Rigid rotations in 3d-Euler: Asymptotic stability of axisymmetric perturbations

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Global solutions for $3d$ Euler

We exhibit a large family of global dynamical solutions to $3d$ (incompressible) Euler ($x \in \mathbb{R}^3$),

$$(\textbf{Euler}) \qquad (\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{U} + \nabla p = 0, \qquad \mathrm{div}(\mathbf{U}) = 0.$$

These are small axisymmetric perturbations of the rigid rotation

$$\mathbf{U} = (-x_2, x_1, 0) + \mathbf{u}(r, z, t).$$

Theorem [Guo-P.-Widmayer]

There exists $\varepsilon > 0$ and a norm X such that axisymmetric initial data $\mathbf{u}(0)$ of size ε lead to a solution of (**Euler**). The solutions scatter linearly and in particular,

$$\|\mathbf{u}(t)\|_{L^\infty} \lesssim 1/t.$$

Euler-Coriolis equation

The rigid rotation

$$\mathbf{U}_\Omega = \Omega(-x_2, x_1, 0), \qquad p_\Omega = \Omega|\mathbf{x}|^2/2,$$

is a steady solution of (**Euler**). Setting $\mathbf{U} = \mathbf{U}_\Omega + \mathbf{u}$ leads to the Euler-Coriolis system

$$(\mathbf{EC}) \qquad (\partial_t + \mathbf{u}\cdot\nabla)\,\mathbf{u} + \Omega\mathbf{e}_3 \times \mathbf{u} + \nabla p = 0, \qquad \mathrm{div}(\mathbf{u}) = 0,$$

and we prove that small axisymmetric initial data for (**EC**) lead to global solutions that disperse.

$\sim$ stability of Couette flow for $2D$ Euler $\mathbf{u}(x,y,t) = (-y,0) + \mathbf{u}(x,y,t)$ [**Bedrossian-Masmoudi**].

Rotating fluid

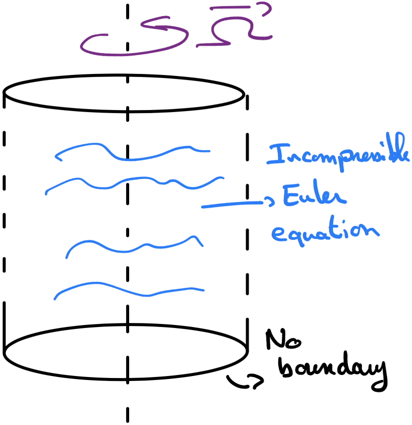

Figure: We consider a fluid rotating uniformly

Relevance of Euler-Coriolis

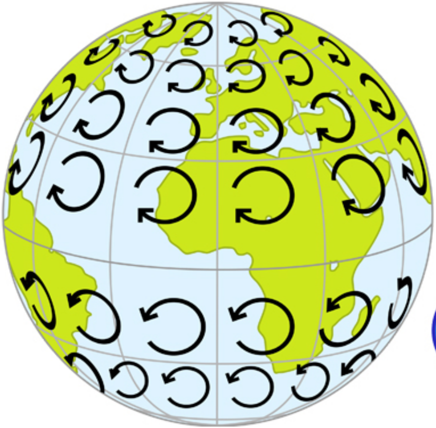

Figure: Image from "The Super Fins"

Many works on Geophysical fluids context

Lots of prior work in many related settings (inviscid, viscous; free boundary, shallow water, boundary layers...),

[Charve, Chemin, Constantin, Dalibard, Desjardins, Dutifroy, Elgindi, Gallagher, Gérard-Varet, Grenier, Ibrahim, Majda, Masmoudi, Pusateri, Saint-Raymond, Schochet, Titi, Widmayer, ...]

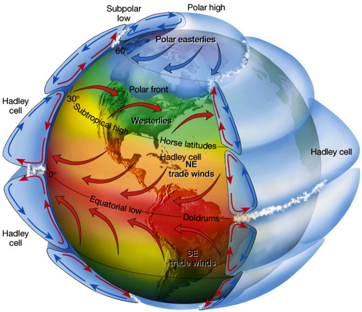

Figure: [Lutgens & Tarbuck]

often

(i) linear models and their associated dispersion relations,

(ii) (limit of) strong forces.

Earlier works on Euler-Coriolis

Connection to two previous results about large rotation $|\Omega| \gg 1$:

1. on a generic torus, [**Babin-Mahalov-Nikolaenko**] (also **Grenier**...) show that general solutions can be decomposed as

$$u(t) = \overline{u}^{2d} + u^{lin} + O(|\Omega|^{-\frac{1}{2}})$$

where u^{lin} solves the Poincaré equation and $\overline{u}^{2d}$ solves a $2d$ Euler equation (normal form).

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2. On $\mathbb{R}^3$, [**Koh-Lee-Takada**] (also **Dutrifoy, Wan-Chen, Angulo-Castillo-Ferreira**...) show long time existence $T \gtrsim \log|\Omega|$ of general, low regularity solutions (Strichartz).

Large rotation

If one considers a large rotation speed: $\Omega = \varepsilon^{-1}$,

$$(\varepsilon\partial_t + \mathbf{k}\times)\, u^\varepsilon + \nabla p^\varepsilon = -\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon, \qquad \mathrm{div}(u^\varepsilon) = 0$$

one obtains a singular dispersive linear operator (in fast time $\tau = \varepsilon^{-1} t$) with a small nonlinearity. [**D, KLT, CW**] obtained time of existence $T \gtrsim \log(1/\varepsilon)$ using the scaling of the Strichartz norms

$$\|e^{it\varepsilon^{-1}\Lambda} f\|_{L^p_t L^q_x} \lesssim \varepsilon^{\frac{1}{p}} \|f\|_X.$$

[**BMN, G**] also obtained long time existence on the Torus using averaging/normal forms which allows to factor out $\partial_t^{-1} = \varepsilon\partial_\tau^{-1}$.

Blow-up and Coriolis

For a related equation (isothermal primitive equations with no viscosity), [**Cao-Ibrahim-Nakanishi-Titi, Collot-I.-Lin, Ghoul-I.-L.-T.**] construct solutions on $\mathbb{T}^3$, $\mathbf{u} = (\mathbf{u}_h, u_z)$ of size $O(1)$ which blow up in finite time.

$$(\partial_t + \mathbf{u} \cdot \nabla)\, \mathbf{u}_h + \mathbf{k} \times \mathbf{u}_h + \nabla_h p = 0, \qquad \mathrm{div}(\mathbf{u}) = 0, \qquad \partial_z p = 0.$$

For the Euler equation without rotation, [**Elgindi, Chen-Hou**] constructed $O(\varepsilon)$ axisymmetric solutions of low-regularity which blow-up in finite time $T = O(\varepsilon^{-1})$(see also [**Wang-Lai-Gomez-Serrano-Buckmaster**]).

These equations are non-dispersive.

Euler-Coriolis: basic properties

We can make a few first observations about the Euler-Coriolis equation

$$(\mathbf{EC}) \qquad (\partial_t + u \cdot \nabla)\, u = -\nabla p - \mathbf{e}_3 \times u, \qquad \mathrm{div}(u) = 0$$

for $x \in \mathbb{R}^3$.

Symmetries

Euler-Coriolis possesses two nontrivial symmetries: scaling and horizontal rotations:

$$u(x,t) \to \lambda^{-1} u(\lambda x, t), \qquad u \to R_\theta^T u \circ R_\theta, \quad R_\theta y = \cos\theta \cdot y - \sin\theta \cdot \mathbf{k} \times y$$

and associated to these, we have two important vector fields

$$(Su)^\alpha = x^\beta \partial_\beta u^\alpha - u^\alpha, \qquad (\Omega u)^\alpha = (x_1\partial_2 - x_2\partial_1)u^\alpha - u_h^\perp,$$

Only 2: not enough to span the tangent space ($\sim$ Euler-Maxwell, gravity-capillary water waves, see [**Deng-Ionescu-P.-Pusateri**]).

Allows to propagate axisymmetry: $\Omega u = 0$.

2 Vector fields

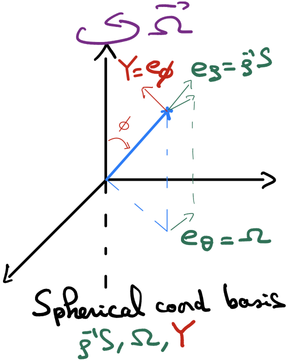

Figure: We will complete the basis of vector fields with a third, non-commuting vector field, Υ.

Poincaré operator

The linearized system is dispersive

$$(\partial_t + \mathbf{k}\times)\, u + \nabla p = 0, \qquad \mathrm{div}(u) = 0,$$

which leads to the scalar dispersive equation

$$(\partial_t + i\Lambda)\, U = 0, \qquad \Lambda(\xi) = \xi_3/|\xi|.$$

Interestingly, a similar dispersion relation also appears in other fluid contexts (e.g. stratified fluids).

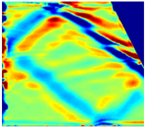

Figure: From Dauxois et al.

Linear dispersion

$\Lambda = |\xi|^{-1}\xi_3$ is 0-homogeneous, so $\xi \cdot \nabla\Lambda = 0$. In fact

$$\nabla\Lambda(\xi) = -\frac{\xi_3}{|\xi|^3}\xi_h + \frac{|\xi_h|^2}{|\xi|^3}\mathbf{e}_3,$$

$$\nabla^2\Lambda = \begin{pmatrix} \partial_r^2\lambda & 0 & \partial_r\partial_z\lambda \\ 0 & \partial_r\lambda/r & 0 \\ \partial_r\partial_z\lambda & 0 & \partial_z^2\lambda \end{pmatrix} \qquad \text{in cyl. coord,}$$

$$\det \nabla^2\Lambda = -\frac{\xi_3|\xi_h|^2}{|\xi|^9}.$$

Hessian degenerate for vertical and horizontal frequencies. Always: a 2×2 minor nondegenerate and this leads to optimal decay

$$e^{it\Lambda}f(0) = \frac{\sin(t)}{t}f(0), \qquad \text{for radial } f.$$

Radial dispersion

Study of dispersive properties of quasilinear dispersive equations with radial dispersion studied extensively [**Alazard, Delort, Deng, Germain, Guo, Gustafson, Ifrim, Ionescu, Masmoudi, Nakanishi, P., Pusateri, Shatah, Stingo, Tataru, Tsai,. . .**]. Radial dispersion relations lead to many simplifications

(i) radial symmetry so rotation vector fields
(ii) various structural info is determined by $1d$ functions (e.g. space-time resonances only occur on spheres)

In more general cases, more pathologies can happen [**Bernicot-Germain**] and here we consider an axisymmetric dispersion relation.

Predispersive solution 1: rigid body motion

A degenerate family of nondispersive solutions is provided by rest solutions in unrotated frame

$$u^{RB}(z,t) = R_t\underline{u}(z), \qquad \underline{u}(z)\cdot\mathbf{k} = 0, \quad R_t\underline{u} = \cos(t)\underline{u} - \sin(t)\underline{u}^\perp.$$

This is far from L^2, but allows to construct predispersive solutions

$$\begin{aligned}
u_h^{app} &:= \varphi(\varepsilon x_h)\left[R_t\underline{u} + \frac{\varepsilon t}{2}u_{[0]} - \frac{\varepsilon}{2}R_{2t}u_{[2]}\right]\\
u^{app,3} &:= -\varepsilon(R_tU(z))\cdot\nabla\varphi(\varepsilon x_h) + O(\varepsilon^2), \qquad \partial_z U(z) = \underline{u}(z)\\
u_{[0]} &:= (\underline{u}\cdot\nabla\varphi)\underline{u} + (\underline{u}^\perp\cdot\nabla\varphi)\underline{u}^\perp - \left[(U\cdot\nabla\varphi)\partial_z\underline{u} + (U^\perp\cdot\nabla\varphi)\partial_z\underline{u}^\perp\right],\\
eq(u^{app}) &= 1_{\{|x_h|\sim\varepsilon^{-1}\}}O(\varepsilon^2).
\end{aligned}$$

Grows secularly for $0 \le t \le \varepsilon^{-1}$ but not small

$$\|u^{app}(0)\|_B = \varepsilon^{-1}\|\underline{u}(z)\|_B + O(1).$$

Predispersive solution: 2d Euler

Another degenerate family of nondispersive solutions is provided by

$$\begin{aligned} u^{2d}(x,y,z,t) &= (v^1(x,y,t), v^2(x,y,t), w(x,y,t)), \\ (\partial_t + \mathbf{v}\cdot\nabla_{x,y})\,\mathbf{v} + \mathbf{v}^\perp + \nabla q = 0, &\qquad \mathrm{div}(\mathbf{v}) = 0, \\ (\partial_t + \mathbf{v}\cdot\nabla_{x,y})\,w = 0. & \end{aligned} \tag{1}$$

With predispersive almost solution

$$u^{app}(x,y,z,t) := \varepsilon\varphi(\varepsilon z)(\mathbf{v}(x,y,\varepsilon t), w(x,y,\varepsilon t))$$

which is divergence free and satisfies

$$\begin{aligned} \partial_t u_h + u\cdot\nabla u_h + \vec{e}_3\times u_h &= \varepsilon^2 \mathrm{div}_h\left\{\varphi(1-\varphi)\mathbf{v}\otimes\mathbf{v}\right\} - \left[\varepsilon\varphi(\mathbf{v}^\perp + \nabla q)\right] \\ \partial_t u_3 + u\cdot\nabla u_3 &= \varepsilon^2 \mathrm{div}_h\left\{\varphi(1-\varphi)\mathbf{v}w\right\} \end{aligned}$$

These solutions can start small

$$\|u^{app}(0)\|_B = \|\mathbf{v}(0)\|_{L^2} + \|w(0)\|_{L^2}.$$

The role of axisymmetry

Axisymmetry simplifies the equations, but more importantly removes a resonant system ($2d$-Euler) which has very different analytical properties (see enemy above).

General strategy

We follow a general strategy for quasilinear dispersive problems

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4. Dispersive analysis 1: improve L^2 control on vector field to stronger $\sim \mathcal{F}L^\infty$ control on vector fields.
5. Dispersive analysis 2: propagate fractional derivatives along a remaining non-commuting vector field (Υ) in L^2.

Parameterization of unknowns

Axisymmetric flow:

$$\mathbf{u} = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z, \qquad \mathrm{div}(\mathbf{u}) = r^{-1}\left[\partial_r(ru_r) + \partial_z(ru_z)\right]$$

3-vector with one elliptic condition: 2 scalar degrees of freedom.
Common parameterization is Stokes' swirl-stream formulation:

$$\mathbf{u} = u_\theta e_\theta + \nabla \times \left(\frac{\psi_\theta}{r} e_\theta\right)$$

It captures the energy decomposition

$$\|u\|_{L^2}^2 = \|u_\theta\|_{L^2}^2 + \|\frac{1}{r}\nabla\psi_\theta\|_{L^2}^2$$

but is a little cumbersome with the Fourier transform. Instead we use a Hodge decomposition of the horizontal vector field

Hodge decomposition

$$\partial_t u^3 + \partial_3 \Delta^{-1}\mathrm{curl}_h(u_h) = \partial_p \Delta^{-1}\partial_\alpha\partial_\beta\left\{u^\alpha u^\beta\right\} - \partial_\alpha\left\{u^\alpha u^3\right\}$$
$$\partial_t \mathrm{curl}_h(u_h) - \partial_3 u^3 = -\in^{jk}\partial_j\partial_l\left\{u^l u^k\right\} - \in^{jk}\partial_j\partial_3\left\{u^3 u^k\right\}$$

and with ($a \sim$ swirl)

$$a := |\nabla_h|^{-1}\mathrm{curl}_h(u_h), \qquad c := |\nabla|\cdot|\nabla_h|^{-1}u^3,$$

$$\mathrm{u}{=}\mathrm{U}_a + U_c, \qquad U_a = -\nabla_h^\perp|\nabla_h|^{-1}a, \quad U_c = \mathbf{\Lambda}\nabla_h|\nabla_h|^{-1}c + \sqrt{1-\Lambda^2}c\ \vec{e}_3.$$

Can parameterize $u \mapsto (a, c) \in \mathbb{R}^2$ with

$$\|u\|_{L^2}^2 = \|a\|_{L^2}^2 + \|c\|_{L^2}^2,$$
$$\|Su\|_{L^2}^2 = \|Sa\|_{L^2}^2 + \|Sc\|_{L^2}^2.$$

RGB Null structures

Expressing u in terms of (a, c), we arrive at

$$\begin{aligned}
\partial_t a - i\Lambda c &= |\nabla| Q_n^1(a,c) + Q_\Omega^1(a,c) \\
&\quad - \in^{pq} |\nabla_h|^{-1} \left\{ |\nabla_h|^{-1} \partial_q a \cdot \partial_p |\nabla_h| a \right\} \\
\partial_t c - i\Lambda a &= |\nabla| Q_n^2(a,c) + Q_\Omega^2(a,c) \\
&\quad - |\nabla|(1-2\Lambda^2)|\nabla_h|^{-1} \in^{jp} \partial_j \left\{ |\nabla_h|^{-1} \partial_p a \cdot \sqrt{1-\Lambda^2} c \right\}
\end{aligned}$$

where Q_n contains a copy of both Λ and $\sqrt{1-\Lambda^2}$ (falling on either unknown) and Q_Ω contains a copy of Ω (falling on either input).

When $\Lambda = 0$, first line contains $2d$ Euler equation (for $\omega = |\nabla_h| a$), second line a passive scalar. It is to remove this system that we use axisymmetry.

Energy estimates

The energy estimates come from the commutation of S with Euler's equation:

$$\partial_t(Su) + \Omega \times (Su) + \nabla(Sp) + \text{div}\,\{S(u \otimes u)\} = 0, \qquad \text{div}(Su) = 0.$$

which allow us to control $S^n u$ in L^2 and in $\dot{H}^{-1}$.

Dispersive analysis

Diagonalizing the linear operator,

$$\mathcal{U}_\pm = a \pm c,$$

we express the solutions as superpositions of linear waves (cf **Bourgain**, **Germain-Masmoudi-Shatah**, **Gustafson-Nakanishi-Tsai**. . .)

$$\mathcal{U}_\pm(x,t) := \int_{\mathbb{R}^3} V_\pm(\xi,t) e^{i[\langle x,\xi\rangle \mp t\Lambda(\xi)]} d\xi$$

and the decay and localization properties of $\mathcal{U}_\pm$ correspond to smoothness properties of $V_\pm(t)$.

Norms

Dispersion is degenerate. To quantify degeneracies, need 2 LP families

$$P_{k,p,q}(\xi) = \varphi(2^{-k}|\xi|)\varphi(2^{-q}\Lambda)\varphi(2^{-p}\sqrt{1-\Lambda^2})$$
$$\simeq P_{k+p}(\xi_1,\xi_2)P_{k+q}(\xi_3)$$

and we control the following quantities for $e^{it\Lambda}V = a + ic$:

(EE) $\quad \|\hat{V}\|_{H^N\cap\dot{H}^{-1}} + \|S^N\hat{V}\|_{L^2\cap\dot{H}^{-1}} \leq 2\varepsilon_1,$

(DA1) $\quad \|S^{N/2}V\|_B \leq 2\varepsilon_1, \qquad \|f\|_B := \sup_{k,p,q} 2^{-p-\frac{q}{2}}\|P_{k,p,q}f\|_{L^2},$

(DA2) $\quad 2^{\beta p}\|\Upsilon^{1+\beta}S^{N/4}P_{k,p,q}V\|_{L^2} \leq 2\varepsilon_1, \qquad \Upsilon f(\xi) = \partial_\phi f(\xi).$

Yet another LP family

We need to propagate fractional regularity in the non-commuting vector field Υ:

(i) One derivative not enough to obtain $1/t$ decay due to norm growth (enough for almost global existence [**G.-Huang.-P.-W.**]).
(ii) Two derivatives only comes with $t^{\frac{1}{2}+}$ growth, which makes the bootstrap difficult to close.
(iii) In principle one can hope to propagate up to $d/2 = 3/2$ copies (accounting for a proper uncertainty-principle).

To use (iii), we propagate fractional derivatives. This requires a *LP*-family adapted to Υ.
For axisymmetric functions, we have that

$$\|\partial_\phi f\|_{L^2} \simeq \|\partial_\phi f\|_{L^2} + \|\partial_\theta f\|_{L^2} \simeq \|\sqrt{-\Delta_{\mathbb{S}^2}} f\|_{L^2}$$

And we can use a *LP*-family tracking down spherical regularity.

$$R_\ell f(x) = \sum_{n\geq 0} \varphi(2^{-\ell}n) \int_{\mathbb{S}^2} f(|x|\vartheta)\mathfrak{Z}_n(\langle\vartheta, \frac{x}{|x|}\rangle) d\nu_{\mathbb{S}^2}(\vartheta), \quad \ell \in \mathbb{N}$$

$$\mathcal{Z}_n(P) = \mathfrak{Z}_n(\langle P, N\rangle), \qquad \mathfrak{Z}_n(x) = \frac{2n+1}{4\pi}\frac{1}{2^n(n!)}\frac{d^n}{dx^n}[(x^2-1)^n].$$

This defines a family of almost projectors

$$f = \sum R_\ell f, \qquad R_\ell R_{\ell'} = 0 \quad \text{if} \quad |\ell - \ell'| \geq 4, \qquad \|f\|_{L^2}^2 \simeq \sum \|R_\ell f\|_{L^2}^2,$$

$$R_\ell \left[R_{\ell_1} f \cdot R_{\ell_2} g\right] = 0 \quad \text{if} \quad \max\{\ell, \ell_1, \ell_2\} \geq \text{med}\{\ell, \ell_1, \ell_2\} + 4$$

R_ℓ commutes with vector fields, usual LP-proj and Fourier transform

$$[\Omega_{ab}, R_\ell] = [S, R_\ell] = [R_\ell, P_k] = [R_\ell, \mathcal{F}] = 0$$

and satisfies a Bernstein property

$$\sum \|\Omega_{ab} R_\ell f\|_{L^r} \simeq 2^\ell \|R_\ell f\|_{L^r}, \qquad 1 \leq r \leq \infty. \tag{2}$$

Spherical LP

Properties of LP-family on spheres worked out by [**Sogge**], but can be reworked by analyzing the spherical convolution (e.g. L^∞-bounds crucial).

Angular regularity plays an important roles in previous works on semilinear dispersive equations [**Herr, Guo, Nakanishi. . .**].

The construction in $2d$ is much easier ($\mathbb{S}^1$ is a group), and was used in works on quasilinear dispersive equations [**Deng-Ionescu-P.-Pusateri**].

Many problems in $3d$ are isotropic and good LP-analysis of angular regularity more flexible than standard vector field analysis.

Linear Decay Estimate

With $\|f\|_D := \sup_{0\le a,b\le 3}\left(\|S^a f\|_B + \|S^b f\|_X\right)$ we have

Linear Decay

Let f be axisymmetric and $t > 0$. We can split

$$P_{k,p,q} e^{it\Lambda} f = I_{k,p,q}(f) + II_{k,p,q}(f)$$

where for any $0 < \beta' < \beta$

$$\|I_{k,p,q}(f)\|_{L^\infty} \lesssim 2^{\frac{3}{2}k - 3k^+} \cdot \min\{2^{2p+q}, 2^{-p-\frac{q}{2}} t^{-\frac{3}{2}}\}\|f\|_D,$$

$$\|II_{k,p,q}(f)\|_{L^2} \lesssim 2^{-3k^+} t^{-1-\beta'} 2^{(-1-2\beta')p} \cdot \mathbb{1}_{2^{2p+q}\gtrsim t^{-1}}\|f\|_D.$$

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- Implies $\|P_k e^{it\Lambda}f\|_{L^\infty} \lesssim t^{-1}2^{\frac{3}{2}k-3k^+}\|f\|_D$.
- faster decay away from degeneracies,
- Proof: adapted stationary phase.

Space time resonance analysis

We are reduced to study nonlinear oscillatory integrals of the form

$$I[V_1, V_2](\xi) = \iint q_m(s) e^{is\Phi(\xi,\eta)} \mathfrak{m}(\xi,\eta) V_1(\xi-\eta, s) V_2(\eta, s) d\eta ds$$

for the quadratic phase

$$\Phi := \Lambda(\xi) \pm \Lambda(\xi-\eta) \pm \Lambda(\eta)$$

and a null multiplier

$$\mathfrak{m}(\xi,\eta) = \Lambda(\zeta_1)\sqrt{1-\Lambda^2(\zeta_2)}\mathfrak{m}(\xi,\eta), \qquad \zeta_j \in \{\xi, \xi-\eta, \eta\}.$$

Bilinear estimates: qualified ST resonances

The first step in the dispersion analysis follows from integrations by parts along commuting vector fields using the fact that

$$|S_\eta \Phi| + |\Omega_\eta \Phi| \gtrsim \frac{\sqrt{1-\Lambda^2(\xi-\eta)}}{|\xi-\eta|^2} |\det(\xi,\eta,\mathbf{e}_3)|$$

and reexpressing $S_\eta = A S_{\xi-\eta} + B\Omega_{\xi-\eta} + C\Upsilon_{\xi-\eta}$.
This allows (loosing one vector field) to upgrade control of vector fields in L^2 to control of vector fields in B.

$$I[V_1,V_2](\xi) = \iint q_m(s) e^{is\Phi(\xi,\eta)} \mathfrak{m}(\xi,\eta) V_1(\xi-\eta,s) V_2(\eta,s) d\eta ds$$

Simplified sample argument

Lemma

Under the bootstrap assumptions there holds that

$$\|\partial_t U_\pm(t)\|_{L^2} \lesssim \epsilon^2 \langle t \rangle^{-\frac{3}{2}+}.$$

We have $\partial_t U_\pm \sim Q_{\mathfrak{m}}(U_\pm, U_\pm)$, so beats the criticality t^{-1}.

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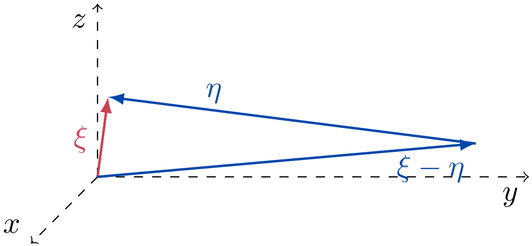
Proof:



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Proof:

- By repeated ibp along vector fields, force $p \sim p_1 \sim p_2$, $q \sim q_1 \sim q_2$.
- In that case, use linear decay (+ null structure). □

Dispersive analysis 2

In order to propagate $\Upsilon^{1+\beta} V$ in L^2, we need to make use of another information: If the phase is smaller than expected, then we can almost always integrate by parts along vector fields: in proper positions,

$$\text{If} \qquad |\Phi| \ll 2^q + 2^{q_1} + 2^{q_2},$$

$$\text{then} \qquad |\det(\xi, \eta, \mathbf{e}_3)| \gtrsim (2^q + 2^{q_1} + 2^{q_2}) \frac{|\xi| \cdot |\xi - \eta| \cdot |\eta|}{(|\xi| + |\xi - \eta| + |\eta|)^2}$$

Note that for our null multipliers,

$$|\mathfrak{m}(\xi, \eta)| \lesssim |\xi| (2^q + 2^{q_1} + 2^{q_2})$$

which almost cancels the first integration by parts.

Interesting open questions

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2. Stability of other steady states $(\nabla^{\perp} g(r), 0)$? ($\sim$ stability of Couette-like shear flows)
3. Adaptation to other equations with rotation (e.g. dynamo equations in MHD)

Thank you for your attention.