Soliton resolution for the energy critical wave equation in six dimensions in the radial case

C. Collot, joint work with T. Duyckaerts, C. Kenig, and F. Merle

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Part 1 : The equation and the results

The focusing critical wave equation

In this talk :

$$\begin{cases} \partial_t^2 u - \Delta u = u^2, \quad \text{or} \quad \partial_t^2 u - \Delta u = |u|u \\ (u(0), \partial_t u(0)) \in \dot{H}^1 \times L^2(\mathbb{R}^6). \end{cases}$$

where $u : [0, T[\times \mathbb{R}^N \to \mathbb{R}$. The general equation :

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u \\ (u(0), \partial_t u(0)) \in \dot{H}^1 \times L^2(\mathbb{R}^N). \end{cases} \tag{NLW}$$

where $N \geq 3$. (NLW) is locally well-posed in $\dot{H}^1 \times L^2(\mathbb{R}^N)$ [Ginibre-Soffer-Velo 92].
Conserved energy :

$$E = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t)|^2 + |\partial_t u(t)| dx - \frac{N-2}{2N} \int_{\mathbb{R}^N} |u(t)|^{\frac{2N}{N-2}} dx.$$

Invariances : If u is a solution and $\lambda > 0$, so is

$$u_{(\lambda),\pm}(t,x) = \pm \frac{1}{\lambda^{N/2-1}} u \left(\frac{\pm t}{\lambda}, \frac{x}{\lambda} \right).$$

(NLW) is energy critical :

$$E(u_{(\lambda),\pm}) = E(u).$$

The ground state

$$-\Delta W = |W|^{\frac{4}{N-2}} W, \quad W \in \dot{H}^1(\mathbb{R}^N). \qquad \text{(Ell)}$$

$\exists$ solutions of (Ell) with arbitrarily large energy [Ding 1986].

Unique nonzero radial solution of (Ell) up to scaling and sign change (ground state) :

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{1-\frac{N}{2}}.$$

Remark : W decays faster than the scaling invariant solution $|W(x)| \ll |x|^{\frac{N-2}{2}}$ as $x \to \infty$, which is false for supercritical case.

For $N = 6$:

$$W(x) = \left(1 + \frac{|x|^2}{24}\right)^{-2} \approx \langle x \rangle^{-4}.$$

Main results

Theorem 1 | Soliton resolution $N = 6$ radial [C.-Duyckaerts-Kenig-Merle '22]

Let u be a radial solution of

$$\partial_t^2 - \Delta u = u^2 \qquad \text{or} \qquad \partial_t^2 - \Delta u = |u|u.$$

Assume it is global for positive times with

$$\sup_{t \geq 0} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty.$$

Then $\exists J \in \mathbb{N}$, $\iota_j \in \{\pm 1\}$ ($\iota_j = 1$ for u^2 nonlinearity),

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t,$$

and v_L a free wave $\partial_t^2 v_L - \Delta v_L = 0$, such that, as $t \to \infty$:

$$(u(t), \partial_t u(t)) = \sum_{j=1}^{J} \left(\frac{\iota_j}{\lambda_j^2(t)} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + (v_L(t), \partial_t v_L(t)) + o(1), \text{ in } \dot{H}^1 \times L^2.$$

$+$ Analogue Theorem when $T(u) < \infty$.

Main results

Theorem 2 | Nonexistence of pure multisoliton radial [C.-Duyckaerts-Kenig-Merle '22]

Let u be a radial solution of

$$\partial_t^2 - \Delta u = u^2 \qquad \text{or} \qquad \partial_t^2 - \Delta u = |u|u.$$

Assume it is global for both time directions with

$$\sup_{t\in\mathbb{R}} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty,$$

and that it is non-radiative for $r \geq R + |t|$ for any $R \in \mathbb{R}$:

$$\lim_{t\to\pm\infty} \int_{|x|\geq R+|t|} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \, dx = 0.$$

Then

$$u = 0, \qquad \text{or} \qquad u = W_{(\lambda)} \qquad \text{or} \qquad u = -W_{(\lambda)}.$$

Pure multisoliton as the assumption implies by previous Theorem

$$(u(t), \partial_t u(t)) = \sum_{j=1}^{J_\pm} \left(\frac{\iota_j^\pm}{\lambda_j^{\pm 2}(t)} W\left(\frac{x}{\lambda_j^\pm(t)}\right), 0 \right) + o_{t\to\pm\infty}(1), \text{ in } \dot{H}^1 \times L^2.$$

Part 2 : Related results

Examples of global solutions

$N = 6$.

[Ginibre-Soffer-Velo '92] $\exists$ scattering solutions

$$(u(t), \partial_t u(t)) = (v_L(t), \partial_t v_L(t)) + o(1), \qquad \text{in } \dot{H}^1 \times L^2.$$

[Krieger-Nakanishi-Schlag '12-'15] $\exists$ manifold of solutions scattering to a ground state

$$(u(t), \partial_t u(t)) = W + (v_L(t), \partial_t v_L(t)) + o(1),$$

[Jendrej '17] $\exists$ a 2-multisoliton

$$(u(t), \partial_t u(t)) = W + \frac{1}{\lambda(t)^2} W\left(\frac{x}{\lambda(t)}\right) + o(1), \qquad \lambda(t) = \kappa^{-1} e^{-\kappa t}, \qquad \kappa = \frac{\sqrt{5}}{2}.$$

General N.

Infinite time concentrating or spreading radial soliton [Krieger-Schlag '07, Krieger Donninger '13]. Non-radial multi-solitons [Martel-Merle '16, Yuan '20 '21]. Radial 2-multisoliton [Jendrej '17].

The soliton resolution conjecture

Soliton resolution conjecture : any global solution of many evolution equation evolves asymptotically into a sum, as $t \to T$, of self-similar solutions (in the sense that their trajectory is the action on the initial datum of a one-dimensional subgroup of the group of invariances).

For (NLW), any solution that remains bounded in the energy space is asymptotically the sum, as $t \to T$, of decoupled solitons and a radiation term.

Origin of the conjecture : numerical experiments [Fermi-Pasta-Ulam 1955], [Zabusky-Kruskal 1965] and completely integrable equations.

For completely integrable systems, the soliton resolution can be proved with the method of inverse scattering [Eckauss-Schuur 1983] for KdV, [Schuur 1986] for mKdV, [Pocovnicu 2011] for Szegö etc...

For typical completely integrable equations, a multisoliton in the future is also a multisoliton in the past, with the same parameters (thus it is a *pure multisoliton*). The collision between solitons is elastic. This is the case for KdV, mKdV, and also the Szegö equation.

This property is believed to be specific to completely integrable equations. See [Martel-Merle '11, '15] for collisions for non-integrable gKdV.

Inelastic collision conjecture : there is no pure multisoliton for (NLW).

Proofs of soliton resolution conjecture

Soliton resolution and inelastic collision :

- Analogue results for (NLW) for $N = 3$ [Duyckaerts-Kenig-Merle '13], $N \geq 5$ [Duyckaerts-Kenig-Merle '19], $N = 4$ [Duyckaerts-Kenig-Martel-Merle '19].
- Co-rotational wave maps [Duyckaerts-Kenig-Martel-Merle '21].
- Radial defocusing 3D wave with a potential [Jia-Liu-Schlag '15].
- Nonlinear waves in 3D outside an obstacle [Duyckearts-Yang '19].

Soliton resolution :

- (NLW) without symmetry assumption for (NLW), for a sequence of times [Duyckaerts-Jia-Kenig-Merle '16].
- Partial result for wave maps [Grinis '17], [Duyckaerts-Jia-Kenig-Merle '16].
- Critical Wave Maps in all equivariant classes [Cote-Kenig-Lawrie-Schlag '15], [Cote '15], [Jendrej-Lawrie '18, '20, '21].
- Damped Klein-Gordon [Burq-Raugel-Schlag '17-18].

Inelastic collision :

- For a nonradial multisoliton of (NLW) in 5D [Martel-Merle '19].
- For a two-bubble wave maps [Jendrej-Lawrie '18].

Part 3 : New far away classification of non-radiative solutions

Non-radiative solutions | Definition

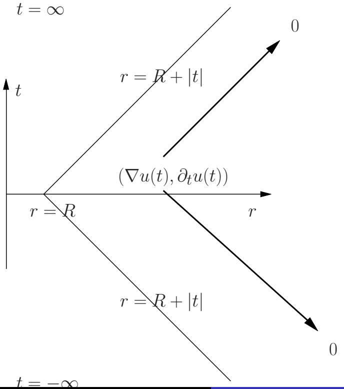
Definition

A radial solution to

$$\partial_t^2 u - \Delta u = u^2$$

is non-radiative for $r \geq R + |t|$ if

$$\lim_{t \to \pm\infty} \int_{|x| \geq R + |t|} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \, dx = 0.$$



Non-radiative solutions | Far away classification

Proposition | Existence [C.-Duyckaerts-Kenig-Merle '22]

Given any $\ell, \ell' \in \mathbb{R}$, there exist polynomials $(A_{i,n})_{0\leq i,\ 0\leq n\leq \lfloor \frac{i+1}{2} \rfloor}$ such that

$$a[\ell,\ell'](t,r) = \sum_{i=0}^{\infty} \sum_{n=0}^{\lfloor \frac{i+1}{2} \rfloor} \frac{\log^n r}{r^{3+i}} A_{i,n}\left(\frac{t}{r}\right) = \frac{\ell t}{r^4} + \frac{\ell'}{r^4} + h.o.t.$$

is well defined for all $r \geq R^*(\ell,\ell')$, where it is a non-radiative solution to :

$$\partial_t^2 a - \Delta a = a^2.$$

Only two functions up to time and scale transformations.

Proposition | Classification [C.-D.-K.-M. '22]

For any $R > 0$, for any solution u to

$$\partial_t^2 - \Delta u = u^2$$

that is non-radiative for $r \geq R + |t|$, there exists $\ell, \ell' \in \mathbb{R}$ and $R' > 0$ such that for all $r \geq R' + |t|$

$$u(t.r) = a[\ell,\ell'](t,r).$$

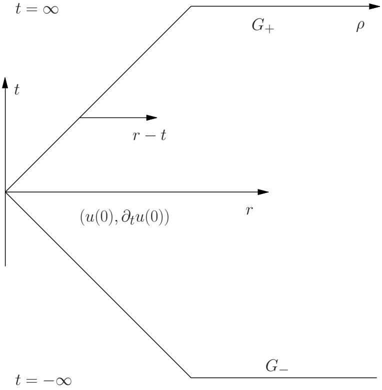
Part 4 : A new channels of energy estimate around the ground state

Radiation fields for free waves

Proposition | [Friedlander 60's, Duyckaerts-Kenig-Merle '16]

For $N \geq 3$, u radial solution on $\mathbb{R}^{1+N}$ of $\begin{cases} \partial_t^2 u - \Delta u = 0, \\ (u(0), \partial_t u(0)) \in \dot{H}^1 \times L^2, \end{cases}$ there exist $G_\pm \in L^2(\mathbb{R})$ radiation profiles :

$$\lim_{t \to \pm\infty} \int_0^\infty \left| r^{\frac{N-1}{2}} (\partial_r u, \partial_t u)(t, r) - (G_\pm, \pm G_\pm)(r - |t|) \right|^2 dr = 0.$$



Recent works [Li-Shen-Wei '21, '22, Delort '21, Cote-Laurent '21].

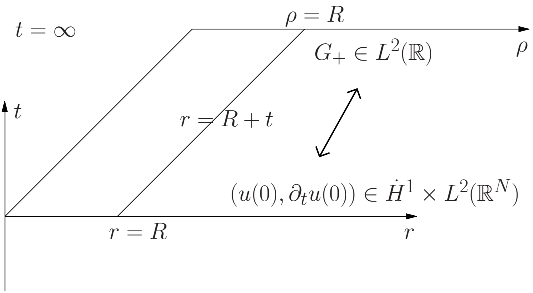
Repartition of energy

The map $(u(0), \partial_t u(0)) \mapsto G_\pm$ is an isometry :

$$\int_{\mathbb{R}^N} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx = C \int_{\mathbb{R}} |G_\pm|^2 d\rho$$

By finite speed of propagation, on exterior wave cones away from origin :

$$\int_{|x|\geq R} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx \gtrsim \int_{\rho \geq R} |G_\pm|^2 d\rho$$



Question : do we have

$$\int_{|x|\geq R} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \geq R} |G_\pm|^2 d\rho \quad ???$$

Channels of energy estimates | Odd dimensions

Proposition | [Duyckaerts-Kenig-Merle '12]

N odd. If $R=0$ then :

$$\int_{\mathbb{R}^N} |\nabla u(0)|^2+|\partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho\geq 0} |G_\pm|^2 d\rho.$$

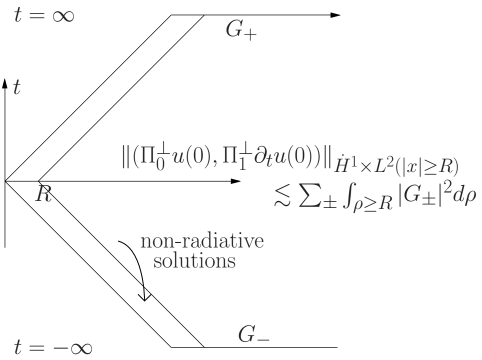
If $R>0$ there exists a finite dimensional set of solutions to $\partial_{tt} v-\Delta v=0$ that are non-radiative for $r\geq R+|t|$:

$$\lim_{t\to\pm\infty} \int_{|x|\geq R+|t|} |\nabla v(t)|^2+|\partial_t v(t)|^2 \, dx=0, \quad \Leftrightarrow \quad G_\pm(\rho)=0 \text{ for } \rho\geq R.$$

For (Π_0^F,Π_1^F) the orthogonal projection outside this set :

$$\int_{|x|\geq R} |\Pi_0^F\nabla u(0)|^2+|\Pi_1^F\partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho\geq 0} |G_\pm|^2 d\rho.$$

Channels of energy estimates | Odd dimensions



Channels of energy estimates | Even dimensions

Proposition | [Cote-Kenig-Schlag '14]

$N=6$. If $R=0$ then :

$$\int_{\mathbb{R}^N} |\partial_t u(0)|^2 \ dx \lesssim \sum_{\pm} \int_{\rho\geq 0} |G_\pm|^2 d\rho \quad \text{holds true},$$

$$\int_{\mathbb{R}^N} |\nabla u(0)|^2 \ dx \not\lesssim \sum_{\pm} \int_{\rho\geq 0} |G_\pm|^2 d\rho \quad \text{fails}.$$

If $R>0$ then $v=r^{-4}$ and $v=tr^4$ are non-radiative for $r\geq R+|t|$:

$$\lim_{t\to\pm\infty} \int_{|x|\geq R+|t|} |\nabla v(t)|^2+|\partial_t v(t)|^2 \ dx=0, \quad \Leftrightarrow \quad G_\pm(\rho)=0 \text{ for } \rho\geq R.$$

For (Π_0^F,Π_1^F) the orthogonal projection outside this set :

$$\int_{|x|\geq R} |\Pi_1^F \partial_t u(0)|^2 \ dx \lesssim \sum_{\pm} \int_{\rho\geq 0} |G_\pm|^2 d\rho \quad \text{holds true},$$

$$\int_{|x|\geq R} |\Pi_0^F \nabla u(0)|^2 \ dx \not\lesssim \sum_{\pm} \int_{\rho\geq 0} |G_\pm|^2 d\rho \quad \text{fails}.$$

Distinction odd/even dimensions reminiscent of Huygens principle. Resonance $r^{-2}-2t^2r^{-4}$.

Channels around a ground state

Ground state positive solution of $-\Delta W = |W|^{\frac{4}{N-2}} W$, potential $V = -\frac{N+2}{N-2} W$.

$$\text{Zero :} \qquad (-\Delta + V)\Lambda W = 0, \qquad \Lambda W = x.\nabla W + \frac{N-2}{2} W.$$

Proposition | [Duyckaerts-Kenig-Merle '19]

$N \geq 3$ odd. $v = \Lambda W$ and $v = t\Lambda W$ are non-radiative solutions of

$$\partial_t^2 v - \Delta v + Vv = 0.$$

For any solution u for Π_i^L the orthogonal projection outside Span(ΛW) :

$$\int_{\mathbb{R}^N} |\Pi_0^\perp \partial_t u(0)|^2 + |\Pi_1^\perp \partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \geq 0} |G_\pm|^2 d\rho.$$

Channels around a ground state $N = 6$

Logarithmic weakening of $\|u\|_{\dot{H}^1(\mathbb{R}^6)}$:

$$\|u\|_Z = \sup_{R>0} \frac{R^{-1}}{\langle \ln R \rangle} \|u\|_{L^2(R \le r \le 2R)}$$

Note $\|u\|_Z \lesssim \sup_{R>0} \ R^{-1}\|u\|_{L^2(R\le r\le 2R)} \lesssim \|u\|_{\dot{H}^1(\mathbb{R}^6)}$

Proposition | [C.-Duyckaerts-Kenig-Merle '22]

$N = 6$. $v = \Lambda W$ and $v = t\Lambda W$ are non-radiative solutions of

$$\partial_t^2 v - \Delta v + Vv = 0.$$

For any solution u for Π_i^L the orthogonal projection outside Span(ΛW) :

$$\|\Pi_0^L u(0)\|_Z^2 + \int_{\mathbb{R}^6} |\Pi_1^\perp \partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \ge 0} |G_\pm|^2 d\rho. \qquad (1)$$

Remark

(4) fails on any finite co-dimensional set for $r \ge R + |t|$, $R > 0$, and fails for $\partial_t^2 u - \Delta u = 0$, counter-examples. $\log^{1/2}$ should be optimal.

Channels around a ground state $N = 6$ | Proof 1

Γ another zero of

$$(-\Delta + V)\Gamma = 0, \qquad \Gamma(r) \approx \begin{cases} r^{-4} & \text{as } r \to 0, \\ 1 & \text{as } r \to \infty. \end{cases}$$

$$\Pi_1^R = \begin{cases} \Pi_{L^2(r\geq R)} \left(\text{Span}\,(\Lambda W)\right)^{\perp} & \text{for } R \geq 1, \\ \Pi_{L^2(r\geq R)} \left(\text{Span}\,(\Gamma\chi, \Lambda W)\right)^{\perp} & \text{for } R < 1. \end{cases}$$

Lemma | Uniform estimates for $\partial_t u$ for $r > |t|$

For any $t_0 \in \mathbb{R}$ and $R \geq |t_0|$:

$$\int_{|x|\geq R} |\Pi_1^R \partial_t u(t_0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \geq 0} |G_{\pm}|^2 d\rho.$$

Step 1. Qualitative property for linearised waves : if the symetrisation $\frac{1}{2}(u(t_0 + t) - u(t_0 - t))$ is non-radiative then $\partial_t u(t_0) = c\Lambda W$.

Step 2. Quantitative property for free waves : Channels of energy estimate of Cote-Kenig-Schlag for $\partial_t^2 u - \Delta u = 0$.

Step 3. Combine in a compactness argument Step 1 for scales ~ 1 and Step 2 for scales ~ 0 and $\sim \infty$.

Channels around a ground state $N=6$ | Proof 2

Lemma | Estimating $u(0)$ from $u(t)$ I

If $\partial_t^2 u - \Delta u + Vu = 0$ then

$$(-\Delta + V)\,(u(0) + h(\partial_t u)) = f(\partial_t u) + \partial_r g(\partial_t u)$$

where

$$\sup_{R>0} R^{-1}\|h(\partial_t u)\|_{L^2(R\le r\le 2R)} + \frac{R}{\langle \ln R\rangle}\|f(\partial_t u)\|_{L^2(R\le r\le 2R)} + \|g(\partial_t u)\|_{L^2(R\le r\le 2R)}$$
$$\lesssim \sup_{R>|t_0|}\left(\int_{|x|\ge R} |\Pi_1^R \partial_t u(t_0)|^2\,dx\right)^{\frac12}.$$

Lemma | Estimating $u(0)$ from $u(t)$ II

If

$$(-\Delta + V)v = f + \partial_r g$$

then

$$\sup_{R>0} \frac{R^{-1}}{\langle \ln R\rangle}\|\Pi_0^L v\|_{L^2(R\le r\le 2R)} \lesssim \sup_{R>0} \frac{R}{\langle \ln R\rangle}\|f\|_{L^2(R\le r\le 2R)} + \frac{1}{\langle \ln R\rangle}\|g\|_{L^2(R\le r\le 2R)}.$$

Channels around a ground state $N=6$ | Proof 2

Lemma | Estimating $u(0)$ from $u(t)$ I

If $\partial_t^2 u - \Delta u + Vu = 0$ then

$$(-\Delta + V)\,(u(0) + h(\partial_t u)) = f(\partial_t u) + \partial_r g(\partial_t u) \tag{2}$$

+ estimates for f, g, h.

Step 1. Generalise the identity for $\partial_t^2 u - \Delta u = 0$:

$$-\Delta\left(u(0) + \int_0^{t_0}(1-\frac{t}{t_0})\partial_t u(t)dt\right) = \frac{\partial_t u(0) - \partial_t u(t_0)}{t_0}.$$

to include potential term V.

Step 2. Decompose in exterior dyadic cells $t_0 = R_k = 2^k$, $u_t(R_k) = a_k \Lambda W + b_k \Gamma + \tilde{u}_t$:

$$(-\Delta + V)\left(u(0) + \int_0^{R_k}(1-\frac{t}{R_k})\partial_t u(t)dt\right) = c_k \Lambda W + \tilde{c}_k \Gamma + \frac{\partial_t \tilde{u}(0) - \partial_t \tilde{u}(R_k)}{R_k}. \tag{3}$$

Step 3. Glue the elliptic equations (3) on the dyadic scale $r \approx R_k$ into a global elliptic equation (2) : use a partition of unity $\sum_k \chi(r/R_k) = 1$, estimate the commutator $[\chi, (-\Delta + V)]$, estimate the differences $|c_{k+1} - c_k| + |\tilde{c}_{k+1} - \tilde{c}_k|$ (appearance of logarithmic loss). $c_0 = 0$ as $(-\Delta + V)$ is self-adjoint, and $\tilde{c}_0 = 0$ from channels of energy estimate.

Channels around a ground state $N = 6$ | Proof 2

Lemma | Estimating $u(0)$ from $u(t)$ II

If $v \in \dot{H}^1$ solves

$$(-\Delta + V)v = f + \partial_r g$$

then

$$\sup_{R>0} \frac{R^{-1}}{\langle \ln R \rangle} \|\Pi_0^L v\|_{L^2(R \le r \le 2R)} \lesssim \sup_{R>0} \frac{R}{\langle \ln R \rangle} \|f\|_{L^2(R \le r \le 2R)} + \frac{1}{\langle \ln R \rangle} \|g\|_{L^2(R \le r \le 2R)}.$$

Use explicit formulas for solving second order ODEs, explicit computations, and notice we have the orthogonality $\int \Lambda W(f + \partial_r g) = 0$ for free as $(-\Delta + V)$ is self-adjoint.

Channels around a multisoliton $N = 6$

For $\vec{\lambda} = (\lambda_1, ..., \lambda_J)$, $0 < \lambda_J \ll ... \ll \lambda_1$:

$$\|u\|_{Z_{\vec{\lambda}}} = \sup_{R>0} \frac{R^{-1}}{\inf_j \langle \ln \frac{R}{\lambda_j} \rangle} \|u\|_{L^2(R \le r \le 2R)}$$

Proposition | [C.-Duyckaerts-Kenig-Merle '22]

$N = 6$. For any solution u of

$$\left(\partial_t^2 - \Delta + \sum_j \frac{1}{\lambda_j^2} V(\frac{r}{\lambda_j}) \right) u = 0,$$

for Π_i^M the orthogonal projection outside $\text{Span}(\Lambda W(\frac{r}{\lambda_1}), ..., \Lambda W(\frac{r}{\lambda_J}))$:

$$\|\Pi_0^M u(0)\|_{Z_{\vec{\lambda}}}^2 + \int_{\mathbb{R}^6} |\Pi_1^M \partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \ge 0} |G_\pm|^2 d\rho + \gamma \left(\int_{\mathbb{R}^6} |\nabla u(0)|^2 + |\partial_t u(0)|^2 dx \right)$$

where $\gamma = \sup_{j<J} \frac{\lambda_{j+1}}{\lambda_j}$.

Channels around a ground state $N = 6$ | Conclusion

New logarithmically weakened channels of energy estimate in six dimensions.

Proposition | [C.-Duyckaerts-Kenig-Merle '22]

$N = 6$ odd. $v = \Lambda W$ and $v = t\Lambda W$ are non-radiative solutions of

$$\partial_t^2 v - \Delta v + V v = 0.$$

For any solution u for Π_i^L the orthogonal projection outside $\text{Span}(\Lambda W)$:

$$\sup_{R>0} \frac{R^{-1}}{\langle \ln R \rangle} \|\Pi_0^L u(0)\|_{L^2(R \le r \le 2R)} + \int_{\mathbb{R}^6} |\Pi_1^\perp \partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \ge 0} |G_\pm|^2 d\rho. \quad (4)$$

Comments :

- Logarithmic weakening is necessary (counter-examples). Linearised and free problems are different.
- The proof combines
  - a uniform estimate for $\partial_t u(t)$ for $r > |t|$ (generalises an estimate for $t = 0$ and $r > 0$ of [Duyckaerts-Kenig-Merle], taking into account non-radiative free waves at $r = 0$)
  - an elliptic equation over a dyadic scale for u_0 with a forcing depending on u_t.
  - estimates for solving this elliptic equation.

Thank you for your attention ! !