Internal modes and Radiation damping for quadratic KG in 3d

Tristan Léger

Princeton University

(joint with Fabio Pusateri)

SITE talk series

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Outline

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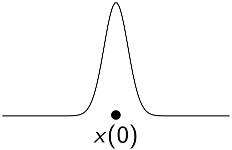

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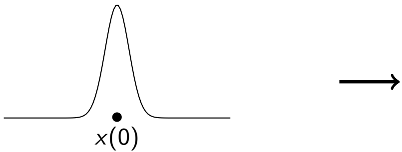

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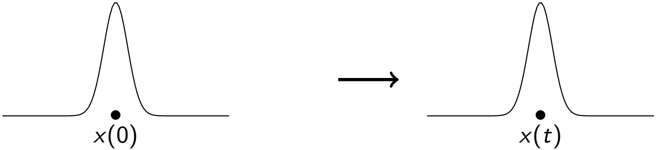

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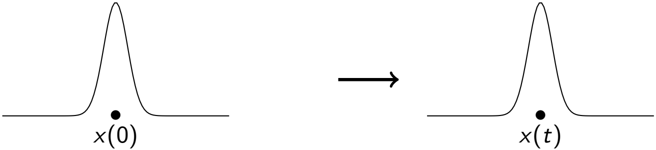

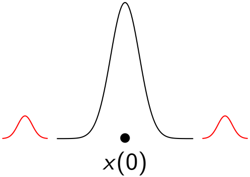

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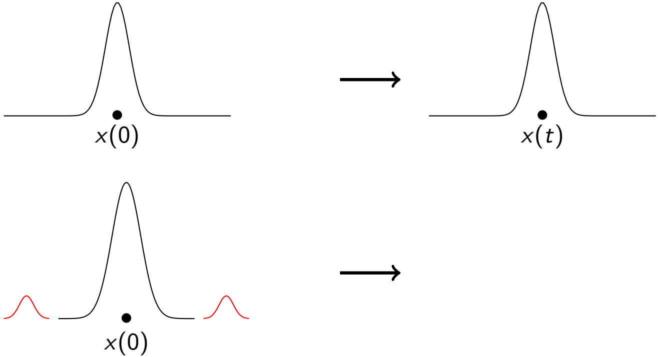

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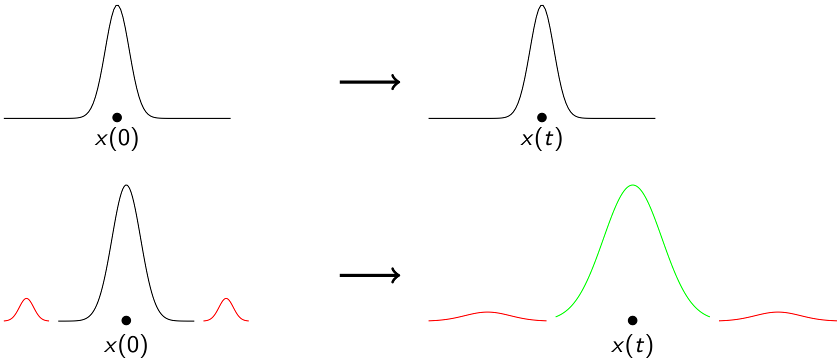

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- We assume $2\lambda > m$ and a 'Fermi Golden rule'

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- More references: Sigal (‘90s); Soffer-Weinstein ($<$ ‘99) on quantum resonance; Tsai-Yau (‘01) on NLS; Bambusi-Cuccagna (‘11) for multiple i.m.; Komech-Kopylova (‘12) in 1d for large powers . . .

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- And how does the full nonlinear solution behave?

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$$|a(t)| \approx \varepsilon(1+\varepsilon^2 t)^{-1/2}, \qquad \|v(t)\|_{L^\infty} \approx \langle t \rangle^{-1},$$
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- The i.m. forces at leading order the radiation equation

- Pass to first order variables

$$A(t) := \frac{1}{2i\lambda} e^{-i\lambda t}(\dot{a} + i\lambda a), \qquad w := (\partial_t + iL)v, \quad L := \sqrt{H+1},$$

$$\begin{cases} \dot{A} & = & -2ie^{-it2\lambda}\overline{A(t)} \int \phi^2 \operatorname{Im} w(t)\, dx + O(A^2)(t) \int \phi^3 + \cdots \\ \partial_t w - iLw & = & e^{it2\lambda} A(t)^2 \mathrm{P_c}(\phi^2) + \cdots \end{cases}$$

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…for some $c_0, c_1 \in \mathbb{R}$,

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- To separate these dynamics from remainders need non-trivial bounds, e.g.

$$\left\| \int_0^t e^{i(t-s)L} w^2 \, ds \right\|_{L^\infty} \lesssim \langle t \rangle^{-(1+)}.$$

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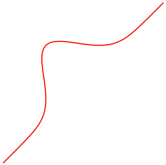

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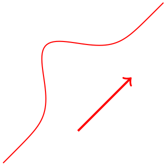

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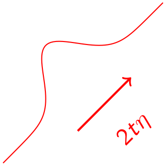

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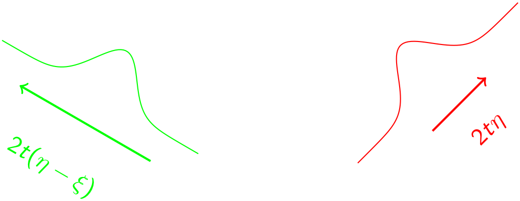

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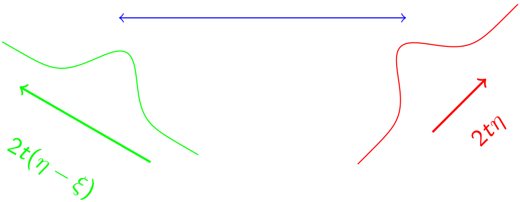

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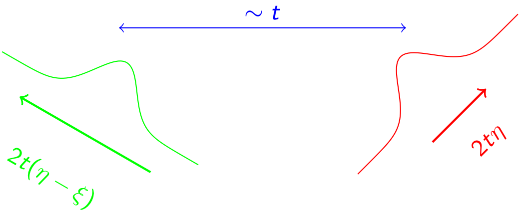

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(Weyl-Kodaira-Titchmarsh, Ikebe, Alsholm-Schmidt, Agmon ...)

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  $d = 1$: Deift-Trubowitz ('80s), Weder ($\sim$ '00) ...

  $d \geq 2$: Agmon, Kato, Kuroda (- '80s), Yajima ('90s), Weder ('03), Beceanu-Schlag ('10s) ...

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  For $V \neq 0$ do not have frequencies can interact more freely.

Nonlinear Spectral Distribution (NSD) and oscillations

$$\widetilde{f}(T,k) - \widetilde{f}(t,0) = -i \int_0^T \iint e^{it(-\langle \xi \rangle + \langle \eta \rangle + \langle \sigma \rangle)} \widetilde{f}(t,\eta) \widetilde{f}(t,\sigma)\, \mu(\xi,\eta,\sigma)\, \mathrm{d}\eta \mathrm{d}\sigma \mathrm{d}t$$

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- Results of this type in Germain-Hani-Walsh '15 for $i\partial_t u + \Delta u + Vu = \overline{u}^2$.

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- Idea: Introduce a decomposition of the spectral measure μ depending on whether we are close or not to the singularity of the kernel, then prove mutlilinear lemmas for each piece.

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- Key new difficulty: Dealing with discrete-continuous interactions.

Final comments

- We are able to deal with an internal mode for quadratic KG in 3d and show that radiation damping still occurs by:
  - Separating damping dynamics from the rest in the ODE for i.m.
  - Control quadratic KG with a potential and singular (frequency) forcing

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- More general take away:

  Can do robust nonlinear Fourier analysis in the presence of a potential.
- In $d = 1$ there has been a lot of recent activity on related problems of PDEs with potentials and low powers, and the stability of solitons / kinks
  - NLS: Germain-P.-Rousset, Naumkin, Delort, Chen-P., Chen ...
  - ϕ^4 and 1d field theories: Kowalczyk-Martel-Munoz, KMM-Van de Bosch, Delort-Masmoudi (i.m. control up to ε^{-4}) ...
  - KG / Sine-G: Lindblad-Luhrmann-Soffer, Germain-P., L-L-S-Schalg, Luhrmann-Schlag ...

Thank You for your attention!