \quad \|u\|_{L^1} = \infty.$$

Asymptotic description of φ by [Giga-Mizoguchi-Senba '11].

- A formal derivation of non self-similar (type II) blowup solutions in the radial setting by [Herrero-Medina-Velázquez '97], [Brenner-Constantin-Kadanoff-Schenkel-Venkataramani '99]:

$$u(x,t) \sim \frac{1}{\lambda^2(t)} w\left(\frac{|x| - R(t)}{\lambda(t)}\right), \quad R(t) \sim (T-t)^{\frac{1}{d}}, \ \lambda \sim R^{d-1}, \ \|u\|_{L^1} < \infty.$$

- Blowup solution can be exhibited with any arbitrary mass by the scaling invariance.

Underlying problem

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

Existence and Stability of blowup solutions.

2. Finite-time blowup in 2DKS

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^2. \qquad \textbf{(2DKS)}$$

2.1 - Statement of the result

Finite time blowup for the 2DKS

$$\partial_t u = \nabla \cdot \left(\nabla u - u\nabla\Phi_u\right)$$

- Type I does not exist, [Senba-Suzuki '11]: $\partial_t u = \Delta u - \nabla u . \nabla \Phi_u + u^2$.
- Type II: "Δ dominates ∂_t" $\rightsquigarrow$ profile, **unknown blowup rates**.

Theorem 1 ([Collot-Ghoul-Masmoudi-Ng., '21]).

- There exists a set $\mathcal{O} \subset L^1 \cap \mathcal{E}$, where $\mathcal{E} = \{u : \sum_{k=0}^2 \|\langle x\rangle^k \nabla^k u\|_{L^2} < +\infty\}$, of initial data u_0 (not necessary radially symmetric) such that

$$u(x,t) = \frac{1}{\lambda^2(t)}\left[Q\left(\frac{x - a(t)}{\lambda(t)}\right) + \varepsilon(x,t)\right],$$

where $a(t) \to \bar{a} \in \mathbb{R}^2$ and $\sum_{k=0}^1 \|\langle y\rangle^k \nabla^k \varepsilon(t)\|_{L^2} \to 0$ as $t \to T$, and λ is given by either

$$\lambda(t) \sim 2e^{-\frac{\gamma+2}{2}}\sqrt{T-t}\exp\left(-\frac{\sqrt{|\log(T-t)|}}{\sqrt{2}}\right), \qquad \textbf{(C1)}$$

or

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{2}}|\log(T-t)|^{-\frac{\ell}{2(\ell-1)}}, \quad \ell \geq 2 \text{ integer}. \qquad \textbf{(C2)}$$

- Case (**C1**) is stable and Case (**C2**) is $(\ell-1)$-codimension stable.

Comments

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

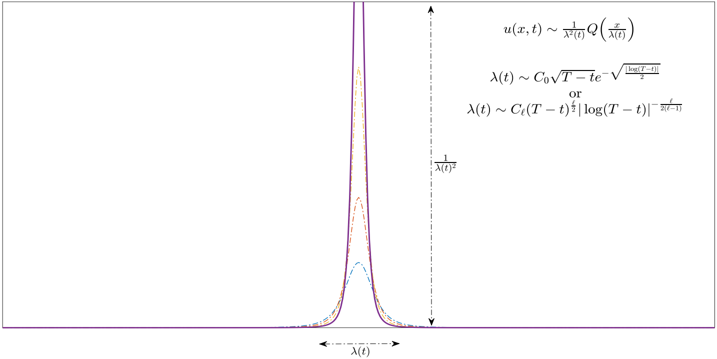


Fig 1: The form of single-point finite time blowup solutions.

Comments

$$\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)$$

■ Existing results: formal level (numerical observation, formal matching asymptotic expansions) and in the radial setting to remove the nonlocal structure difficulty, i.e. $u(x,t) = u(r,t)$,

$$m(r) = \int_0^r u(\zeta)\zeta d\zeta, \quad u(r) = \frac{\partial_r m(r)}{r}, \quad \partial_r\Phi_u(r) = -\frac{m(r)}{r}, \quad r = |x|,$$

$$\partial_t u = \frac{1}{r}\partial_r\left(r\partial_r u - ru\partial_r\Phi_u\right) \quad \Longrightarrow \quad \boxed{\partial_t m = \partial_r^2 m - \frac{\partial_r m}{r} + \frac{\partial_r m^2}{2r}}$$

Refs: [Herrero-Velázquez '96 & '97], [Velázquez '02], [Schweyer-Raphael '14], [Dyachenko-Lushnikov-Vladimirova '13], ...

■ The new result: full nonradial setting, refined description of the stable blowup mechanism, new (unstable) blowup dynamics, a nature approach via spectral analysis/robust energy-type method.

2.2 - A formal derivation of blowup

Formal matched asymptotic expansions

$$\boxed{\partial_t u = \nabla \cdot \left(\nabla u - u \nabla \Phi_u\right)}$$

- Formal analysis via matched asymptotic expansions [Velázquez '02]: working with the *self-similar variables*

$$u(x,t) = \frac{1}{T-t} w(z,\tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t),$$

$$\boxed{\partial_\tau w = \nabla \cdot \left(\nabla w - w \nabla \Phi_w\right) - \frac{1}{2} \nabla \cdot (zw)}$$

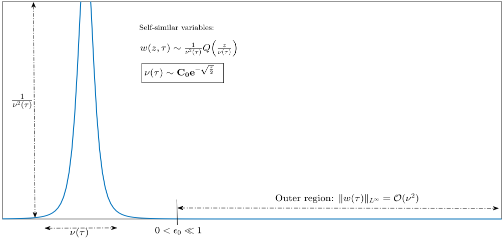


Fig 2: Understanding of the matched asymptotic expansions

Matched asymptotic expansions $\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw)$

- Inner approximate solution: $w^{\text{inn}}(z,\tau) = \dfrac{1}{\nu(\tau)^2} P\Big(\dfrac{z}{\nu}, \tau\Big),$

$$\nu^2 \partial_\tau P = \nabla \cdot (\nabla P - P \nabla \Phi_P) + \sigma(\tau) \nabla \cdot (yP), \quad \sigma = \nu \nu_\tau - \frac{\nu^2}{2}.$$

- Expanding P: $P(y,\tau) = Q(y) + \sigma(\tau) T_1(y) + T_2(y,\tau)$, where

$$\mathscr{L}_0 T_1 = -\nabla \cdot (yQ), \quad \mathscr{L}_0 T_2 = \nu^2 \sigma_\tau T_1 - \sigma^2(\tau) \nabla \cdot (y T_1) + lot.$$

$$\mathscr{L}_0 f = \nabla \cdot (\nabla f - f \nabla \Phi_Q - Q \nabla \Phi_f).$$

- Inner expansion: for $\nu \ll |z| < \epsilon_0$,

$$w^{inn}(z,\tau) = \underbrace{\frac{8\nu^2}{|z|^4}}_{Q} \underbrace{-\frac{4\sigma}{|z|^2}}_{\sigma T_1} \underbrace{-\sigma_\tau \Big[\log |z| - \log \nu - \frac{5}{4}\Big] + \frac{\sigma^2}{\nu^2}}_{T_2} + lot.$$

Matched asymptotic expansions $\boxed{\partial_\tau w = \nabla \cdot (\nabla w - w\nabla \Phi_w) - \frac{1}{2}\nabla \cdot (zw)}$

- Outer approximate solution: $w^{out} = \mathcal{O}(\nu^2)$, $\quad \dfrac{\partial \Phi_Q}{\partial |z|} \sim -\dfrac{4}{|z|}$ for $|z| \to 0$,

$$\partial_\tau w^{\text{out}} = \Delta w^{\text{out}} + \frac{4}{|z|}\frac{\partial w^{\text{out}}}{\partial |z|} - \frac{1}{2}\nabla \cdot \left(zw^{\text{out}}\right) := \mathscr{H} w^{out}.$$

- Expanding w^{out}: $\quad w^{out} = \nu^2 W_1 + \nu_\tau \nu W_2$, where $\mathscr{H} W_1 = 0, \quad \mathscr{H} W_2 = 2W_1$.
- Outer expansion: for $\nu \ll |z| < \epsilon_0$,

$$w^{out}(z,\tau) = \nu^2 \underbrace{\left[\frac{8}{|z|^4} + \frac{2}{|z|^2}\right]}_{W_1} + \nu_\tau \nu \underbrace{\left[-\frac{4}{|z|^2} + \log|z| - \frac{3}{4} - \frac{\log 4}{2} + \frac{\gamma}{2}\right]}_{W_2} + lot.$$

- Matching expansions yields the leading ODE:

$$\sigma_\tau \log \nu + \frac{5}{4}\sigma_\tau + \frac{\sigma^2}{\nu^2} = -\left(\frac{3}{4} + \frac{\log 4}{2} - \frac{\gamma}{2}\right)\nu\nu_\tau \quad \Longrightarrow \quad \boxed{\nu(\tau) = C_0 e^{-\sqrt{\frac{\tau}{2}}}}$$

- Analysis of the stability was formally done by [Velazquez '02] at the linear level.

Existing rigorous analysis

$$\boxed{\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)}$$

- Rigorous analysis via modulation techniques [Schweyer-Raphael '14]: working with the *blowup variables*:

$$u(x,t) = \frac{1}{\lambda^2} v(y,s), \quad y = \frac{x}{\lambda}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad \Big(\lambda(t) > 0 \text{ unknown}\Big),$$

$$\boxed{\partial_s v = \Delta v - \nabla \cdot (v\nabla\Phi_v) - b\nabla \cdot (yv)} \quad b = -\lambda_t \lambda.$$

- Approximate solution:

$$v^{app}(y;b) = Q(|y|) + bT_1(|y|) + S_2(|y|;b),$$

$$\mathscr{L}_0 T_1 = \nabla.(yQ), \quad \mathscr{L}_0 S_2 = b^2 \nabla.(yT_1) + b_s T_1 + lot.$$

Improving S_2 in the blowup zone $|y| \sim \frac{1}{\sqrt{b}}$ leads to the leading ODE ($m_{T_1} \sim c_1 \ln r$)

$$b_s = -\frac{2b^2}{|\log b|} \quad \Longrightarrow \quad \lambda(t) = \sqrt{T-t}\, e^{-\sqrt{\frac{|\log(T-t)|}{2}} + \mathcal{O}(1)}.$$

- Control of the remainder $\varepsilon = v - v^{app}$: based on the special structure

$$\mathscr{L}_0 \varepsilon = \nabla \cdot \Big(Q\nabla \mathscr{M}\varepsilon\Big), \quad \mathscr{M}\varepsilon = \frac{\varepsilon}{Q} - \Phi_\varepsilon.$$

Requirement: radial + L^1 smallness + a complicated treatment for $b\nabla.(y\varepsilon)$.

2.3 - Strategy of the new constructive proof

Strategy of the new constructive proof

$$\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)$$

- Self-similar variables:

$$u(x,t) = \frac{1}{T-t} w(z,\tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \frac{d\tau}{dt} = \frac{1}{T-t},$$

$$\partial_\tau w = \nabla \cdot (\nabla w - w\nabla\Phi_w) - \frac{1}{2}\nabla \cdot (zw).$$

- Blowup variables: $\|w(\tau)\|_{L^\infty} \to \infty$ as $\tau \to \infty$,

$$w(x,t) = \frac{1}{\nu^2} v(y,\tau), \quad y = \frac{z}{\nu},$$

where $\nu(\tau) \to 0$ as $\tau \to \infty$ is an unknown parameter function:

$$\nu^2 \partial_\tau v = \nabla \cdot (\nabla v - v\nabla\Phi_v) + \sigma(\tau)\nabla \cdot (zv), \quad \sigma(\tau) = \mathcal{O}(\nu^2).$$

$\Longrightarrow$ A suggestion of the leading term in the expansion of $v \sim Q$ since $\sigma \to 0$ as $\tau \to \infty$.

The linearized problem

- Linearized problem: $w(z,\tau) = Q_\nu(z) + \eta(z,\tau)$, where $Q_\nu(z) = \frac{1}{\nu^2} Q\left(\frac{z}{\nu}\right)$ and η solves

$$\partial_\tau \eta = \mathscr{L}^\nu \eta + \left(\frac{\nu_\tau}{\nu} - \frac{1}{2}\right) \nabla \cdot (z Q_\nu) - \nabla \cdot \Big(\eta \Phi_\eta\Big), \qquad \nu \to 0 \text{ unknown},$$

$$\boxed{\mathscr{L}^\nu \eta = \underbrace{\nabla \cdot \big(\nabla \eta - \eta \nabla \Phi_{Q_\nu} - Q_\nu \nabla \Phi_\eta\big)}_{\equiv \mathscr{L}_0^\nu \eta} - \frac{1}{2} \nabla \cdot (z\eta)}$$

- Structure of $\mathscr{L}_0^\nu$:

$$\mathscr{L}_0^\nu \eta = \nabla \cdot \big(Q_\nu \nabla \mathscr{M}^\nu \eta\big), \quad \mathscr{M}^\nu \eta = \frac{\eta}{Q_\nu} - \Phi_\eta.$$

($\mathscr{M}^\nu$ comes from the linearization of the energy functional $\mathcal{F}$ around Q_ν).

A key proposition of the linear analysis in 2DKS

Proposition 2 ([Collot-Ghoul-Masmoudi-Ng., 2020]).

- In the radial setting and in terms of the partial mass, $\mathscr{L}_\nu$ becomes a local operator,

$$\mathrm{spec}(\mathscr{L}^\nu)|_{rad} = \left\{ \alpha_{n,\nu} = 1 - n - \frac{1}{2|\ln \nu|} + \mathcal{O}\left(\frac{1}{|\ln \nu|^2}\right), \quad n \in \mathbb{N} \right\}.$$

The analysis of eigenproblem has been done through a matched asymptotic expansions technique, where the eigenfunction $\varphi_{n,\nu}$ is built from iterative kernels of the linearized operator (think of Neumann series).
$\rightsquigarrow$ spectral analysis to control the radial part in L^2_ω.

- For the nonradial part $\rightsquigarrow$ energy methods: dissipation + coercivity

$$\int_{\mathbb{R}^2} \mathscr{L}^\nu(u\sqrt{\rho})\mathscr{M}^\nu(u\sqrt{\rho}) \leq -c_0 \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{Q_\nu}\rho, \quad \text{with} \quad \rho(z) = e^{-\frac{|z|^2}{4}}.$$

(up to appropriate orthogonality conditions)

Approximate solution

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw)$$

- The approximate solution: for $\ell \geq 1$ integer,

$$w^{app}(z,\tau) = Q_\nu(z) + \underbrace{a_\ell(\tau)\Big[\varphi_{\ell,\nu}(|z|) - \varphi_{0,\nu}(|z|)\Big]}_{\text{modification driving the law of blowup}}.$$

A suitable projection onto $\varphi_{\ell,\nu}$ and compatibility condition:

$$(\ell = 1,\ \text{stable}) \quad \frac{\nu_\tau}{\nu} = \frac{1}{4\ln\nu} + \frac{e_2}{|\ln\nu|^2} \quad \Longrightarrow \quad \boxed{\nu = C_0 e^{-\sqrt{\frac{\tau}{2}}}}$$

$$(\ell \geq 2,\ \text{unstable}) \quad \frac{\nu_\tau}{\nu} = \frac{1-\ell}{2} + \frac{\ell+1}{4\ln\nu} \quad \Longrightarrow \quad \boxed{\nu = C_\ell e^{\frac{(1-\ell)\tau}{2}} \tau^{\frac{\ell}{2(1-\ell)}}}$$

- The linearized equation: $\varepsilon = w - w^{app}$,

$$\partial_\tau \varepsilon = \mathscr{L}^\nu \varepsilon + \mathit{Error} + \mathit{SmallLinear} + \mathit{Nonlinear}.$$

What happens at the nonlinear analysis?

- **The main issue:** The perturbation ε can be large near the origin, and the only control in L^2_ω does not allow for a use of dissipation. In particular the direction $\nabla.(z\, Q_\nu)$, which is the kernel of $\mathscr{L}_0^\nu = \nabla.\big(Q_\nu \nabla \mathscr{M}^\nu \cdot\big)$, becomes the leading part of ε in the zone $|z| \sim \nu$.

- **The treatment:** Recall that $0 = \Delta Q_\lambda - \nabla.(Q_\lambda \nabla \Phi_{Q_\lambda})$ for any $\lambda > 0$,

$$0 = \frac{d}{d\lambda}\Big[\Delta Q_\lambda - \nabla.(Q_\lambda \nabla \Phi_{Q_\lambda})\Big]_{\lambda=\nu} \implies \mathscr{L}_0^\nu\big[\nabla.(z\, Q_\nu)\big] = 0.$$

We introduce $\tilde{\nu} \sim \nu$ and impose a local orthogonality condition to eliminate $\nabla.(z\, Q_\nu)$. It's crucial that the key proposition still holds true for the linearized operator $\mathscr{L}^{\tilde{\nu}}$ up to an admissible error, from which we are able to close the nonlinear analysis.

- **An expectation:** Such an idea can be successfully applied to other problems in some critical regimes.

3. Collapsing-ring blowup for $d \geq 3$

Radial blowup solutions

$$\boxed{\partial_t u = \nabla \cdot \left(\nabla u - u \nabla \Phi_u\right), \ d \geq 3}$$

- Basis features: mass conservation, scaling symmetry $u_\gamma(x,t) = \gamma^2 u(\gamma x, \gamma^2 t)$, $\forall \gamma > 0$,
$\Longrightarrow$ L^1-supercritical, $L^{\frac{d}{2}}$-critical.
- Radial setting: $u(x,t) = u(r,t)$, $r = |x|$,

$$\boxed{\partial_t u = \partial_r^2 u + \frac{d-1}{r}\partial_r u + \frac{1}{r^{d-1}}\partial_r(u\, m_u)} \quad m_u(r,t) = \int_0^r u(\zeta,t)\zeta^{d-1}d\zeta.$$

- Traveling solutions blowing up in finite time:

$$u(r,t) = \frac{1}{\lambda^2(t)} w\left(\frac{r - R(t)}{\lambda(t)}\right), \quad 0 < \lambda(t) \ll R(t) \to 0 \ \text{ as } t \to T.$$

- Blowup solutions with arbitrary mass $M < +\infty$, different from the 2D case.

A formal derivation of blowup

$$\partial_t u = \partial_r^2 u + \frac{d-1}{r}\partial_r u + \frac{1}{r^{d-1}}\partial_r(u\, m_u)$$

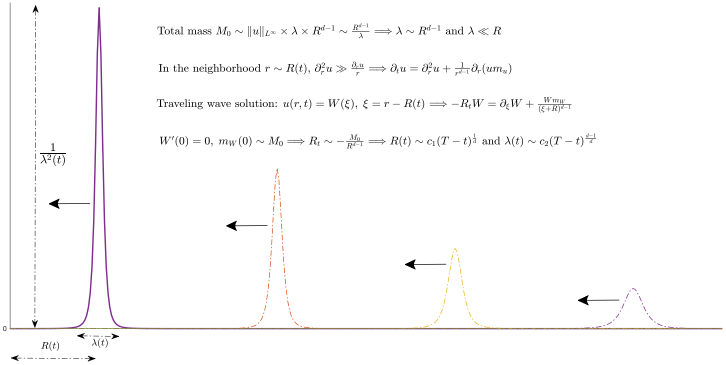


Fig 3: Illustration of a collapsing-ring blowup solution

Traveling blowup solution in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

Theorem 3 ([Collot-Ghoul-Masmoudi-Ng., 2021]).

- There exists a set $\mathcal{O} \subset W^{1,\infty}(\mathbb{R}_+)$ of initial data $m_u(0)$ such that

$$m_u(r,t) = M(t)\left[Q\left(\frac{r-R(t)}{\lambda(t)}\right) + m_\varepsilon(r,t)\right], \quad Q(\xi) = \frac{e^{\frac{\xi}{2}}}{1+e^{\frac{\xi}{2}}},$$

where $\|m_\varepsilon(t)\|_{W^{1,\infty}(\mathbb{R}_+)} \to 0$ as $t \to T$,

$$M \to M_0, \quad \lambda = \frac{R^{d-1}}{M}, \quad R(t) \sim \left[(d/2)M(T-t)\right]^{\frac{1}{d}}.$$

- The constructed solution is stable under small perturbation in $\mathcal{O}$.

Traveling shock solutions in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

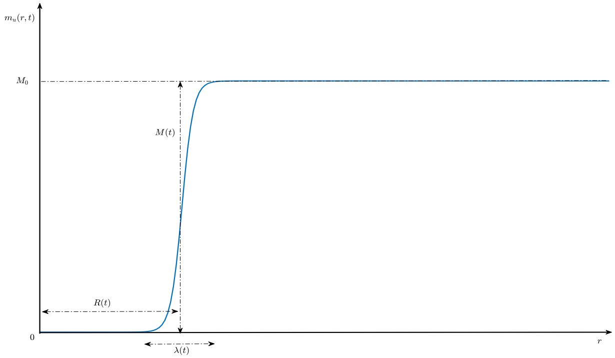


Fig 4: Illustration of a traveling shock solution to the partial mass equation.

A numerical simulation for $d = 3$

Fig 5: (horizontally zoomed solution) The initial data $m_u(r,0) = MQ\left(\frac{r - M^{\frac{1}{3}}\epsilon}{M^{-\frac{1}{3}}\epsilon^2}\right)$, *where* $M = 27$ *and* $\epsilon = 0.7$. *With* $\epsilon = 0.7$, *the theoretical blowup time is* $T = \epsilon^3 \approx 0.343$. *Maple solver gives an approximation of the blowup time by saying "could not compute solution for* $t > 0.32$: *Newton iteration is not converging".*

Change of variables $\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}$

- Inviscid variables (fix the shock location):

$$m_u(r,t) = M(t) m_w(\zeta,\tau), \quad \zeta = \frac{r}{R(t)}, \quad \frac{d\tau}{dt} = \frac{M(t)}{R(t)^d}, \quad \text{and} \quad \nu = \frac{R^{d-1}}{M},$$

to fix the location of the shock at $\zeta = 1$,

$$\partial_\tau m_w = \left(\frac{m_w}{\zeta^{d-1}} - \frac{1}{2}\zeta\right)\partial_\zeta m_w + \nu \Delta_{\zeta,2-d}\, m_w + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\zeta \partial_\zeta m_w - \frac{M_\tau}{M} m_w.$$

- Blowup variables (zoom at the shock):

$$m_w(\zeta,\tau) = m_v(\xi,s), \quad \xi = \frac{\zeta - 1}{\nu}, \quad \frac{ds}{d\tau} = \frac{1}{\nu},$$

where m_v solves the new equation

$$\partial_s m_v = \partial_\xi^2 m_v + m_v \partial_\xi m_v - \frac{1}{2}\partial_\xi m_v + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\partial_\xi m_v - \frac{M_s}{M} m_v + l.o.t$$

- The blowup profile is connected to the traveling solution to Burgers equation:

$$Q'' - \frac{1}{2}Q' + QQ' = 0, \quad \lim_{\xi \to -\infty} Q(\xi) = 0.$$

The linearized problem $\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}$

- Introducing $m_q(\xi, s) = m_v(\xi, s) - Q(\xi)$ yields

$$\partial_s m_q = \mathscr{L}_0 m_q + L(m_q) + NL(m_q) + \Psi,$$

where $\mathscr{L}_0 = \partial_\xi^2 - (1/2 - Q)\partial_\xi + Q'$ is the linearied operator appearing in the study of stability of traveling wave solutions to Burgers equation.

$$\left\langle \mathscr{L}_0 g, g \right\rangle_{L^2_{\omega_0}} \leq -\delta_0 \|g\|^2_{H^1_{\omega_0}} + C\left\langle g, Q' \right\rangle^2_{L^2_{\omega_0}}, \quad \omega_0 = Q^{-1} e^{\frac{\xi}{2}}.$$

- Introducing $m_\varepsilon(\zeta, \tau) = m_w(\zeta, \tau) - Q_\nu(\zeta)$ yields

$$m_{\varepsilon,1} = \partial_\zeta m_\varepsilon, \qquad \partial_\tau m_{\varepsilon,1} = \mathscr{A}_1 m_{\varepsilon,1} + \mathcal{P} m_{\varepsilon,1} + E, \quad \zeta \geq 1,$$

where

$$\mathscr{A}_1 = -\left(\frac{d-1}{\zeta^d} + \frac{1}{2}\right) + \left(\frac{1}{\zeta^{d-1}} - \frac{1}{2}\zeta\right)\partial_\zeta + \nu\partial_\zeta^2.$$

An observation (constructive approach): $0 < \kappa \ll 1$,

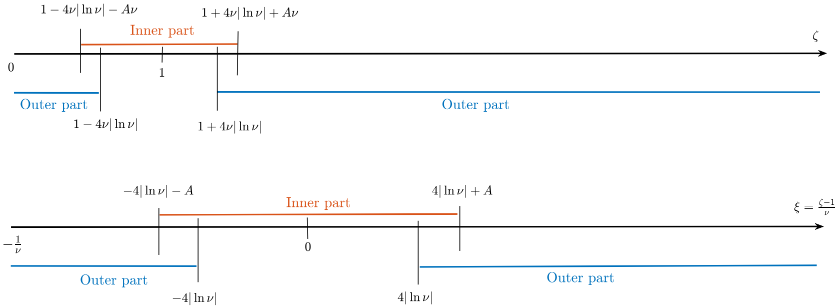
$$\phi_1 = e^{-\kappa\tau} e^{-\frac{3}{8}\left(\frac{|\zeta - 1| - 4\nu|\ln\nu|}{\nu}\right)}, \qquad \partial_\tau \phi_1 - \mathscr{A}_1\phi_1 \geq \frac{c_0}{\nu}\phi_1, \quad \zeta \in [1, 2^{\frac{1}{d}}).$$

A design of the bootstrap regime

- Inner-outer estimates: $A \gg 1$, $0 < \kappa \ll 1$,

$$\|\chi_{4|\ln \nu|+A} m_q(\tau)\|_{L^2_{\omega_0}} \lesssim e^{-\kappa\tau}, \qquad \|\partial_\zeta m_\varepsilon(\tau)\|_{L^\infty(|\zeta-1|\geq 4\nu|\ln\nu|)} \lesssim e^{-\kappa\tau}$$

- The coercivity of $\mathscr{L}_0$ to control the inner norm.
- A delay estimate for a transport-type equation helps to construct $\phi_1(\zeta,\tau)$.



4. Conclusion & Perspectives

Interesting problems

- multiple-collapse phenomena/ interaction-collision of multi-solitons;
- *classification* of blowup dynamics (rates & profiles);
- Numerical methods for blowup problems (detection, rates & profiles).

A multiple-collapse phenomenon in the 2D Keller-Segel system