Soliton resolutim for equivariant wave maps

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Wave maps: generalize free waves to manifold-valued maps.

$$\psi : \mathbb{R}^{1+d}_{t,x} \longrightarrow (\mathcal{M}, g) \hookrightarrow (\mathbb{R}^N, \langle \cdot , \cdot \rangle_{\mathbb{R}^N})$$

- Lagrangian: $\mathcal{L}(\psi) = \frac{1}{2} \int_{\mathbb{R}^{1+2}_{t,x}} \left(-|\partial_t \psi|_g^2 + |\nabla \psi|_g^2 \right) dx\, dt$

- Euler-Lagrange eqn: $\partial_t^2 \psi - \Delta \psi \perp T_\psi \mathcal{M}$

$$\rightsquigarrow \boxed{\partial_t^2 \psi - \Delta \psi = S(\psi)(\partial^\alpha \psi, \partial_\alpha \psi)}$$

2nd Fund. form

* key pt: • nonlinearity $\longleftrightarrow$ geometry of $\mathcal{M}$

Canonical example: $d=2$, $\mathcal{M}=\mathbb{S}^2$

$$(*)\quad \begin{cases} \partial_t^2 \psi - \Delta\psi = \psi\left(|\nabla\psi|^2 - |\partial_t\psi|^2\right) \\ \psi|_{t=0} = \psi_0 : \mathbb{R}^2 \longrightarrow \mathbb{S}^2 \,;\quad \partial_t\psi|_{t=0} = \dot{\psi}_0 : \mathbb{R}^2 \longrightarrow T_{\psi_0}\mathbb{S}^2 \end{cases}$$

90's - 00's : Shatah-Tahvildar-Zadeh; Shatah-Struwe; Grillakis
Klainerman-Machedon/Selberg;

$\rightsquigarrow$ Thm: (Tataru; Tao '01) smooth $(\psi_0, \dot{\psi}_0)$ w/

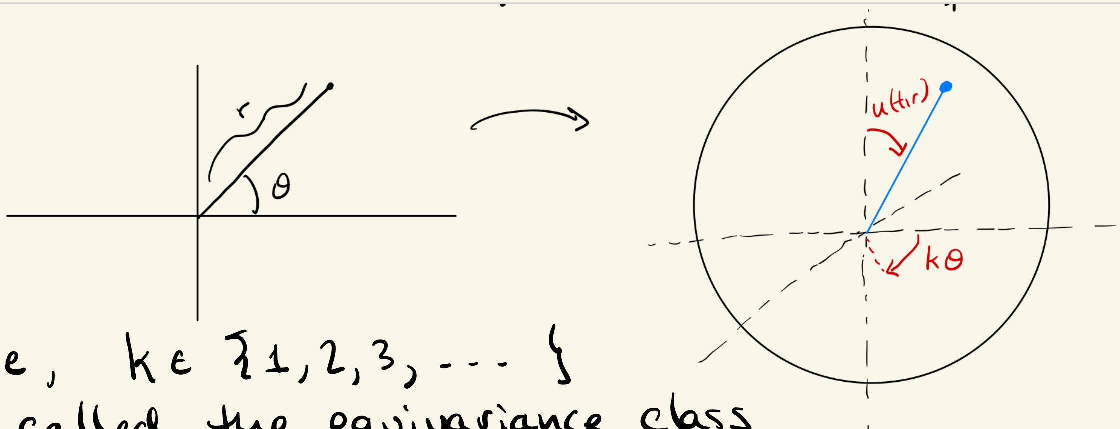
$\mathcal{E}(\psi_0, \dot{\psi}_0) \ll 1 \Rightarrow (*)$ admits global-in-time smooth evolution $\psi(t)$ that disperses

$$\mathcal{E}(\psi, \partial_t\psi) = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_t\psi|_g^2 + |\nabla\psi|_g^2\, dx$$

Equivariant wave maps

- We study the dynamics (long-time behavior) of large solutions, but only in a special case:

$$\psi(t, r\cos\theta, r\sin\theta) = \left(\sin u(t,r)\cos k\theta,\ \sin u(t,r)\sin k\theta,\ \cos u(t,r)\right)$$



- Here, $k \in \{1, 2, 3, \dots\}$ is called the equivariance class

- Reduced Euler-Lagrange eqn: $u = u(t,r)$, $k \in \{1,2,3,\dots\}$

$$(WM_k) \quad \boxed{\partial_t^2 u - \Delta u + \frac{k^2}{r^2}\frac{\sin 2u}{2} = 0}$$

- Energy:

$$E(\vec{u}) = \pi \int_0^\infty \left[(\partial_t u)^2 + (\partial_r u)^2 + \frac{k^2}{r^2}\sin^2 u \right] r\,dr$$

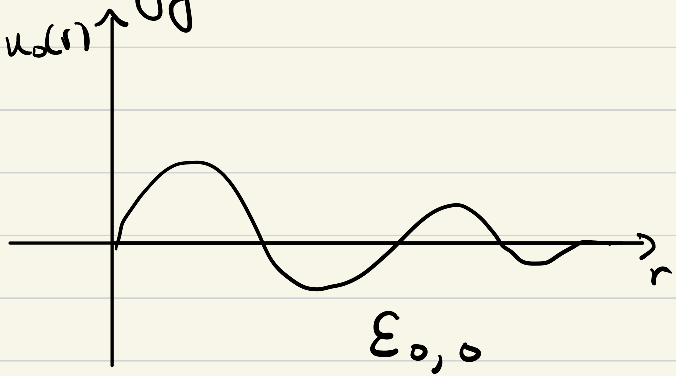
- Notation: $\vec{u}(t) = (u(t), \partial_t u(t))$

- Scaling: $\lambda > 0$; $\vec{u}$ a solution, then so is

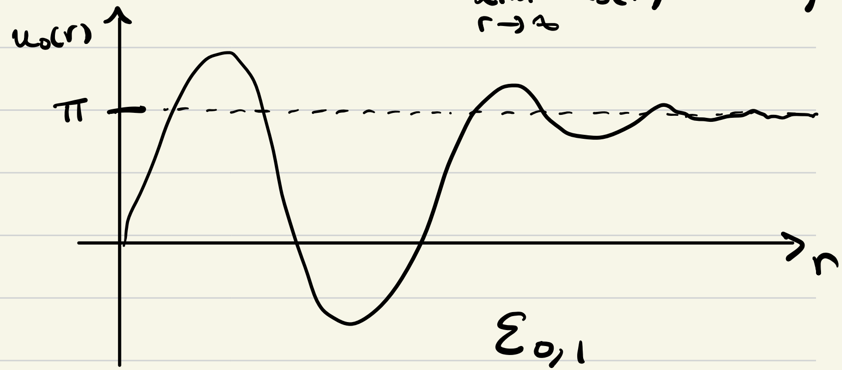
$$\vec{u}_\lambda(t,r) = \left(u(t/\lambda, r/\lambda), \frac{1}{\lambda}\partial_t u\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right)\right)$$

- Energy criticality: $E(\vec{u}_\lambda) = E(\vec{u})$

- Energy sectors: $\mathcal{E}_{m,n} := \left\{ \vec{u}_0 : E(\vec{u}_0) < \infty, \ \lim_{r\to 0} u_0(r) = m\pi, \ \lim_{r\to\infty} u_0(r) = n\pi \right\}$



$\mathcal{E}_{0,0}$



$\mathcal{E}_{0,1}$

<u>Thm</u>: (Shatah, Tahvildar-Zadeh; Shatah, Struwe 90's)

- (WM$_k$) is locally well-posed in each $\mathcal{E}_{m,n}$, $\forall k$
- If $\vec{u}_0 \in \mathcal{E}_{m,m}$ and $E(\vec{u}_0) \ll 1$ then $\vec{u}(t)$ is global-in-time and scatters; i.e., $\exists$ linear wave $\vec{u}_L^*(t)$ s.t.

$$\| \vec{u}(t) - (m\pi + u_L^*(t), \partial_t u_L^*(t)) \|_{\mathcal{E}} \longrightarrow 0 \quad \text{as } t \to \infty$$

Harmonic maps

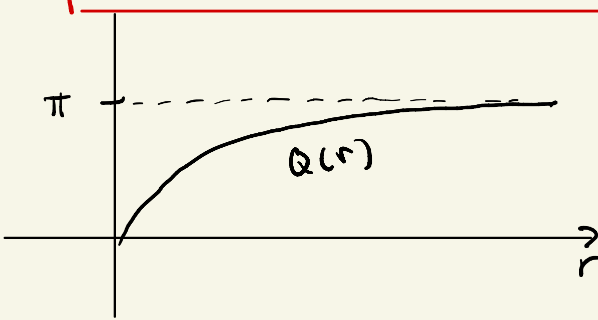
• Stationary solutions (solitons): $Q(r) := 2\arctan(r^k)$.

* on $\mathcal{E}_{m,m}$ $\rightsquigarrow$ $m\pi$
* on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow$ $m\pi + Q_\lambda(r)$
* on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow$ $m\pi - Q_\lambda(r)$

* Critical pts of potential energy

Q = polar angle of deg $= k$ Harmonic map.

• $\boxed{Q_\lambda(r) := Q(r/\lambda)}$, $\lambda > 0$ $\longleftarrow$ 1-parameter family for each k



* key role in dynamics of large solutions

Comments

1. $Q(r) = 2\arctan(r^k)$ is an example of a topological soliton.

- Q is polar angle of $\deg = k$ Harmonic map

$$\mathbb{C}_\infty \ni z \longmapsto z^k \in \mathbb{C}_\infty \quad \text{(stereographic projection)}$$

- other examples:

  1d - kinks in classical scalar field models

  2d - Ginzburg-Landau Vortices; Harmonic maps

  3d - Magnetic Monopoles; Skyrmions

  4d - Yang Mills Instantons

- math physics: particle-like objects (position, momentum, mass, energy.)
  - w/ discrete top. invariant.

2. (WM_k) : $\partial_t^2 u - \Delta u + \frac{k^2}{r^2} \frac{\sin 2u}{2} = 0$

is an example of a nonlinear dispersive PDE that is energy critical.

ex. (NLW) $\partial_t^2 u - \Delta u - |u|^{\frac{4}{D-2}} u = 0$, $u = u(t, x)$
$x \in \mathbb{R}^d$

(NLS) $i \partial_t u + \Delta u + |u|^{\frac{4}{D-2}} u = 0$

- good candidates to understand dynamics of large solutions since conserved energy gives meaningful notion of "size"

Main questions

Obs: (WM_k) admits Hamiltonian formulation:

$$\vec{u} = (u, \partial_t u) \quad \text{solves}$$

$$(WM_k) \iff \partial_t \vec{u} = J \circ DE(\vec{u})$$

Question: Describe the asymptotic dynamics of all solutions. ω-limit sets? phase portrait?

* why is this <u>not</u> totally hopeless? Dispersion

Soliton Resolution Conjecture

Loosely: Every solution to "such" dispersive PDEs asymptotically resolves into:

- a superposition of decoupled solitons (coherent traveling waves) plus a radiative term (eg. a linear wave)

* conjecture inspired by
  1. numerical simulations (Zabusky-Kruskal)
  2. theory of completely integrable systems
  3. bubbling theory in parabolic setting (e.g. harmonic map flow)

Theorem (Soliton Resolution; Jendrej-L. '21)

Let $\vec{u}(t)$ be any finite energy wave map, $k \in \{1,2,3,\dots\}$

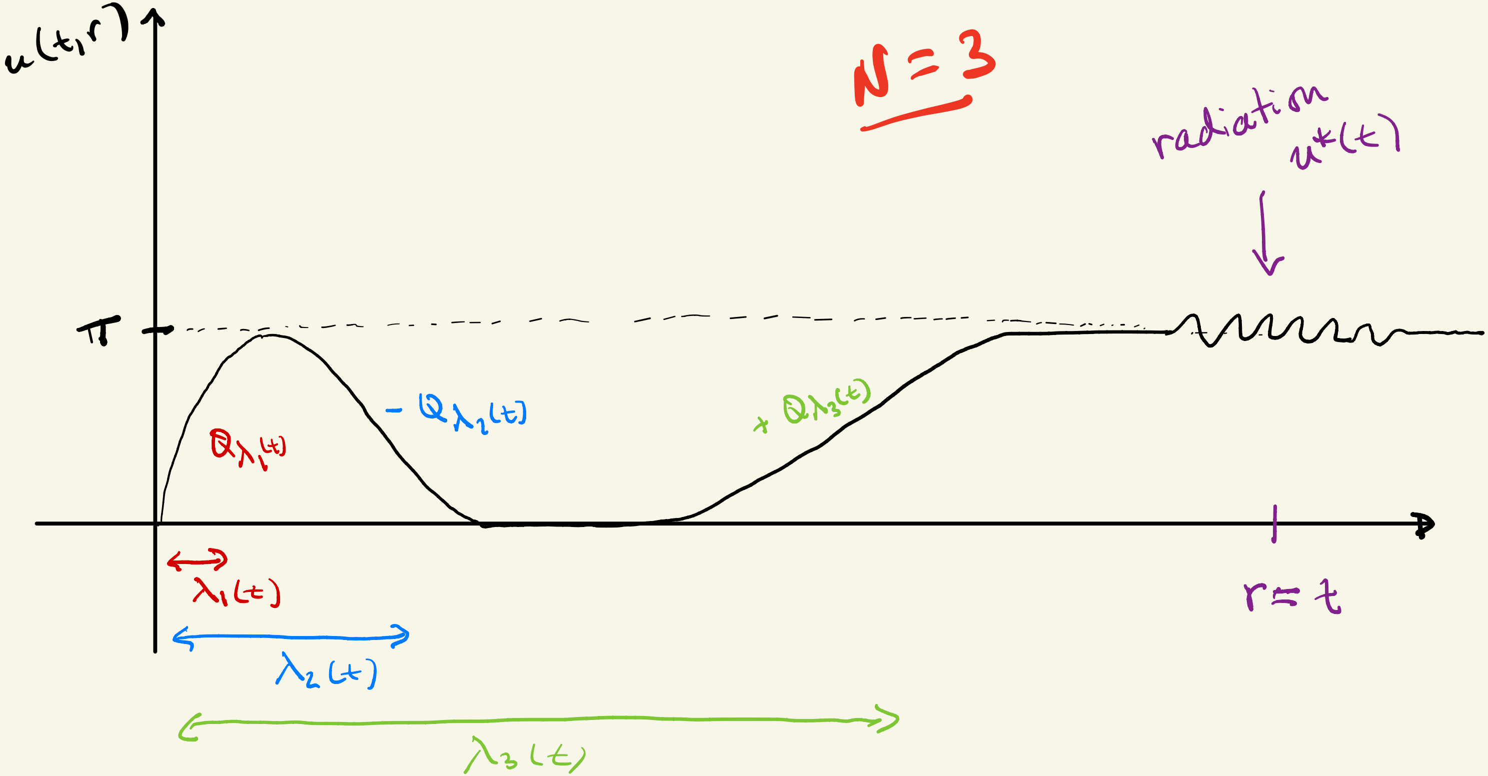
1. (Blow-up; $T_+ < \infty$) $\exists\, m, \ell \in \mathbb{Z}$, $N \geq 1$, $\vec{\iota} \in \{-1,1\}^N$, scales $0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll T_+ - t$, radiation $\vec{u}^*_0 \in \mathcal{E}_{0,\ell}$ st.

$$\vec{u}(t) = m\pi + \sum_{j=1}^{N} \iota_j \left(\vec{Q}_{\lambda_j(t)} - \pi \right) + \vec{u}^*_0 + o_{\mathcal{E}}(1) \text{ as } t \to T_+$$

2. (Global sol'n; $T_+ = \infty$) $\exists\, m \in \mathbb{Z}$, $N \geq 0$, $\vec{\iota} \in \{-1,1\}^N$, scales $0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll t$, and a linear wave $\vec{u}^*_L(t) \in \mathcal{E}_{0,0}$ st.

$$\vec{u}(t) = m\pi + \sum_{j=1}^{N} \iota_j \left(Q_{\lambda_j(t)} - \pi \right) + \vec{u}^*_L(t) + o_{\mathcal{E}}(1) \text{ as } t \to \infty$$

Schematic picture



Existence of Multi-bubbles?

- Depends on equivariance class $k \in \{1, 2, 3, \ldots\}$

  - $\underline{k \geq 2}$: Jendrej '16 ; Jendrej - L. '18 - '20
    constructed + studied "2-bubble" solutions $(N=2)$
    (these are global-in-time)

  - $\underline{k = 1}$: <u>No</u> solutions w/ $N > 1$ known to exist. (why?)
    - Rodriguez '18 (no 2-bubbles)

  - <u>NLW</u> : - known w/ $N = 2$ in dim $d = 6$ (Jendrej '16)
    - dim 3, 4, 5 : not known to exist

- $k = 1, 2, 3, \ldots$: Blow-up solutions w/ $N = 1$ bubble
  Krieger - Schlag - Tataru ; Rodnianski - Sterbenz ; Raphaël - Rodnianski

- Blow-up w/ $N \geq 2$ bubbles? unknown.

Timeline of related results

1. (2010) Duyckaerts, Kenig, Merle: proved soliton resolution for (NLW) in dim $D=3$, radial, along a sequence of times $t_n \to T_+$
   - 1st result treating possibility of many solitons
   - ideas here crucial to our proof.
   - ~(2016) D-Jia-K-M - NLW → non-radial
2. (2012) Duyckaerts, Kenig, Merle: full soliton resolution conjecture for critical NLW, radial, $D=3$

$$\partial_t^2 u - \Delta u = u^5$$

* 1st result of this type for any eqn that is not completely integrable
* method of ENERGY CHANNELS

3. (2018) Jendrej - L.: Classified all equivariant wave maps w/ $E \leq 2E(Q)$, $k \geq 2$.
1st result of this type in region of phase space where existence of multi-solitons was confirmed (by Jendrej '16).

* modulation analysis: size of soliton-soliton interaction is crucially used.

4. (2019) Duyckaerts, Kenig, Merle: full resolution for radial NLW in ODD space dimensions $D \geq 5$

5. (2021). Duyckaerts, Kenig, Martel, Merle: soliton resolution for equivariant wave maps w/ $k = 1$

- method of energy channels
- also treats NLW, $D = 4$

6. (2021). our contribution: different method of proof

- no-return analysis – size of inter-soliton interaction is crucial.
  - unified approach: can handle all equivariance classes $k \geq 1$

7. (2022) Collot, Duyckaerts, Kenig, Merle: NLW in $D=6$

No elastic collisions

<u>Def</u>: a sol'n $\vec{u}(t)$ is a pure multi-bubble in forward time if $\vec{u}^* \equiv 0$ (no radiation); ie

$$u(t) = m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} + o_\varepsilon(1) \quad \text{as } t \to \infty$$

<u>Cor</u>: (cor of pf of Thm) $\nexists$ any sol'n $\vec{u}(t)$ that is a pure N-bubble in both forward and backward time if $N > 1$.

+ any "collision" emits radiation.

<u>Thm</u> (Jendrej '16; Jendrej-L. '18-20) $k \geq 2$, $\exists!$ pure 2-bubble $\vec{u}_{(2)}(t)$ in forward time. And, $\vec{u}_{(2)}(t)$ scatters freely as $t \to -\infty$ (totally inelastic collision).

* contrast w/ completely integrable eqn's. (KdV, sine-Gordon)

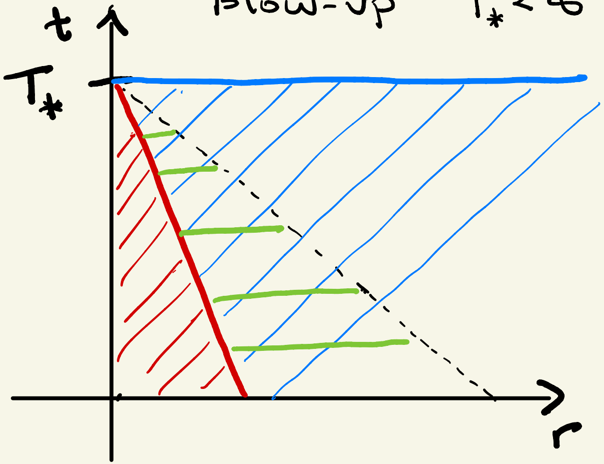
Outline of proof

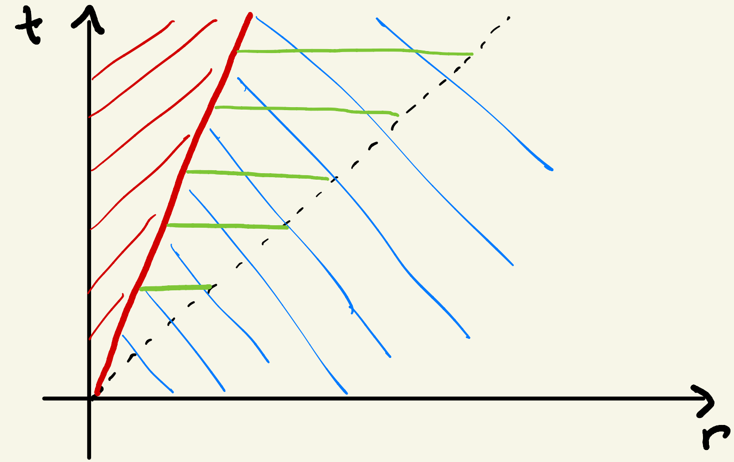
1) Identify radiation $u^*(t)$ continuously in time
   - Christodoulou, Shatah, Tahvildar-Zadeh - 90's
   Duyckaerts, Kenig, Merle, Côte, Kenig, L. Schlag

2) Show resolution holds along at least one sequence of times $t_n \longrightarrow T_*$
   - Côte, Kenig, L., Schlag '12, Côte '14, Jia-Kenig '15
   * use fundamental technique of Duyckaerts, Kenig, Merle
     - NLW ~ '10
   - method of profile decompositions (Bahouri-Gérard '98)
     (concentration compactness)

3) Show resolution holds continuously in time
   - our contribution

Radiation $u^*(t)$





Shatah, Tahvildar-Zadeh: $E(\vec{u}(t); \text{—}) \longrightarrow 0$ as $t \to T_*$

- convergence in continuous time in Blue region

$$\lim_{t \to T_*} E(\vec{u}(t); r \text{ in Blue}) = E(\vec{u}^*)$$

Sequential Decomposition

- Multi-bubble configuration: $\vec{\iota} \in \{-1, 1\}^N$, $\vec{\lambda} \in (0, \infty)^N$

$$\vec{Q}(m, \vec{\iota}, \vec{\lambda}) := \left(m\pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_j} - \pi), \ 0 \right)$$

- distance to multi-bubble:

$$d(t) := \inf_{\vec{\iota}, \vec{\lambda}} \left(\| \vec{u}(t) - \vec{u}^*(t) - \vec{Q}(m, \vec{\iota}, \vec{\lambda}) \|_{\mathcal{E}}^2 + \sum_{j=1}^{N} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k \right)^{1/2}$$

$$(\lambda_{N+1} := t)$$

<u>Theorem</u>: (Côte '14 $k=1$, Jia-Kenig '15 $k=2$) $\exists$ seq. of times $t_n \longrightarrow T_*$ s.t.

$$d(t_n) \longrightarrow 0 \text{ as } n \to \infty$$

Ideas for Proof of Step 3.

- Goal: show $d(t) \longrightarrow 0$ as $t \longrightarrow T_*$

* we need to rule out following scenario:

$\vec{u}(t)$ close to a multi-bubble $\rightsquigarrow$ $\vec{u}(t)$ far from a multi-bubble $\rightsquigarrow$ $\vec{u}(t)$ close to a multi-bubble

"collision"

t_n "collision (time) interval" t_{n+1}

* we need a "<u>no-return" lemma</u>

- inspired by Nakanishi-Schlag, Krieger-Nakanishi-Schlag, Duyckaerts-Merle: <u>single</u>, <u>linearly unstable</u> soliton

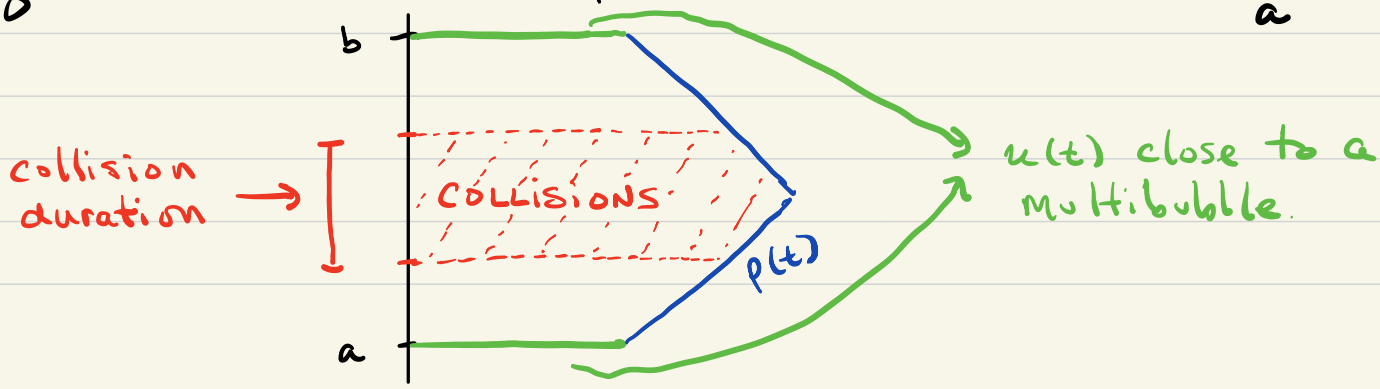
<u>KEY IDEA:</u> No linear instability. USE multi-soliton interactions (Jendrej-L. '18)

Virial identity (as Lyapunov functional)

$$\frac{d}{dt}\langle \partial_t u \mid r\partial_r u \, \chi_{\rho(t)} \rangle_{L^2} = -\int_0^{\rho(t)} |\partial_t u(t)|^2 r\,dr + \text{Error}(\vec{u}, \rho)$$

- integrate btwn times $t=a$, $t=b$

$$\int_a^b \int_0^{\rho(t)} |\partial_t u(t)|^2 r\,dr\,dt \lesssim \rho(a)d(a) + \rho(b)d(b) + \int_a^b (\text{Error})\,dt$$



Expect: $\int_a^b \int_0^{\rho(t)} |\partial_t u(t)|^2 \, r\,dr\,dt \gtrsim$ collision duration

* need "Compactness Lemma"

Delicate issue: collision duration $\gtrsim$ spatial scale $\rho(a), \rho(b)$??

INTERIOR vs. EXTERIOR BUBBLES:

- $K \in \{1, 2, \dots, N\}$, $[a,b] \subset [T_0, T_*)$, $0 < \varepsilon < \eta$

Definition 1: We say $[a,b]$ is a collision interval with N−K exterior bubbles if:

(i) $d(a) \leq \varepsilon$, $d(b) \leq \varepsilon$ and $\exists\, c \in [a,b]$ w/ $d(c) > \eta$

(ii) $\exists$ curve $r = \rho_K(t)$ such that in region $r \geq \rho_K(t)$ $\vec{u}(t)$ is ε-close to $N-K$ bubbles + radiation.

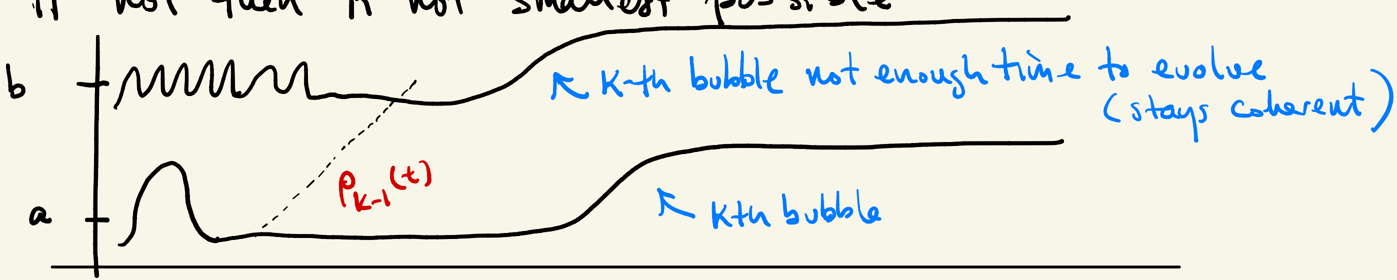
Def 2: Let K be the smallest number such that there exists $\eta > 0$, a sequence $\varepsilon_n \to 0$, and a sequence of collision intervals $[a_n, b_n]$ w/ parameters (ε_n, η, K)

Lemma 1. $\exists\, C = C(\vec{u}) > 0$ and $\varepsilon \geq 0$ s.t. if $[a,b]$ is a collision interval w/ parameters (ε, η, K) then

$$b - a \geq C \max\left(\mu_K(a), \mu_K(b) \right)$$

where μ_K = scale of K-th bubble

pf: if not then K not smallest possible



Ejection Lemma on "modulation intervals"

- Error in virial ineq satisfies $\quad \text{Error}(t) \lesssim d(t)$

"<u>Lemma</u>" (Ejection) if $d(t)$ starts to grow at t_0
then $\forall \; t^* \geq t_0$

$$\int_{t_0}^{t^*} d(t)\,dt \leq C_0 \, d(t^*)^{2/k} \mu_k(t^*) \qquad (k \geq 2)$$

$$\int_{t_0}^{t^*} d(t)\,dt \leq C_0 \, d(t^*)^2 \left(-\log d(t^*)\right)^{1/2} \mu_k(t^*) \quad k=1$$

and $$\tfrac{1}{2}\mu_k(t^*) \leq \mu_k(t) \leq 2\mu_k(t^*)$$
$$\forall \; t \in [t_0, t^*]$$

- $\mu_k(t)$ = scale of largest "interior" bubble.

Modulation

- If $\vec{u}(t) - \vec{u}^*(t)$ is close to an N-bubble
  $\Rightarrow$ dynamics dominated by non-linear inter-soliton interactions

  * key idea: collision analysis of multi-solitons to control growth of $d(t)$

* obtain differential inequalities on scales $\lambda_j(t)$

$$\lambda_j''(t) \simeq -\iota_j \iota_{j+1} \omega^2 \frac{\lambda_j(t)^{k-1}}{\lambda_{j+1}(t)^k} + \iota_j \iota_{j-1} \omega^2 \frac{\lambda_{j-1}(t)^{k-1}}{\lambda_j(t)^k}$$

$$d(t) \simeq \sum_{j \in A} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^{k/2} \qquad A := \{ j : \iota_j \iota_{j+1} = -1 \}$$

* in practice challenging to justify this system.

* method of refined modulation parameters; Jendrej-L. '18
  inspired by Raphaël-Szeftel '11 (different context)

  - localized "virial" correction to $\lambda_j'(t)$

$$\beta_j(t) = \lambda_j'(t) + \underbrace{\text{"small correction"}}_{\substack{\text{can be thought} \\ \text{of as "removing} \\ \text{oscillations"}}}$$