Non-uniqueness of Leray solutions of the forced Navier-Stokes equations

Dallas Albritton   Elia Brué   Maria Colombo

Institute for Advanced Study

Long Time Behavior and Singularity Formation in PDEs – Part IV
Research Center on Stability, Instability, and Turbulence

January 18, 2022

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+ \tag{NS}$$

For each $u_0 \in L^2$ ($\operatorname{div} u_0 = 0$) and $f \in L^1_t L^2_x$, there exists a global-in-time *Leray-Hopf solution* (Leray, Acta 1934), (Hopf, Math. Nachr. 1951):

$$u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathbb{R}^3 \times \mathbb{R}_+)$$

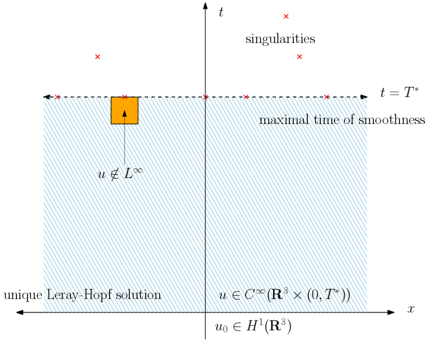
- $u \in C_{\mathrm{w}}([0, +\infty); L^2)$ and $u(\cdot, 0) = u_0$
- solves (NS) for some pressure $p \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}_+)$
- satisfies *energy inequality* for all $t > 0$:

$$\frac{1}{2} \int |u(x,t)|^2 \, dx + \int_0^t \int |\nabla u|^2 \, dx \, ds \leq \frac{1}{2} \int |u_0(x)|^2 \, dx + \int_0^t \int f \cdot u \, dx \, ds \,.$$

Suitable weak solutions further satisfy the *local energy inequality*:

$$(\partial_t - \Delta)\frac{1}{2}|u|^2 + |\nabla u|^2 + \operatorname{div}\left[\left(\frac{1}{2}|u|^2 + p\right)u\right] \leq f \cdot u$$

Partial regularity (CKN, CPAM 1982): $\mathcal{P}_1(\mathrm{Sing}[u]) = 0$ (when $f \in L^{5/2+}$)



Weak-strong uniqueness: If $u_0 \in H^1$ and $f = 0$, there exists a unique strong solution, and Leray-Hopf solutions agree with the strong solution.

The Navier-Stokes equations are used for *predicting* fluid flows.

- If there is no blow-up, then there is perhaps no need to consider Leray-Hopf solutions.
- If there is blow-up, then the solution may be continued as a suitable Leray-Hopf solution. *Is it unique?*

> The other strange thing is the failure of the uniqueness proof in three dimensions.
>
> ---
>
> It is hard to believe that the initial value problem of viscous liquids for $n = 3$ should have more than one solution, and more attention should be paid to the settling of the uniqueness question.

— **Hopf** (translation by Klöckner)

Ladyzhenskaya's example

In 1969, Ladyzhenskaya constructed an example of non-uniqueness for (NS) within a Leray–Hopf-type class.

Though striking, it is in a self-similarly shrinking domain with force and non-standard boundary conditions (inhomogeneous, for the stream function).

> We note a certain "exoticness" of the domain Q_T in which our example has been constructed does not imply loss of the uniqueness theorems just mentioned.
>
> ---
>
> The example described here can provoke "displeasure" for only one reason. It has been constructed for boundary conditions of type (18) but not for adhesion conditions.
>
> ---
>
> The examples presented here are interesting to me in that they refute the entrenched opinion on the "naturalness" for nonstationary problems of physics and mechanics of the class of solutions which have finite energy norm.

— **Ladyzhenskaya** (translated)

I. The program of Jia, Šverák, and Guillod.

- (Jia-Šverák, Inventiones 2014, JFA 2015)
  Could non-uniqueness arise due to bifurcations from or within the class of large self-similar solutions?
- (Guillod-Šverák, arXiv 2017) Compelling numerical evidence
- Leray solutions, but no "proof."

(I will explain in a few slides.)

II. Convex integration.

- (Buckmaster-Vicol, Annals 2019): There exists $\beta > 0$ such that for any non-negative smooth function $e(t) : [0, T] \to \mathbb{R}_{\geq 0}$, there exists $v \in C^0_t([0, T]; H^\beta_x(\mathbb{T}^3))$ a weak solution of the Navier-Stokes equations, such that $\int |v(x, t)|^2 \, dx = e(t)$ for all $t \in [0, T]$.
- (Cheskidov-Luo) Non-uniqueness in $L^q_t L^\infty_x$, $q < 2$, $d \geq 2$ (arXiv 2020) and $C_t L^p_x$, $p < 2$, $d = 2$, respectively (arXiv 2021)
- Proof, but no Leray solutions.

Theorem (A.-Brué-Colombo, arXiv 2021)

There exist two distinct suitable Leray–Hopf solutions to the Navier–Stokes equations with identical body force $f \in L^1_t L^2_x$ *and* $u_0 \equiv 0$.

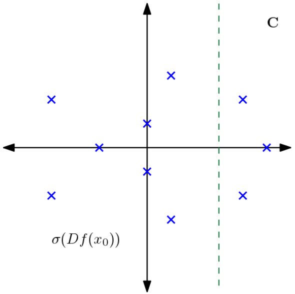
Non-uniqueness is an extreme form of instability.

- *Instability*: A nearby trajectory is driven away (exponentially) quickly.
- *Non-uniqueness*: An infinitesimally close trajectory is driven away instantaneously.

Consider the ODE

$$\dot{x} = f(x) \quad \ldots \quad \text{equilibrium } f(x_0) = 0$$

$$\text{Linearized equation :} \quad \dot{y} = (Df)(x_0)y$$



Unstable manifold M_u:

Contains all the trajectories which approach x_0 backward-in-time at a certain rate. $\dim M_u = \dim E_u$.

Generalization to semilinear parabolic PDEs: (Henry 1981)

Scaling symmetry and dimensional analysis:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)$$

$$[x] = L, \quad [t] = L^2, \quad [u] = L^{-1}, \quad [f] = L^{-3}, \quad \text{since } [\nu] = L^2/T = 1$$

Self-similarity variables:

$$\xi = \frac{x}{\sqrt{t}}, \quad \tau = \log t \in \mathbb{R} \ (!)$$

$$u(x,t) = \frac{1}{\sqrt{t}} U(\xi,\tau), \quad f(x,t) = \frac{1}{t^{\frac{3}{2}}} F(\xi,\tau) \quad \text{(convention)}$$

Navier-Stokes in self-similarity variables:

$$\partial_\tau U \underbrace{- \frac{1}{2} (1 + \xi \cdot \nabla_\xi) U}_{\text{additional terms}} - \Delta U + U \cdot \nabla U + \nabla P = F, \quad \operatorname{div} U = 0 . \qquad \text{(NS-SS)}$$

$U = O(1)$ smooth and decaying $\iff$ u in critical spaces, e.g., $L^\infty_t L^{3,\infty}_x$

U steady solution of (NS-SS) $\iff$ u forward self-similar solution of (NS)

Linearized operator in function space X:

$$-\boldsymbol{L}_{\mathrm{ss}} = -\frac{1}{2}\left(1+\xi\cdot\nabla_\xi\right)U - \Delta U + \mathbb{P}\left(\bar{U}\cdot\nabla U + U\cdot\nabla\bar{U}\right),$$

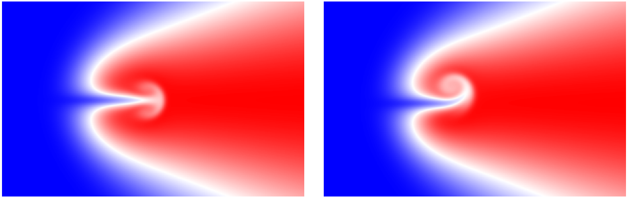
$u_{0,\sigma} = \sigma \times a_0$ (-1-homogeneous initial datum of size σ)

$\bar{U}_\sigma =$ self-similarity profile of corresponding solution

Theorem (Jia-Šverák, JFA 2015)

Suppose (A) transcritical bifurcation or (B) Hopf bifurcation. Then, upon truncating properly, there exist two distinct Leray-Hopf solutions with identical compactly supported data u_0, and $|u_0| = O(1/|x|)$ at $x = 0$.

Numerical evidence of $\mathbb{Z}_2$-symmetry-breaking bifurcation:



$|x|U_\sigma \cdot e_\theta$ for two solutions when $\sigma \approx 300$ (Guillod-Šverák, arXiv 2017)

Theorem (A.-Brué-Colombo, arXiv 2021)

There exist $\bar{U} \in C_0^\infty(\mathbb{R}^3)$ $(\mathrm{div}\, \bar{U} = 0)$ *satisfying*

1. *(Linear instability)* $\boldsymbol{L}_{\rm ss}$ *has an unstable eigenvalue* λ *with non-trivial eigenfunction* $\eta \in H^k$ *for all* $k \geq 0$*:*

$$\boldsymbol{L}_{\rm ss}\eta = \lambda\eta \quad \textit{and} \quad a := \mathrm{Re}\,\lambda > 0.$$

Define

$$U^{\rm lin}(\cdot,\tau) = \mathrm{Re}(e^{\tau\lambda}\eta)\,, \ \textit{which solves}\ \partial_\tau U^{\rm lin} = \boldsymbol{L}_{\rm ss} U^{\rm lin}\,.$$

2. *(Nonlinear instability) There exist* $T \in \mathbb{R}$ *and* $U^{\rm per} : \mathbb{R}^3 \times (-\infty, T] \to \mathbb{R}^3$*,*

$$\|U^{\rm per}(\cdot,\tau)\|_{H^k} \lesssim_k e^{2\tau a} \quad \forall \tau \in (-\infty, T]\,, \forall k \geq 0$$

such that

$$\bar{u}, \quad u = \bar{u} + u^{\rm lin} + u^{\rm per}$$

are the desired distinct suitable Leray-Hopf solutions with zero initial velocity and identical force $\bar{f}$*, whose similarity profile is*

$$\bar{F} := -\frac{1}{2}(1 + \xi \cdot \nabla_\xi)\bar{U} - \Delta\bar{U} + \bar{U}\cdot\nabla\bar{U}\,.$$

Elements of the proof

Claim: It is enough to find a decaying unstable steady state of the (forced) Euler equations in three dimensions,

$$\partial_t u + u \cdot \nabla u + \nabla p = f \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ .$$

with, more specifically, an unstable eigenvalue.

Heuristic:

$$-\boldsymbol{L}_{\mathrm{ss}}^{(\beta)} = \underbrace{-\frac{1}{2}\left(1 + \xi \cdot \nabla_\xi\right) U - \Delta U}_{\text{perturbative when } \beta \gg 1} + \beta \mathbb{P}\left(\bar{U} \cdot \nabla U + U \cdot \nabla \bar{U}\right)$$

Remarkably, no suitable steady state was known, and a key component and major difficulty of our proof is its construction.

We "lift" a two-dimensional instability into dimension three. Consider

$$\text{Shear flows:} \quad \bar{u}(x_1, x_2) = (b(x_2), 0)$$

$$\text{Vortices:} \quad \bar{u}(x) = \zeta(r)x^{\perp}, \quad \bar{\omega}(x) = g(r), \quad x \in \mathbb{R}^2, \ r = |x|$$

Linearized Euler equations around vortex:

$$\partial_t \omega + \underbrace{\zeta(r)\partial_\theta \omega + (u \cdot e_r)g'(r)}_{=:-L_{\mathrm{st}}\omega} = 0, \quad u = \mathrm{BS}_{\mathrm{2d}}[\omega] \ \text{(Biot-Savart law)}$$

Remarks on stability (see (Drazin-Reid, Chapters 3-4))

- *Rayleigh's stability criterion* (Rayleigh 1880): If $g'(r) < 0$ for all $r > 0$, then there are no unstable eigenvalues (spectral stability).
- Stability of these vortices is by *inviscid damping*. Nonlinear stability is a well known open problem.

Unstable vortices

Theorem (Vishik, arXiv '18)

There exist $m \geq 2$ and a smooth, decaying, radially symmetric vorticity profile $\bar{\omega}$ such that the linearized operator $L_{\mathrm{st}} : D(L_{\mathrm{st}}) \subset L^2_m \to L^2_m$,

$$-L_{\mathrm{st}}\omega := \bar{u} \cdot \nabla \omega + u \cdot \nabla \bar{\omega} \,, \quad u = \mathrm{BS}_{2\mathrm{d}}[\omega] \,,$$

has an unstable eigenvalue. Moreover, the velocity profile $\bar{u}$ can be chosen to be compactly supported (ABC 2021).

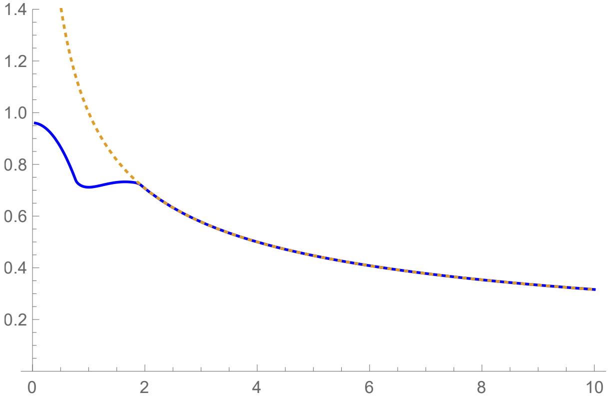
$$L^2_m = \{m\text{-fold rotationally symmetric functions}\}, \quad m \geq 2$$

(removes ambiguity in the Biot-Savart law on L^2)

Remarks

- Instability of shears explored in (Tollmien 1934), rigorously in (Lin, SIMA 2002), see also influential paper (Fadeev 1971).
- Vishik's mechanism is the same and, to our knowledge, is the *only known mechanism* for generating unstable eigenvalues.

Example vorticity profile



Solid blue: Background vorticity profile $\bar{\omega}(x) = g(r)$.
Dashed orange: Power law vortex $r^{-\alpha}$.

The vortex will satisfy

(i) g has exactly two zeros

(ii) $\zeta' < 0$ (decreasing "differential rotation")

Rewrite the eigenvalue equation $L_{\mathrm{st}}\omega = \lambda\omega$ as

Rayleigh's stability equation:

$$(\Xi(s) - c)\left(\frac{d^2}{ds^2} - m^2\right)\phi - A(s)\phi = 0\,.$$

- $s = \log r$... exponential coordinates
- $\psi(e^s) = \phi(s)e^{im\theta}$... separation of variables
- $A(s) = e^s g'(e^s)$... $r \times$ the radial derivative of vorticity
- $\Xi(s) = \zeta(e^s)$... "differential rotation"
- $\lambda = -imc$... unstable eigenvalue $\iff \mathrm{Im}\, c > 0$

Constructing the unstable vortex, cont.

Rayleigh's stability equation:

$$(\Xi(s) - c)\left(\frac{d^2}{ds^2} - m^2\right)\phi - A(s)\phi = 0\,.$$

- Let $a < b$ be the two zeros of A.
- Let $c_a = \Xi(a)$ and $c_b = \Xi(b)$ be the "critical values."

You can divide by $\Xi(s) - c$ and consider a *steady Schrödinger equation* when

- $\mathrm{Im}\; c \neq 0$ (stable or unstable eigenvalue), or
- $c = c_a, c_b$ (neutral eigenvalue)

Rayleigh's stability equation, divided:

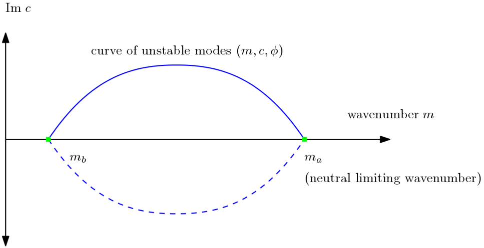
$$-\frac{d^2\phi}{ds^2} + m^2\phi + \frac{A(s)}{\Xi(s) - c}\phi = 0$$

Define the Schrödinger operators

$$L_a := -\frac{d^2}{ds^2} + \frac{A(s)}{\Xi(s) - c_a}, \quad L_b \text{ similarly}$$

Consider $-m_a^2 < -m_b^2$ the *bottom of the spectra* (choice of A).

Define the *neutral limiting modes* (m_a, c_a, ϕ_a) and (m_b, c_b, ϕ_b).



Unstable modes are found bifurcating from the neutral modes.

The unstable vortex is a key ingredient in

Theorem (Sharpness of the Yudovich class, Vishik, arXiv '18)

Let $T > 0$. For every $p \in (2, +\infty)$, there exist two distinct finite-energy weak solutions $\bar{u}, u$ of the Euler equations on $\mathbb{R}^2 \times (0, T)$, further satisfying $\bar{\omega}, \omega \in L^\infty_t(L^1 \cap L^p)_x$, with identical body force

$$f \in L^1_t L^2_x, \quad \mathrm{curl}\, f \in L^1_t(L^1 \cap L^p)_x$$

and zero initial velocity (see also (ABCDGJK, arXiv '21)).

Theorem (A.-Colombo, in preparation)

For each $\beta \in (0,2)$, there exists two distinct Leray solutions of the hypodissipative Navier-Stokes equations in the plane,

$$\partial_t u + u \cdot \nabla u + (-\Delta)^{\frac{\beta}{2}} u + \nabla p = f, \quad \operatorname{div} u = 0 \ \text{in}\ \mathbb{R}^2 \times \mathbb{R}_+ .$$

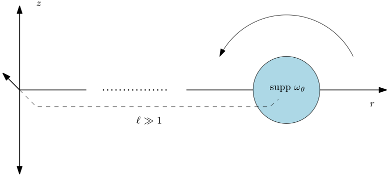
with identical initial velocity $u_0 \in L^2$ and body force $f \in L^1_t L^2_x$.

Perspective: Do not develop further spectral theory than in (Vishik '18). Choose similarity variables in which viscosity is perturbative:

$$\partial_\tau U + U \cdot \nabla U + \underbrace{e^{\tau\gamma}(-\Delta)^{\frac{\beta}{2}} U}_{\text{perturbative when } \tau \ll -1} + \nabla P = F, \quad \operatorname{div} U = 0 \ \text{in}\ \mathbb{R}^2 \times \mathbb{R} .$$

$\beta \in (0, 1]$: quasilinear regime
$\beta \in (1, 2)$: semilinear regime

Steam vortex ring above Mount Etna, Italy



Vortex ring in axisymmetric coordinates

Our unstable three-dimensional object will be a *vortex ring*.

A few mathematical works on vortex rings:

- Inviscid (Fraenkel 1970), ..., (Abe, arXiv 2020)
- Viscous (Gallay–Šverák, Ann. Sci. ÉNS 2019) (Bedrossian–Germain–Harrop-Griffiths, arXiv 2018)

As $r \to +\infty$, the axisymmetric Euler equations without swirl formally converge to the two-dimensional Euler equations.

I. Axisymmetric vorticity equation ($\omega = -\omega^\theta(r,z)e_\theta$, $\psi = \psi^\theta(r,z)e_\theta$):

$$\partial_t \omega^\theta + u \cdot \nabla \omega^\theta - \frac{u^r}{r}\omega^\theta = 0$$

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \partial_z^2\right)\psi^\theta = \omega^\theta$$

$$u = -\partial_z \psi^\theta e_r + \left(\partial_r + \frac{1}{r}\right)\psi^\theta e_z.$$

II. Two-dimensional vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

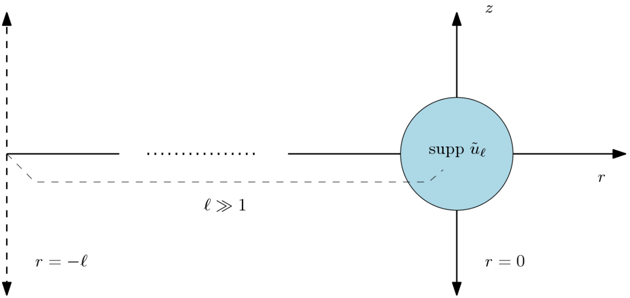
$$\Delta_{x,y}\psi = \omega, \quad u = \nabla^\perp \psi$$

Sketch of proof

$\tilde{u}$... truncation of Vishik's unstable vortex

$$\tilde{u}_\ell = \tilde{u} + v_\ell, \quad \tilde{\omega}_\ell = \mathrm{curl}_\ell\, \tilde{u}_\ell := -\partial_z \tilde{u}^r_\ell + \partial_r \tilde{u}^z_\ell\,.$$

v_ℓ is a correction to make $\tilde{u}_\ell$ divergence-free in the physical variables.



Consider

$$-\boldsymbol{L}_\ell\omega := \underbrace{\tilde{u}_\ell\cdot\nabla\omega + \frac{1}{2}(\mathrm{div}_{2d}\, v_\ell)\omega}_{=:-\boldsymbol{M}_\ell\omega} + \underbrace{\mathrm{BS}_\ell[\omega]\cdot\nabla\tilde{\omega}_\ell}_{=:-\boldsymbol{K}_\ell\omega}$$

$$\underbrace{-(r+\ell)^{-1}\mathrm{BS}_\ell[\omega]^r\tilde{\omega}_\ell - (r+\ell)^{-1}\tilde{u}^r_\ell\omega - \frac{1}{2}(\mathrm{div}_{2d}\, v_\ell)\omega}_{=:-\boldsymbol{S}_\ell\omega}$$

and

$$-\boldsymbol{L}_\infty\omega := \underbrace{\tilde{u}\cdot\nabla\omega}_{=:-\boldsymbol{M}_\infty\omega} + \underbrace{\mathrm{BS}_{2d}[\omega]\cdot\nabla\tilde{\omega}}_{=:-\boldsymbol{K}_\infty\omega}$$

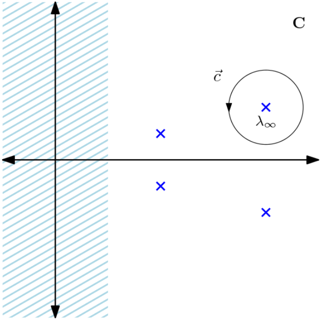
on *weighted function spaces* $L^2_{\gamma,\ell} := L^2(\mathbb{R}^2_{r>-\ell};\gamma\, dr\, dz)$.

- $\gamma\equiv 1$ on support of vortex and grows polynomially away.
- constrast with $L^2(\mathbb{R}^2_{r>-\ell};(r+\ell)\, dr\, dz)$ and $L^2(\mathbb{R}^2;\ dr\, rz)$.

Proposition (Axisymmetric instability)

Let λ_∞ be an unstable eigenvalue of $\boldsymbol{L}_\infty$. For all $\varepsilon \in (0, \operatorname{Re}\lambda_\infty)$ and $\ell \gg_{\tilde{u},\varepsilon,\lambda_\infty} 1$, $\boldsymbol{L}_\ell$ has an unstable eigenvalue λ_ℓ with $|\lambda_\ell - \lambda_\infty| < \varepsilon$.

Spectral projection $\mathbf{Pr}_\ell \omega := \frac{1}{2\pi i} \int_{\vec{c}} R(\lambda, \boldsymbol{L}_\ell)\omega \, d\lambda$



Claim: $R(\lambda, \boldsymbol{L}_\ell P_\ell)\omega \to R(\lambda, \boldsymbol{L}_\infty)\omega$ in L^2_γ, $\forall \omega \in C_0^\infty(\mathbb{R}^2)$, uniformly on $\vec{c}$.

We extend by zero via $P_\ell \omega := \mathbf{1}_{r>-\ell}\omega$.

Decompose the operator $\lambda - \boldsymbol{L}_\ell P_\ell$ as

$$
\begin{aligned}
&\lambda - \boldsymbol{M}_\ell P_\ell - \boldsymbol{K}_\ell P_\ell - \boldsymbol{S}_\ell P_\ell \\
&\quad = \lambda - \boldsymbol{M}_\ell P_\ell - \boldsymbol{K}_\infty - \underbrace{\boldsymbol{K}_\infty (P_\ell - I)}_{=:\mathrm{I}_\ell} - \underbrace{(\boldsymbol{K}_\ell P_\ell - \boldsymbol{K}_\infty) P_\ell}_{=:\mathrm{II}_\ell} - \boldsymbol{S}_\ell P_\ell \\
&\quad = \underbrace{(\lambda - \boldsymbol{M}_\ell P_\ell - \boldsymbol{K}_\infty)}_{\text{resolvent converge}} [I - \underbrace{R(\lambda, \boldsymbol{M}_\ell P_\ell + \boldsymbol{K}_\infty)}_{\text{uniformly bounded}} \underbrace{(\mathrm{I}_\ell + \mathrm{II}_\ell + \boldsymbol{S}_\ell P_\ell)}_{\to 0 \text{ in operator norm}}]
\end{aligned}
$$

Term-by-term:

- $\mathrm{I}_\ell := \boldsymbol{K}_\infty (P_\ell - I)$: Projects onto $\{r < -\ell\}$, applies 2d Biot-Savart law, and restricts to $r = O(1)$. Small due to weights.
- $\mathrm{II}_\ell := (\boldsymbol{K}_\ell P_\ell - \boldsymbol{K}_\infty) P_\ell$: Captures difference between the two Biot-Savart laws at $r = O(1)$. Apply cut-offs and subtract the two PDEs for the stream functions. This is the most technical step.
- $\boldsymbol{S}_\ell$: Stretching terms have $(r + \ell)^{-1}$ in front.

The remainder of the proof

It remains to

- perturb the instability from 3d Euler equations to 3d Navier-Stokes equations in self-similarity variables: Add on $-\Delta - 1 - \xi \cdot \nabla_\xi / 2$;
- ensure instability in velocity rather than vorticity formulation (bootstrap decay of the unstable eigenfunction); and
- exhibit a non-trivial trajectory on the unstable manifold via fixed point argument: Write

  $$U = \bar{U} + U^{\mathrm{lin}} + U^{\mathrm{per}},$$

  where $U^{\mathrm{lin}} = \mathrm{Re}(e^{\tau\lambda}\eta)$ and η is a most unstable eigenfunction. Solve for $U^{\mathrm{per}} = O(e^{2\tau \mathrm{Re}\,\lambda})$.

$\square$

Questions about non-uniqueness:

- *Is it possible to remove the force?* (Potential interactions with computer-assisted proof.)
- *Is it possible to rigorously exhibit non-unique continuations of blow-ups?* (Lessons from CGL, online lecture by Šverák, 2020) ($\exists$ "reverse bubbling" in HMHF...)
- *What are the implications of non-unique continuation of blow-up (or "extreme instability") for physical theories?*

Questions about coherent structures and (in)stability:

- There should be many unstable profiles $\bar{U}$. *How generic are they? Is there an easier way to find them?*

Thank you!