Multi-soliton dynamics for non linear Klein-Gordon equations

Raphaël Côte, Strasbourg University
collaboration with Yvan Martel, Xu Yuan, Lifeng Zhao

Université de Strasbourg
IRMA Institut de Recherche Mathématique Avancée
Investissements d'avenir
CNRS

Nonlinear dispersive PDEs

Schrödinger type models	$i\partial_t u + \Delta u + \sigma \lvert u\rvert^{p-1} u = 0$
Wave type models	$\partial_{tt} u - \Delta u + \sigma \lvert u\rvert^{p-1} u = 0$
Korteweg-de Vries type models	$\partial_t u + \partial_{xxx} u + \partial_x(\sigma \lvert u\rvert^{p-1} u) = 0$

Plane wave solutions $e^{i\xi\cdot(x-\omega t)}$ of the *linear* part of the equation satisfy a dispersion relation

$$\omega = \omega(\xi).$$

An equation is dispersive when this dependency is not trivial; different frequencies are transported at different speed.

Linear solutions of dispersive PDEs tend to 0 (typically in L^∞, by stationnary phase) even though L^2 and $\dot{H}^s$ norms are preserved, and so spread.

Nonlinearity: driven by the signum $\sigma \in \{\pm 1\}$:

- $\sigma = -1$ defocusing $\rightsquigarrow$ “helps dispersion”
- $\sigma = +1$ focusing $\rightsquigarrow$ “offsets dispersion”, supposedly richer dynamics.

Main question: can we describe the behavior of solutions for large time?

Nonlinear Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0, \quad u(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

- Model in Quantum Field Physics for a self-interacting, nonlinear scalar field.
- Nonlinear wave type equation; focusing semi linear nonlinearity. Mass term (no scaling).
- Damping parameter $\alpha \geqslant 0$.
  If $\alpha = 0$, no damping, we denote the equation (NLKG).
  If $\alpha > 0$, there is damping, we denote the equation (dNLKG).
- Finite speed of propagation; speed of light is 1.
- (NLKG) is invariant under Lorentz transformations.

We focus on H^1 subcritical power nonlinearities ($p = 3$ is the most physically relevant):

$$f(u) = |u|^{p-1}u \quad \text{for } 1 < p < \frac{N+2}{N-2} \quad (p > 1 \text{ for } N = 1 \text{ or } 2)$$

Let $F(u) = \int_0^u f(v)dv = \frac{|u|^{p+1}}{p+1}$. Vector formulation: $\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, $\partial_t \vec{u} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \vec{u} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix}$.

Local well posedness and conservation laws

LWP. (NLKG) and (dNLKG) are locally well-posed in $H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$, for any $s \geqslant 1$ (Ginibre-Velo 85 and Nakamura-Ozawa 01 in dimension $N = 2$).

For any $\vec{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ there exists a unique solution

$$\vec{u}(t) \in \mathcal{C}((-T_-, T_+,), H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N))$$

such that $\vec{u}(0) = (u_0, u_1)$, and the map $\vec{u}_0 \mapsto \vec{u}$ is smooth. If $T_+ < +\infty$, $\limsup_{t \to T_+} \|\vec{u}(t)\|_{H^1 \times L^2} = +\infty$.

In fact, (NLKG) and (dNLKG) are well posed in $H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ for $s \geqslant s_p$, $s_p < 1$.

Identities. Energy and momentum

$$\mathcal{E}[\vec{u}](t) = \int \left[\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}|u(t,x)|^2 - F(u(t,x))\right] dx$$

$$\mathcal{P}[\vec{u}](t) = \frac{1}{2}\int \partial_t u(t,x) \nabla u(t,x) dx$$

$$\mathcal{E}(\vec{u})(t) - \mathcal{E}[\vec{u}](s) = -2\alpha \int_s^t \|\partial_t u(\tau)\|_{L^2}^2 d\tau \qquad \mathcal{P}[\vec{u}](t) = e^{-2\alpha t}\mathcal{P}[\vec{u}](0).$$

Large time behavior

Existence of finite time blow up solution

- ODE blow up.
- $\mathcal{E}[\vec{u}_0] < 0$. (Viriel type / convexity argument).

Soliton resolution conjecture

Global solutions should generically behave for large times as a sum of decoupled solitons (non linear objects), up to some dispersive term.

For (NLKG), solitons are special travelling wave solutions.

For (dNLKG), there is no travelling wave solutions, only stationary solutions.

Stationary solutions

Bound states

$$\mathcal{B} = \{q \in H^1(\mathbb{R}^N) : q \text{ is a nontrivial solution of } -\Delta q + q - f(q) = 0\}.$$

Bound states are $\mathcal{C}^3$ and exponentially decaying (along with their derivatives) with uniform rate $e^{-\omega_0|x|}$. Consider the potential part of the energy

$$E(u) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - F(u)\right] dx.$$

Ground states are bound states that minimize the functional E:

$$\mathcal{G} = \{q_{\mathrm{GS}} \in \mathcal{B} : E(q_{\mathrm{GS}}) \leqslant E(q) \text{ for all } q \in \mathcal{B}\}.$$

Classification of ground states [Kwong, Serrin-Tang]:

There exists $Q \in H^1(\mathbb{R}^N)$ radial, $Q > 0$ such that

$$\mathcal{G} = \{Q(x - x_0) : x_0 \in \mathbb{R}^N\}.$$

One has the asymptotic for $r \geqslant 1$:

$$Q(r) = \kappa r^{-\frac{N-1}{2}} e^{-r} + O(r^{-\frac{N+1}{2}} e^{-r}), \quad \kappa > 0.$$

Excited states

If $N = 1$, $\mathcal{B} = \mathcal{G}$. [ODE methods; Boussinesq] $Q(x) = \left(\dfrac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}$.

If $N \geqslant 2$, $\mathcal{G} \subsetneq \mathcal{B}$. Bound states which are not ground states are called excited states.

Existence of

- radial nodal excited states [Shooting method; Berestycki-Lions 83].
- non-radial excited states [del Pino-Musso-Pacard-Pistoia 11, 13].
- radial, positive excited states for non pure power nonlinearity f (in dimension 3). [Dàvila-del Pino-Guerra 12].

Traveling waves

Lorentz boosts. (NLKG) is invariant under Lorentz boosts (but (dNLKG) is not). Given velocity $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{R}^N$, where $|\beta| < 1$ (we denote by $|\cdot|$ the euclidian norm on $\mathbb{R}^N$) and

$$\Lambda_\beta := \begin{pmatrix} \gamma & -\beta_1\gamma & \cdots & \beta_d\gamma \\ -\beta_1\gamma & & & \\ \vdots & & I_d + \frac{\gamma-1}{|\beta|^2}\beta\beta^\top & \\ -\beta_d\gamma & & & \end{pmatrix} \in \mathcal{M}_{N+1}(\mathbb{R}), \quad \gamma := \frac{1}{\sqrt{1-|\beta|^2}}.$$

If u solution to (NLKG), then $v(t,x) = u(\Lambda_\beta(t,x)^\top)$ solution to (NLKG).

Traveling waves. Solutions of the form $\vec{u}(t,x) = \vec{p}(x - \beta t)$. They are the Lorentz boosts of bound states: if $q \in \mathcal{B}$, then

$$\vec{R}(t,x) = \begin{pmatrix} q_\beta(x-\beta t) \\ -\beta \cdot \nabla q_\beta(x-\beta t) \end{pmatrix}, \quad q_\beta(x) = q(\Lambda_\beta(x)), \quad \Lambda_\beta(x) = x + (\gamma-1)\frac{\beta(\beta\cdot x)}{|\beta|^2}.$$

is a traveling wave solution to (NLKG), and all traveling wave solutions are of this form.

Solitons

Solitons are traveling waves based on ground-states, that is, solutions of the form

$$\vec{Q}_\beta(t,x) = \begin{pmatrix} Q(\Lambda_\beta(x - x_0 - \beta t)) \\ -\beta\cdot\nabla Q(\Lambda_\beta(x - x_0 - \beta t)) \end{pmatrix}.$$

Stability (dNKLG) solitons are (linearly) unstable.

Consider the linearized problem for $\vec{u} = (Q,0) + \vec{\varepsilon}$ with $\vec{\varepsilon} = (\varepsilon,\eta)$

$$\partial_t\vec{\varepsilon} = \begin{pmatrix} 0 & 1 \\ L & -2\alpha \end{pmatrix}\vec{\varepsilon}, \quad \text{with} \quad L = -\Delta + 1 - f'(Q). \tag{1}$$

Lemma
$\sigma_{ess}(L) = [1+\infty)$, $\ker(L) = Span(\partial_{x_i}Q,\ i = 1,\dots,N)$ *and* L *has a unique negative eigenvalue* $-\nu_0^2$ ($\nu_0 > 0$) *with eigenfunction* Y, $\mathcal{C}^3$ *with exponential decay.*

Define

$$\nu^\pm = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{and} \quad \vec{Y}^\pm = (Y, \nu^\pm Y).$$

Then $\vec{\varepsilon} = e^{\nu^\pm t}\vec{Y}^\pm$ is solution to (1).

Multi-solitons

Multi-solitons are special solutions of (NKLG) or (dNKLG) which behave as a sum of decoupled solitons for large times:

$$u(t,x) \sim \sum_{k=1}^{K} q_k(\Lambda_{\beta_k}(x-\beta_k(t)) \quad \text{as } t \to +\infty,$$

$$\text{where} \quad |\beta_k(t)-\beta_j(t)| \to +\infty \text{ for } k \neq j.$$

- In some sense, such a solution does not disperse.
- If $|\beta_k(t)-\beta_j(t)| \sim ct$, one speaks of weak interaction.
  If $|\beta_k(t)-\beta_j(t)| = o(t)$, one speaks of strong interaction (unstable objects).
- Multi-solitons are important in the description of long time dynamics in view of the soliton resolution conjecture.
- C.-Muñoz 14, C.-Martel 18: Constructions of weakly interacting multi-solitons for (NLKG).
- Multi-solitons were constructed in various contexts: Merle 90 L^2 critical (NLS); Martel 05, L^2 subcritical and critical (gKdV); Martel-Merle 06, L^2 subcritical (NLS); C.-Martel-Merle 11, L^2 supercritical (NLS) and (gKdV); Ming-Rousset-Tzvetkov 14, water-wave system ...
  Strong interaction: Raphael-Martel 16 (NLS); Nguyen 18 (gKdV) ...

Soliton resolution for global solutions of (dNKLG) in 1D

Theorem (C.-Martel-Yuan 20)

Let $\vec{u}(t)$ be any finite energy global *solution of (dNKLG), with $N = 1$, $p > 2$ and $\alpha > 0$. Then,*

- *either $\vec{u}(t) \to \sigma(Q(\cdot - \ell), 0)$ as $t \to +\infty$ (where $\sigma \in \{-1, 0, 1\}$ and $\ell \in \mathbb{R}$), and in this case $\|\vec{u}(t) - \sigma(Q(\cdot - \ell), 0)\|_{H^1 \times L^2} = O(e^{-\delta t})$ for some $\delta = \delta(\alpha) > 0$.*
- *or there exist $K \geqslant 2$, $\sigma = \pm 1$, $\ell \in \mathbb{R}$ and functions $z_k : [0, \infty) \to \mathbb{R}$, for all $k = 1, \dots, K$ such that for all $t \in [0, \infty)$,*

$$\left\| u(t) - \sigma \sum_{k=1}^{K} (-1)^k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \quad (2)$$

$$\forall k = 1, \dots, K, \quad z_k(t) = \left(k - \frac{K+1}{2}\right) \ln t + \tau_k + \ell + O(t^{-\beta}), \quad (3)$$

where τ_k are universal constants and $\beta = \beta(p) > 0$.

Theorem (C.-Martel-Yuan 20)

Given $K \geqslant 2$, $\sigma = \pm 1$ and $\ell \in \mathbb{R}$, there exist solutions of (dNLKG) satisfying (2) *and* (3).

Comments on the soliton resolution conjecture

- Formulated by Zabusky-Kruskal 65 for (KdV).
- In integrable cases, inverse scattering method (for smooth, decaying initial data) $\rightsquigarrow$ applies to (KdV), (mKdV).

- Recent progress on wave type equations.
  Duyckaerts-Kenig-Merle (12-21) for the radial energy critical non linear wave equation.
  C.-Kenig-Lawrie-Schlag (14-17), Jendrej-Lawrie (21) for equivariant wave maps.
- Seems currently out of reach for (NLKG).

- For 1D (dNLKG), the description is much sharper: rate of convergence + position of the soliton's centers (universal). Also, the reciprocal of the soliton resolution holds.
  The description for long time is complete.
- Extension to higher dimensions? Need to
  - understand the dynamics around an excited state.
  - understand the ODE system for the solitons' centers.

Description of 2-solitons

We say that a solution u of $(dNKLG)$ is a 2-solitary wave if it is defined for large times and there exists $t_n \to +\infty$, and for $i = 1, 2$, $\sigma_i \in \{\pm 1\}$ and $z_{i,n} \in \mathbb{R}^N$ such that $|z_{1,n} - z_{2,n}| \to +\infty$ and

$$\|u(t_n) - (\sigma_1 Q(\cdot - z_{1,n}) + \sigma_2 Q(\cdot - z_{2,n})\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \to 0.$$

Theorem (Description of 2-solitary waves, C.-Martel-Yuan-Zhao 19)

For any 2-solitary wave u of (dNKLG) with $N \geqslant 1$, $2 < p < \frac{2N}{N-2}$, $\alpha > 0$, there exist $\sigma = \pm 1$, $T > 0$ and $t \in [T, \infty) \mapsto (z_1(t), z_2(t))$ such that for all $t \in [T, \infty)$,

$$\|u(t) - \sigma(Q(\cdot - z_1(t)) - Q(\cdot - z_2(t)))\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{4}$$

and for some constant $c_0 = c_0(N)$, $\omega_\infty \in \mathbb{S}^{N-1}$ and $z_\infty \in \mathbb{R}^N$, and $\theta < \min(p-1, 2)$

$$z_k(t) = z_\infty + \frac{(-1)^k}{2}\left(\log t - \frac{1}{2}(N-1)\log\log t + c_0\right)\omega_\infty + O(t^{1-\theta}). \tag{5}$$

Initial data leading to 2-solitons

Theorem (Classification of 2-solitary waves, C.-Martel-Yuan-Zhao 19)

There exist $C, \delta > 0$ and a Lipschitz map

$$H : (\mathbb{R}^N \setminus \bar{\mathcal{B}}_{\mathbb{R}^N}(10|\log\delta|)) \times \mathcal{B}_{H^1\times L^2}(\delta) \to \mathbb{R}^2, \quad (L, \vec{v}) \mapsto H(L, \vec{v})$$

such that $|H(L,\vec{v})| < C\left(e^{-\frac{L}{2}} + \|\vec{v}\|_{H^1\times L^2}\right)$, and with the following property. Given any $L, \vec{v}, h_1, h_2$ such that

$$|L| > 10|\log\delta|, \quad \|\vec{v}\|_{H^1\times L^2} < \delta, \quad |h_1| + |h_2| < \delta,$$

the solution $\vec{u}$ of (dNLKG) with initial data

$$\vec{u}(0) = \left(\vec{Q} + h_1\vec{Y}^+\right)\left(\cdot - \frac{L}{2}\right) - \left(\vec{Q} + h_2\vec{Y}^+\right)\left(\cdot + \frac{L}{2}\right) + \vec{v}$$

is a 2-solitary wave if and only if $(h_1, h_2) = H(L, \vec{v})$.

This result essentially means that locally around the sum of two sufficiently separated solitons with opposite signs, the initial data of 2-solitary waves form a codimension-2 Lipschitz manifold (the unstable directions being directed by $\vec{Y}^+$ translated around each soliton).

Description of the flow around 2-solitons

Theorem (Ishizuka-Nakanishi 21)
There exist $\delta > 0$ such any solution solution u of (dNKLG) with initial data

$$\vec{u}(0) = (Q(\cdot - z_1) - Q(\cdot - z_2), 0) + \vec{\varphi}$$

where $|z_1 - z_2| \geqslant 1/\delta$ and $\|\vec{\varphi}\|_{H^1 \times L^2} < \delta$ either

- *blows up in finite time,*
- *converges to $(0, 0)$,*
- *converges to $(\pm Q(\cdot - z_\infty), 0)$,*
- *is a 2-soliton,*

and all cases are possible.

Theorem (Burq-Raugel-Schlag 15)
Let $2 \leqslant N \leqslant 6$, and $\vec{u}$ be a radial solution to (dNLKG). Then either $\vec{u}$ blows up in finite time, or $\vec{u}$ tends to $(0,0)$ or to a bound state $(q,0)$ in $H^1 \times L^2$ as $t \to +\infty$.

Let $q \in \mathcal{B}$ be a bound state, $\Omega_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}$ (for $1 \leqslant i < j \leqslant N$) be the angular derivatives, and consider

$$\mathcal{Z}_q = \operatorname{Span}\left\{\partial_{x_n} q, n = 1, \cdots, N; \Omega_{ij} q, 1 \leqslant i < j \leqslant N\right\}. \tag{6}$$

One always has $\mathcal{Z}_q \subset \ker L_q$ where $L_q = -\Delta + 1 + f'(q)$.

We say that the bound state q is *degenerate* if $\mathcal{Z}_q \subsetneq \ker \mathcal{L}_q$.
We say that it is a *degree-1 excited state* if furthermore, there exists $\phi \in H^1$ such that

$$\ker L_q = \mathcal{Z}_q \oplus \operatorname{Span}\{\phi\} \quad \text{and} \quad E'''(q) \cdot (\phi, \phi, \phi) \neq 0, \tag{7}$$

Theorem (C.-Yuan 21)
Let q be a degree-1 excited state. Then, there exists a global solution $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0,+\infty), H^1 \times L^2)$ of (dNKLG) such that

$$\|u(t) - q\|_{H^1} + \|\partial_t u(t)\|_{L^2} \sim t^{-1} \quad \text{as} \quad t \to +\infty.$$

Back to our main goal: (dNKLG) in 1D

Theorem (C.-Martel-Yuan 21)

Let $\vec{u}(t)$ be any finite energy global *solution of (dNKLG), with $N = 1$, $p > 2$ and $\alpha > 0$. Then,*

- *either $\vec{u}(t) \to \sigma(Q, 0)$ as $t \to +\infty$ (where $\sigma \in \{-1, 0, 1\}$), and in this case $\|\vec{u}(t) - \sigma(Q, 0)\|_{H^1 \times L^2} = O(e^{-\delta t})$ for some $\delta = \delta(\alpha) > 0$.*
- *or there exist $K \geqslant 2$, $\sigma = \pm 1$, $\ell \in \mathbb{R}$ and functions $z_k : [0, \infty) \to \mathbb{R}$, for all $k = 1, \ldots, K$ such that for all $t \in [0, \infty)$,*

$$\left\| u(t) - \sigma \sum_{k=1}^{K} (-1)^k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{2}$$

$$\forall k = 1, \ldots, K, \quad z_k(t) = \left(k - \frac{K+1}{2}\right) \ln t + \tau_k + \ell + O(t^{-\beta}), \tag{3}$$

where τ_k are universal constants and $\beta = \beta(p) > 0$.

Theorem (C.-Martel-Yuan 21)

Given $K \geqslant 2$, $\sigma = \pm 1$ and $\ell \in \mathbb{R}$, there exist solutions of (dNLKG) satisfying (2) *and* (3).

How do multi-soliton appear

Theorem (Bubbling theorem, Lions 88, Benci-Cerami 88)

Let $(u_n)_n$ be a sequence of H^1 functions such that $E(u_n)$ is bounded and

$$-\Delta u_n + u_n - f(u_n) \to 0 \quad in \quad H^{-1}.$$

Then $(u_n)_n$ is a bounded sequence in H^1, and there exists a subsequence still denoted u_n, $K \geqslant 0$ bound states $(q_k)_{k=1,\dots,K}$ and shifts $z_{k,n} \in \mathbb{R}$ such that $|z_{k,n} - z_{j,n}| \to +\infty$ for any $k \neq j$ and

$$u_n - \sum_{k=1}^{K} (q_k(\cdot - z_{k,n}), 0) \to 0 \quad in \quad H^1.$$

[Based on concentration-compactness arguments]

Global 1D solutions are bounded and consequences

Proposition
Let $N = 1$, $\alpha \geqslant 0$. Let $\vec{u} \in \mathcal{C}([0,+\infty), H^1 \times L^2)$ be a global solution.

Then $\vec{u}$ is bounded: $\sup_{t \geqslant 0} \|\vec{u}\|_{H^1 \times L^2} < +\infty$.

Based on ideas of Cazenave 85: convexity arguments on

$$M(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 ds$$

$$W(t) = \frac{1}{2}\left(\|\partial_t u(t)\|_{L^2}^2 + \|\partial_x u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\right)$$

Proposition (Feireisl 98)
Let $N \in \mathbb{N}$, $\alpha > 0$. Let $\vec{u} \in \mathcal{C}([0,+\infty), H^1 \times L^2)$ be a global bounded solution of (dNLKG): $\sup_{t \geqslant 0} \|\vec{u}\|_{H^1 \times L^2} < +\infty$.

Then $\partial_t u(t) \to 0$ in L^2, $\partial_{tt} u \to 0$ in H^{-1} (and $\partial_t u \in L^2([0,+\infty), L^2)$).

In particular the bubbling theorem applies for any sequence $t_n \to +\infty$.

At this point, there exist a sequence $t_n \to +\infty$, $K \geqslant 1$, $\sigma_1, \dots, \sigma_K \in \{\pm 1\}$, $z_{k,n}$ such that $z_{k+1,n} - z_{k,n} \to +\infty$ for any $k = 1, \dots, K-1$ and

$$\vec{u}(t_n) - \sum_{k=1}^{K} \sigma_k (Q(\cdot - z_{k,n}), 0) \to 0 \quad \text{in } H^1 \times L^2.$$

Recall the instability direction

$$L = -\Delta + 1 - f'(Q), \quad LY = -\nu_0^2 Y.$$

Define

$$\nu^\pm = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \zeta^\pm = \alpha \pm \sqrt{\alpha^2 + \nu_0^2}.$$

$$\vec{Y}^\pm = \begin{pmatrix} Y \\ \nu^\pm Y \end{pmatrix}, \quad \vec{Z} = \begin{pmatrix} \zeta^\pm Y \\ Y \end{pmatrix}$$

Then $e^{\nu^\pm t} Y^\pm$ is a solution to the linearized flow

$$\partial_t \vec{\varepsilon} = \begin{pmatrix} 0 & 1 \\ -L & -2\alpha \end{pmatrix} \vec{\varepsilon},$$

and if $\vec{\varepsilon}$ is solution to this linearized problem, then

$$a^\pm := \langle \vec{\varepsilon}, \vec{Z}^\pm \rangle \quad \text{satifies} \quad \dot{a}^\pm = \nu^\pm a^\pm.$$

Notations

Given $\sigma_k \in \{\pm 1\}$, functions $z_k(t)$ and $\ell_k(t)$ such that $\sum_k |\ell_k| \ll 1$ and $z_{k+1} - z_k \gg 1$, we define

$$Q_k = \sigma_k Q(\cdot - z_k), \quad \vec{Q}_k = (Q_k, -\ell_k \partial_x Q_k) \qquad R = \sum_{k=1}^{K} Q_k, \quad \vec{R} = \sum_{k=1}^{K} \vec{Q}_k$$

$$Y_k = \sigma_k Y(\cdot - z_k), \quad \vec{Y}_k^{\pm} = \sigma_k \vec{Y}^{\pm}(\cdot - z_k) \qquad G = f(R) - \sum_{k=1}^{K} f(Q_k)$$

Proposition (Modulation)
Let $\vec{u}$ be a solution of (dNLKG) on $[T_1, T_2]$ such that

$$\sup_{t \in [T_1, T_2]} \left\{ \inf_{\xi_{k+1} - \xi_k > \log \gamma} \left\| u(t) - \sum_{k=1}^{K} \sigma_k Q(\cdot - \xi_k) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \right\} < \gamma,$$

for some small $\gamma > 0$. Then, there exist unique $\mathcal{C}^1$ functions $(z_k(t), \ell_k(t))_{k \in \{1,\dots,K\}} \in \mathbb{R}^{2K}$, such that $\vec{u}$ decomposes as

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \sum_{k=1}^{K} \vec{Q}_k + \vec{\varepsilon}, \quad \vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}, \quad \langle \varepsilon, \partial_x Q_k \rangle = \langle \eta, \partial_x Q_k \rangle = 0.$$

Bounds on the modulation

1. *Equation of $\vec{\varepsilon}$.*

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_\varepsilon \\ \partial_t \eta = \partial_x^2 \varepsilon - \varepsilon + f(R+\varepsilon) - f(R) - 2\alpha\eta + \mathrm{Mod}_\eta + G \end{cases}$$

where $\mathrm{Mod}_\varepsilon = \sum_{k=1}^{K} (\dot{z}_k - \ell_k)\, \partial_x Q_k, \qquad \mathrm{Mod}_\eta = \sum_{k=1}^{K} \left(\dot{\ell}_k + 2\alpha\ell_k\right) \partial_x Q_k - \sum_{k=1}^{K} \ell_k \dot{z}_k \partial_x^2 Q_k.$

2. *Control of the geometric parameters.* For $k = 1, \ldots, K$,

$$|\dot{z}_k - \ell_k| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2,$$

$$\left|\dot{\ell}_k + 2\alpha\ell_k\right| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-(z_{l+1}-z_l)}.$$

3. *Control of the exponential directions.* Recall $a_k^\pm = \langle \vec{\varepsilon}, \vec{Z}_k^\pm \rangle$,

$$|\dot{a}_k^\pm - \nu^\pm a_k^\pm| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-(z_{l+1}-z_l)}.$$

Bounds on the modulation

1. *Equation of $\vec{\varepsilon}$.*

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_\varepsilon \\ \partial_t \eta = \partial_x^2 \varepsilon - \varepsilon + f(R+\varepsilon) - f(R) - 2\alpha\eta + \mathrm{Mod}_\eta + G \end{cases}$$

where $\quad \mathrm{Mod}_\varepsilon = \sum_{k=1}^{K} (\dot{z}_k - \ell_k)\, \partial_x Q_k, \qquad \mathrm{Mod}_\eta = \sum_{k=1}^{K} \left(\dot{\ell}_k + 2\alpha \ell_k\right) \partial_x Q_k - \sum_{k=1}^{K} \ell_k \dot{z}_k \partial_x^2 Q_k.$

2. *Control of the geometric parameters.* For $k = 1, \ldots, K$,

$$|\dot{z}_k - \ell_k| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2,$$

$$\left| \dot{\ell}_k + 2\alpha \ell_k + \kappa \sigma_k \left[\sigma_{k-1} e^{-(z_k - z_{k-1})} - \sigma_{k+1} e^{-(z_{k+1} - z_k)} \right] \right| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-\theta(z_{l+1} - z_l)}.$$

3. *Control of the exponential directions.* Recall $a_k^\pm = \langle \vec{\varepsilon}, \vec{Z}_k^\pm \rangle$,

$$|\dot{a}_k^\pm - \nu^\pm a_k^\pm| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-(z_{l+1} - z_l)}.$$

Energy dissipation

For $\mu > 0$ small to be chosen, we denote $\rho = 2\alpha - \mu$. Consider the nonlinear energy functional

$$\mathcal{E} = \int \left\{ (\partial_x \varepsilon)^2 + (1 - \rho\mu)\varepsilon^2 + (\eta + \mu\varepsilon)^2 - 2[F(R + \varepsilon) - F(R) - f(R)\varepsilon] \right\}.$$

Proposition (Energy estimates)

There exists $\mu > 0$ such that in the context of the previous proposition, the following holds.

1. Coercivity and bound.

$$\mu \|\vec{\varepsilon}\|^2_{H^1 \times L^2} - \frac{1}{2\mu} \sum_{k=1}^{K} \left((a_k^+)^2 + (a_k^-)^2 \right) \leq \mathcal{E} \leq \frac{1}{\mu} \|\vec{\varepsilon}\|^2_{H^1 \times L^2}.$$

2. Time variation.

$$\frac{d}{dt}\mathcal{E} \leqslant -2\mu\mathcal{E} + \frac{1}{\mu} \|\vec{\varepsilon}\|_{H^1 \times L^2} \left[\|\vec{\varepsilon}\|^2_{H^1 \times L^2} + \sum_{k=1}^{K} |\ell_k|^2 + \sum_{k=1}^{K-1} e^{-(z_{k+1} - z_k)} \right].$$

This is enough to prove exponential rate of convergence when $K = 0, 1$.

Notation (bis)

Shifted center	$y_k = z_k + \frac{\ell_k}{2\alpha}, \quad r_k = y_{k+1} - y_k$
Distance related	$\mathcal{K}_+ = \{k = 1, \ldots, K-1 : \sigma_k = \sigma_{k+1}\}, \quad F_+ = \sum_{k \in \mathcal{K}_+} e^{-r_k},$
	$\mathcal{K}_- = \{k = 1, \ldots, K-1 : \sigma_k = -\sigma_{k+1}\}, \quad F_- = \sum_{k \in \mathcal{K}_-} e^{-r_k}$
Damped components	$\mathcal{F} = \mathcal{E} + \mathcal{G}, \quad \mathcal{G} = \sum_{k=1}^{K} \lvert\ell_k\rvert^2 + \frac{1}{2\mu} \sum_{k=1}^{K} (a_k^-)^2$
	$\mathcal{N} = \left[\lVert\vec{\varepsilon}\rVert_{H^1 \times L^2}^2 + \sum_{k=1}^{K} \lvert\ell_k\rvert^2 \right]^{\frac{1}{2}}, \quad b = \sum_{k=1}^{K} (a_k^+)^2$
Previous estimates	$\mathcal{N}^2 \lesssim \mathcal{F} + \frac{b}{2\mu} \lesssim \mathcal{N}^2$
in the new variables	$\lvert\dot{b} - 2\nu^+ b\rvert \lesssim \mathcal{N}^3 + \mathcal{N}(F_+ + F_-)$
	$\frac{d}{dt}\mathcal{F} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3 + \mathcal{N}(F_+ + F_-)$

Dynamics of the solitons' centers

$$\dot{y}_k = -\frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} + \frac{\kappa}{2\alpha}\sigma_k\sigma_{k+1}e^{-r_k} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big),$$

so that

$$\dot{r}_k = -\frac{\kappa}{\alpha}\sigma_k\sigma_{k+1}e^{-r_k} + \frac{\kappa}{2\alpha}\sigma_{k+1}\sigma_{k+2}e^{-r_{k+1}} + \frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big).$$

Then

$$\dot{F}_+ = -\sum_{k\in\mathcal{K}_+}\dot{r}_k e^{-r_k} = \frac{\kappa}{2\alpha}S_+ + O\big(F_+(\mathcal{N}^2 + F_+^\theta + F_-^\theta)\big),$$

where

$$S_+ = \sum_{k\in\mathcal{K}_+}\Big[2e^{-2r_k} - \sigma_{k+1}\sigma_{k+2}e^{-(r_k+r_{k+1})} - \sigma_{k-1}\sigma_k e^{-(r_{k-1}+r_k)}\Big]$$

Key observation. $S_+ \geqslant \tilde{\lambda}\sum_{k\in\mathcal{K}_+} e^{-2r_k}$ for some $\tilde{\lambda} > 0$.

Indeed, if $k \in \mathcal{K}_+$: if $k-1 \in \mathcal{K}_-$, then the contribution in S_+ is positive (same if $k+1 \in \mathcal{K}_-$).

So $S_+ \geqslant X_+^T A X_+$ where $X_+ = (e^{-r_k})_{k\in\mathcal{K}_+}$ and $A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$ is definite positive.

(Same for F_-). Then

$$\frac{d}{dt}\left(\frac{1}{F_+}\right) \leqslant -\lambda + \frac{1}{\lambda F_-}(\mathcal{N}^2 + F_+^\theta + F_-^\theta)$$
$$\frac{d}{dt}\left(\frac{1}{F_-}\right) \geqslant \lambda - \frac{1}{\lambda F_-}(\mathcal{N}^2 + F_+^\theta + F_-^\theta).$$

Considering a slight modification of the $F_\pm$, one can turn the $\mathcal{N}^2$ terms around.

Main bootstrap

Proposition
There exists a universal constant $\delta > 0$ such that the following holds. Let $\vec{u}$ be a global solution of (dNLKG) and, let $T_\delta \gg 1$ (depending on $\vec{u}$) be such that $\vec{u}$ admits a decomposition in a neighborhood of T_δ, with

$$\mathcal{N}(T_\delta)^2 + b(T_\delta) + F_+(T_\delta) + F_-(T_\delta) \leqslant \delta^2.$$

Then, for all $t \geqslant T_\delta$, it holds

$$\mathcal{N}(t)^2 + b(t) + F_+(t) + F_-(t) \lesssim \delta^2, \quad \text{and} \quad F_-(t) \leqslant 2\left(\delta^{-2} + \lambda(t - T_\delta)\right)^{-1}.$$

Lemma (Expansion of the energy)
For times on which $\vec{u}$ admits a decomposition, one has (for some $c_1 > 0$ explicit constant):

$$E(\vec{u}) = KE(Q,0) - c_1\kappa F_+ + c_1\kappa F_- + O(\mathcal{N}^2 + F_+^\theta + F_-^\theta),$$

$$2\alpha \int_t^\infty \|\partial_t u(s)\|_{L^2}^2 ds = c_1 F_-(t) - c_1\kappa F_+(t)) + O(\mathcal{N}^2(t) + F_+^\theta(t) + F_-^\theta(t)).$$

Key point. $\displaystyle\int_{T_\delta}^{+\infty} \|\partial_t u\|_{L^2}^2 ds \leqslant \delta^2.$

Ideas of proof

Let $C \gg 1$ be some constant (independent on δ) to be chosen later.

Consider the maximal time $T_* > T_\delta$ such for all $t \in [T_\delta, T^*]$

$$\mathcal{N} \leqslant C\delta, \quad b \leqslant C\delta^2, \quad F_+ \leqslant \delta^{3/2}, \quad F_- \leqslant \delta^{3/2}. \tag{BS}$$

Goal: show that $T_* = +\infty$ by improving all the estimates in (BS).

In the following proof, the implied constants in $\lesssim$ are independent of δ, T_* and C.

Estimate on $\mathcal{F}, \mathcal{G}$

Integrating $\dot{\mathcal{F}} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3 + \mathcal{N}(F_+ + F_-)$ yields $\mathcal{F} \lesssim C^3\delta^3 + C\delta^{5/2} \leqslant \delta^2$ if δ is small enough. Same for $\mathcal{G}$.

Estimate on $\mathcal{N}$ Coercivity gives $\mathcal{N}^2 \lesssim \mathcal{F} + b \lesssim C\delta^2$ so that $\mathcal{N} \lesssim \sqrt{C}\delta$.

Estimate on b We prove that $b = \sum_k (a_k^+)^2 \leqslant C\delta^2/2$.

By contradiction: assume that $b < C\delta^2/2$ on $[T_\delta, t_2[$ and $b(t_2) = C\delta^2/2$.

Let $t_1 \in [T_\delta, t_2[$ such that $b(t_1) = C\delta^2/4$ and $b > C\delta^2/4$ on $]t_1, t_2]$.

On $[t_1, t_2]$ there holds $\dfrac{\dot b}{b} = 2\nu^+ + O(C^2\delta + \delta^{1/2})$. Integrating yields $t_2 - t_1 = \dfrac{\ln 2}{2\nu^+} + O(\delta^{1/2})$.

Therefore $\displaystyle\int_{t_1}^{t_2} b(s)ds \gtrsim C\delta^2$.

On the other hand, $\displaystyle\int_{t_1}^{t_2} \|\eta\|_{L^2}^2 ds \lesssim \int_{t_1}^{t_2} (\|\partial_t u\|_{L^2}^2 + \mathcal{G})ds \lesssim \delta^2$ and

$$a_k^+ = \frac{\zeta^+}{\zeta^-} a_k^- + \frac{\zeta^- - \zeta^+}{\zeta^-}\langle \eta, Y_k \rangle,$$

so that $\displaystyle\int_{t_1}^{t_2} b(s)ds \lesssim \int_{t_1}^{t_2} (\mathcal{G} + \|\eta\|_{L^2}^2)ds \lesssim \delta^2$, a contradiction for C large.

Estimate on F_-

The expansion on the energy gives $F_+ \leqslant F_- + O(\mathcal{N}^2 + F_-^\theta + F_+^\theta)$ so that $F_+^\theta \leqslant \mathcal{N}^2 + O(F_-^\theta)$

$$\frac{d}{dt}\left(\frac{1}{F_-}\right) \geqslant \lambda + \frac{1}{\lambda F_-}(\mathcal{N}^2 - F_+^\theta - F_-^\theta) \geqslant \lambda - O(F_-^{\theta-1}) \geqslant \frac{\lambda}{2}.$$

In particular, F_- is decreasing and integrating, we get $F_-(t) \leqslant 2\left(\delta^{-2} + \lambda(t - T_\delta)\right)^{-1}$.

Estimate on F_+

We use again $F_+ \leqslant F_- + O(\mathcal{N}^2) \lesssim C\delta^2 \leqslant \delta^{3/2}/2$.

All estimates have been improved: therefore $T_* = +\infty$.

Consequences

Proposition (Alternating solitons)

If $K \geqslant 2$, then $\sigma_k = -\sigma_{k+1}$ for all $k = 1, \dots, K-1$, that is $F_+ = 0$.

Idea of proof. By contradiction: assume that K_+ is not empty, so that $F_+ > 0$. The evolution on F_+ now writes

$$\frac{d}{dt}\left(\frac{1}{F_+}\right) \leqslant -\lambda + C\left(F_+^{\theta-1} + \frac{1}{t^\theta}\frac{1}{F_+}\right) \leqslant -\frac{\lambda}{2} + \frac{1}{t^\theta}\frac{1}{F_+}.$$

Integrating yields

$$\frac{1}{F_+} \leqslant C - \frac{\lambda}{2} t \to -\infty \quad \text{absurd.}$$

One can then refine of the argument in the bootstrap (specifically on b), to derive:

Proposition (Convergence to the sum of solitons)

There exists $T > 0$ such that the decomposition of $\vec{u}$ satisfies, for all $t \geqslant T$

$$F_-(t) \lesssim t^{-1}, \quad \mathcal{N}(t) \lesssim t^{-1}.$$

Evolution of the solitons' centers

$$\dot{y}_k = \frac{\kappa}{2\alpha}\left(e^{-(y_k - y_{k-1})} - e^{-(y_{k+1} - y_k)}\right) + O\left(t^{-\theta}\right).$$

Explicit solution to the unperturbed ODE system:

$$\bar{y}_k(t) := \left(k - \frac{K+1}{2}\right)\log t + \tau_k, \quad \text{for } k = 1, \dots, K,$$

$$\sum_{k=1}^{K} \tau_k = 0, \quad e^{-(\tau_{k+1} - \tau_k)} = \frac{2\alpha}{\kappa}\gamma_k \quad \text{where} \quad \gamma_k = \frac{k(K-k)}{2}.$$

Observe that $\frac{1}{K}\sum_{k=1}^{K} y_k = \bar{y}_\infty + O(t^{-\theta+1})$, denote

$$\xi_k = y_k - \bar{y}_k - \bar{y}_\infty, \quad \boldsymbol{\xi} = (\xi_1, \dots, \xi_k).$$

Proposition
$\boldsymbol{\xi}$ is bounded: for all $t \geqslant T$, $|\boldsymbol{\xi}(t)| \leqslant M$. ($|\cdot|$ euclidian norm on $\mathbb{R}^K$)

Follows from convexity type arguments.

Evolution of the solitons' centers (continued)

$$\dot{\xi} = t^{-1}\Phi(\xi) + O(t^{-\theta}).$$

1. $|\xi(t)| \leqslant M$
2. The unperturbed system admits a Lyapunov functional up to fixing the center of mass: all solutions with center 0 converge to 0.
3. $D\Phi(0)\boldsymbol{e}_1 = 0$ and
$$\forall \zeta \in \boldsymbol{e}_1^{\perp}, \quad (D\Phi(0)\zeta, \zeta) \leqslant -|\zeta|^2.$$
4. $(\xi, \boldsymbol{e}_1) = O(t^{1-\theta})$.

Fix $\varepsilon > 0$ small so that $\Phi \sim D\Phi(0)$ on $B(0, \varepsilon)$.

By continuity of the flow, there exist t_1 such that $|\xi(t_1)| < \varepsilon$. Then a Gronwall argument yields

$$\xi(t) = O(t^{1-\theta}).$$

Lemma (Modulation of initial data)
Given $(z_k, \ell_k)_{k=1,\dots,K} \in \mathbb{R}^{2K}$ such that $r = \min(z_{k+1} - z_k, k = 1, \dots, K-1)$ is large enough, there exist linear maps $B, V \in \mathcal{M}_K(\mathbb{R})$ such that

$$\|B - \beta\, Id\,\| \lesssim e^{-\frac{1}{2}r}, \quad \|V_j\| \lesssim e^{-\frac{1}{2}r}, \quad \textit{where} \quad \beta := \frac{1}{2\sqrt{\alpha^2 + \nu_0^2}},$$

and such that the function

$$W : \mathbb{R}^{k+1} \to \mathbb{R}, \quad (\boldsymbol{a}, x) \mapsto \sum_{k=1}^{K} \Big(B_k(\boldsymbol{a}) Y_k(x) + V_k(\boldsymbol{a}) \partial_x Q_k(x) \Big)$$

satisfies $\langle W(\boldsymbol{a}), \partial_x Q_k \rangle = 0$, $\langle W(\boldsymbol{a}), Y_k \rangle = \beta a_k$ for all k.

Thus, setting $\vec{W}(\boldsymbol{a}) = (W(\boldsymbol{a}), \nu^+ W(\boldsymbol{a}))$, there hold $\langle \vec{W}(\boldsymbol{a}), \vec{Z}_k^+ \rangle = a_k$.

Proposition
Fix $\delta > 0$ small enough.

Given $(z_k(0))_{k=1,\dots,K}$ with $r(0) \geqslant 5\ln(\delta)$, $(\ell_k(0))_{k=1,\dots,K} \in B(0,\delta)$, $\vec{\varepsilon}_\perp(0) \in B_{H^1\times L^2}(\delta)$ such that

$$\forall k = 1, \dots, K, \quad \langle \varepsilon_\perp(0), \partial_x Q_k \rangle = \langle \eta_\perp(0), \partial_x Q_k \rangle = \langle \vec{\varepsilon}_\perp(0), \vec{Z}_k^+ \rangle = 0,$$

there exists $\boldsymbol{a}_*^+(0) \in B(0, \delta^{5/4})$ such that the solution u_* to (dNLKG) with initial data

$$\vec{u}_*(0) = \sum_{k=1}^{K} (-1)^k (Q(\cdot - z_k(0)), 0) + W(\boldsymbol{a}_*^+(0)) + \vec{\varepsilon}_\perp(0) \text{ is a } K\text{-soliton}.$$

To fix ℓ, apply a space shift.

Proof. For any $\boldsymbol{a}^+(0) \in B(0, \delta^{5/4})$, consider the solution $u = u_{a^+}$ to (dNLKG) with initial data

$$\vec{u}(0) = \sum_{k=1}^{K} (-1)^k (Q(\cdot - z_k(0)), 0) + W(\boldsymbol{a}^+(0)) + \vec{\varepsilon}_\perp(0).$$

Decompose u as before, then $\mathcal{N}(0), F_-(0) \lesssim \delta^2$, $b(0) = |\boldsymbol{a}^+(0)|$.

Boostrap: $\mathcal{N} \leqslant \delta^{3/4}$, $F_- \leqslant \delta^{3/2}$, $b(0) \leqslant \delta^{5/2}$. Denote the exit time $T_* = T_*(\boldsymbol{a}^+(0)) \geqslant 0$.

The analysis performed before shows that

- $\mathcal{N} \lesssim \delta \rightarrow$ it bootstraps.
- $F_- \lesssim \delta^2 \rightarrow$ it bootstraps.

Therefore, if $T_* < +\infty$, then $b(T_*) = \delta^{5/2}$, and we have transversality:

$$\dot{b}(T_*) \geqslant 2\nu^+ b - O(\delta^3) \geqslant \nu^+ \delta^{5/2} > 0.$$

Contradiction argument. Assume that for all $\boldsymbol{a}^+(0) \in B(0, \delta^{5/4})$, $T_*(\boldsymbol{a}^+(0) < +\infty$. Consider the map

$$\Phi : B_{\mathbb{R}^K}(0, \delta^{5/4}) \rightarrow \mathbb{S}_{\mathbb{R}^K}(0, \delta^{5/2}), \quad \boldsymbol{a}^+(0) \mapsto \boldsymbol{a}^+(T_*((\boldsymbol{a}^+(0)).$$

Then, by transversality

- Φ is continuous.
- If $|\boldsymbol{a}^+(0)| = \delta^{5/4}$, $T_* = 0$ and $\Phi(\boldsymbol{a}^+(0)) = \boldsymbol{a}^+(0)$.

This contradicts the no-retractation theorem.

Hence, there exists $\boldsymbol{a}_*^+(0) \in B(0, \delta^{5/4})$ such that $T_* = +\infty$. $\square$