Rigorous Justification of the Viscous Primitive Equations and Their Global Regularity Coupled to Moisture Dynamics

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Long Time Behavior and Singularity Formation in PDEs – Part IV
SITE New York University - Abu Dhabi
16-20 January, 2022

Rayleigh Bénard Convection / Boussinesq Approximation

- Conservation of Momentum

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0}\nabla p + f\vec{k} \times \vec{u} = gT\vec{k}$$

- Incompressibility

$$\nabla \cdot \vec{u} = 0$$

- Heat Transport and Diffusion

$$\frac{\partial}{\partial t}T - \kappa \Delta T + (\vec{u} \cdot \nabla)T = 0$$

The Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = \vec{f}$$

$$\nabla\cdot\vec{u} = 0$$

Plus Boundary conditions, say periodic in the box

$$\Omega = [0, L]^3$$

Navier-Stokes Equations

- The Three-dimensional Case

 * Global existence of the weak solutions

 * Short time existence of the strong solutions

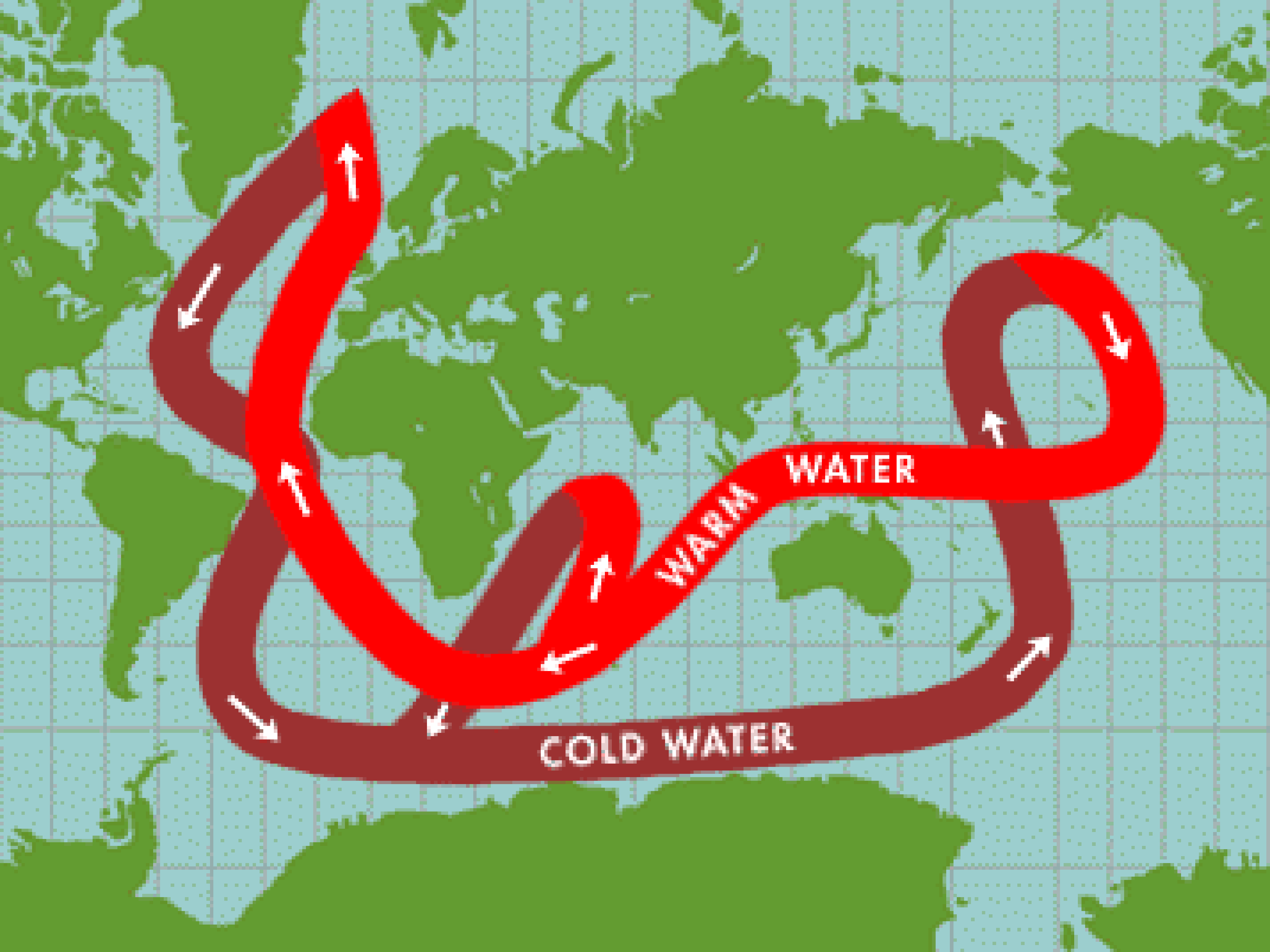
 * Uniqueness of the strong solutions

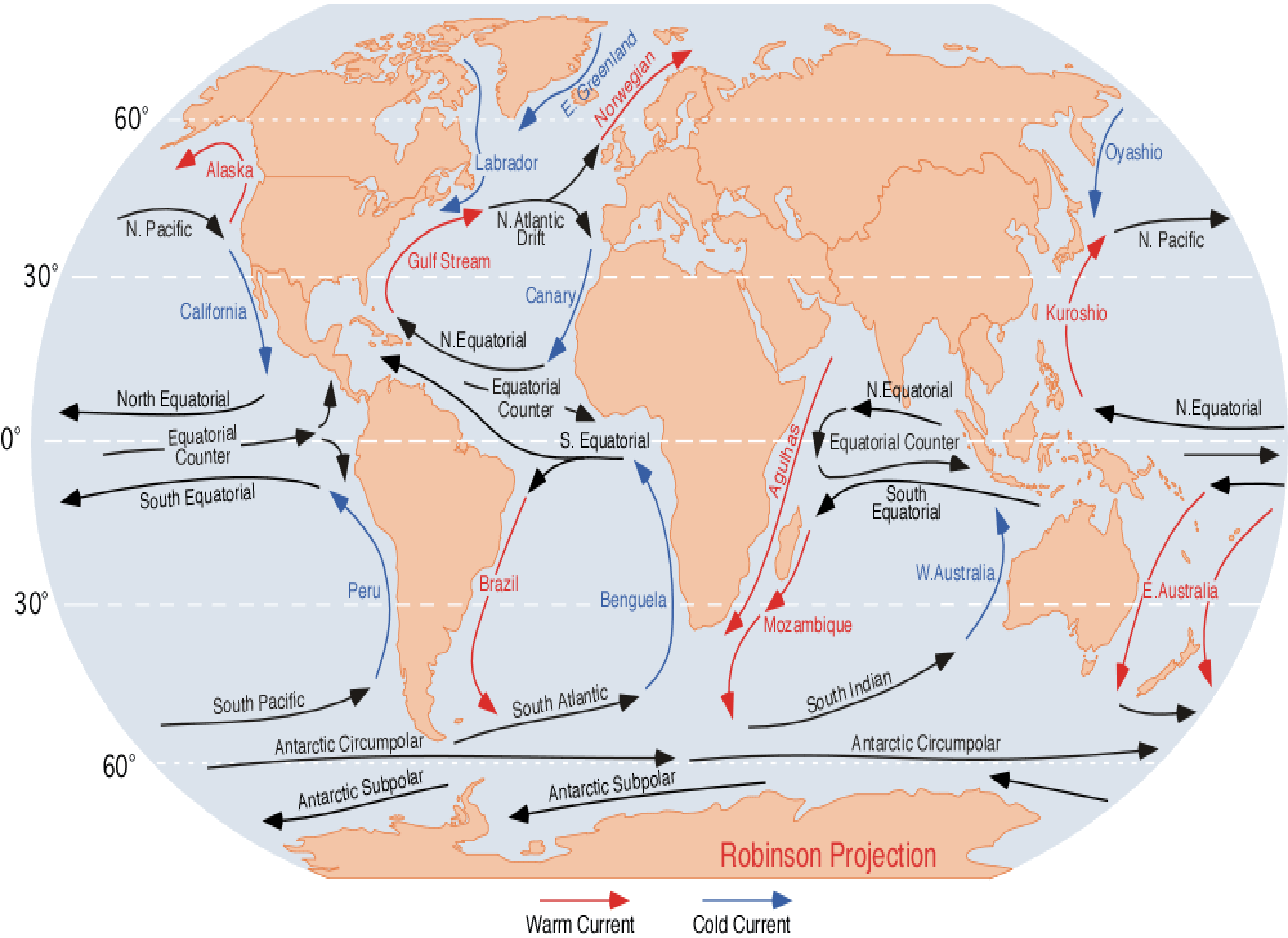
- Open Problems:

 * Uniqueness of Leray-Hopf weak solutions.

 * Global existence of the strong solution.

Large Scale Oceanic Circulations


WARM WATER
COLD WATER


60°
30°
0°
30°
60°
E. Greenland
Norwegian
Labrador
Alaska
Oyashio
N. Pacific
N. Atlantic Drift
N. Pacific
Gulf Stream
California
Canary
Kuroshio
N.Equatorial
Equatorial Counter
N.Equatorial
N.Equatorial
North Equatorial
S. Equatorial
Equatorial Counter
Equatorial Counter
South Equatorial
Agulhas
South Equatorial
Peru
Brazil
Benguela
W.Australia
E.Australia
Mozambique
South Pacific
South Atlantic
South Indian
Antarctic Circumpolar
Antarctic Circumpolar
Antarctic Subpolar
Antarctic Subpolar
Robinson Projection
Warm Current
Cold Current

Rayleigh Bénard Convection / Boussinesq Approximation

- Conservation of Momentum

$$\frac{\partial}{\partial t}\vec{u} - \upsilon\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho_0}\nabla p + f\vec{k}\times\vec{u} = gT\vec{k}$$

- Incompressibility

$$\nabla\cdot\vec{u} = 0$$

- Heat Transport and Diffusion

$$\frac{\partial}{\partial t}T - \kappa\Delta T + (\vec{u}\cdot\nabla)T = 0$$

Bénard Convection/Boussinesq Approximation

$$\frac{\partial}{\partial t}v_H - \upsilon\left(\Delta_H + \frac{\partial^2}{\partial z^2}\right)v_H + (v_H \cdot \nabla_H)v_H + w\frac{\partial}{\partial z}v_H + \frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = 0$$

$$\frac{\partial}{\partial t}w - \upsilon\left(\Delta_H + \frac{\partial^2}{\partial z^2}\right)w + (v_H \cdot \nabla_H)w + w\frac{\partial}{\partial z}w + \frac{1}{\rho_0}\frac{\partial}{\partial z}p + Tg = 0$$

$$\nabla_H \cdot v_H + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial t}T - \kappa\Delta T + (v_H \cdot \nabla_H)T + w\frac{\partial}{\partial z}T = \rho_0 Q$$

Here $(v_H, w) = \vec{u}.$

Typical Scales in the Ocean

- horizontal distance $L \sim 10^6$ m
- horizontal velocity $U \sim 10^{-1}$ m/s
- depth $H \sim 10^3$ m
- Coriolis parameter $f \sim 10^{-4}$ 1/s
- gravity $g \sim 10$ m/s^2
- density $\rho_0 \sim 10^3$ kg/m^3

Calculating the typical values

- Typical vertical velocity $W = UH/L \sim 10^{-4}$ m/s
- Typical pressure $P = \rho_0 g H \sim 10^{7}$ Pa
- Typical time scale $T = L/U \sim 10^{7}$ s

Scale Analysis of Vertical Motion – The Ideal Case

$$\frac{\partial}{\partial t}w + (v_H \cdot \nabla_H)w + w\frac{\partial}{\partial z}w + \frac{1}{\rho_0}\frac{\partial}{\partial z}p + Tg = 0$$

$$\frac{W}{\mathrm{T}} + \frac{UW}{L} + \frac{W^2}{H} + \frac{P}{H\rho_0} + Tg = 0$$

$$10^{-11} + 10^{-11} + 10^{-11} + 10 + 10 = 0$$

Hydrostatic Balance

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p + Tg = 0$$

The Primitive Equations of Large Scale Oceanic and Atmospheric Dynamics

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \nabla_H p + f \vec{k} \times v_H$$
$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p + gT = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = Q + K_h \Delta_H T + K_v T_{zz}$$

- Introduced by Richardson (1922) for weather prediction
- J.L. Lions, R. Temam, S. Wang (1992) Gave some asymptotic derivation of the PE.
- J. Li and E.S. Titi (2017) Rigorous justification of the PE, with global in time rate of convergence.

Results

- C. Cao and E.S.T. [*Annals of Mathematics* (2007)]. [arXiv March 1, 2005].

  * The global existence of the weak solutions (Galerkin method/announced)

  * The global existence and uniqueness of the strong solutions.

Primitive equations with full dissipation

- Global weak: Lions – Temam – Wang (Nonlinearity 1992A, 1992B, J. Math. Pures Appl. 1995);
- Conditional uniqueness:
  - z-weak solutions: Bresch et al (Differential Integral Equations 2003),
  - continuous initial data: Kukavica–Pei–Rusin–Ziane (Nonlinearity 2014),
  - certain discontinuous initial data: Li–Titi (2015)
- Local strong: Guillén-González, Masmoudi and Rodríguez-Bellido (Differential Integral Equations 2001);
- Global strong (2D): Bresch – Kazhikhov – Lemoine (SIAM J. Math. Anal. 2004);
- Global strong (**3D**): Cao – Titi (arXiv 2005/Ann. Math. 2007), Kobelkov (C. R. Math. Acad. Sci. Paris 2006), Kukavica – Ziane (C. R. Math. Acad. Sci. Paris 2007, Nonlinearity 2007), Hieber–Kashiwabara (Arch. Rational Mech. Anal. 2016)

Mathematical justification of the hydrostatic approximation (small aspect ratio limit)

Hydrostatic approximation

In the context of the large-scale ocean and atmosphere, an important feature is

$$\begin{aligned}\text{Aspect ratio } &= \frac{\text{the depth}}{\text{the width}}\\ &\simeq \frac{\text{several kilometers}}{\text{several thousands kilometers}}\\ &\ll 1.\end{aligned}$$

Aspect ration goes to zero $\Longrightarrow$

Hydrostatic Approximation

Formal small aspect ratio limit

Consider the anisotropic Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu_1 \Delta_H u - \nu_2 \partial_z^2 u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \quad \text{in } M \times (0, \varepsilon),$$

where $u = (v, w)$, with $v = (v^1, v^2)$, and M is a domain in $\mathbb{R}^2$.

Suppose that $\nu_1 = O(1)$ and $\nu_2 = O(\varepsilon^2)$. Changing of variables:

$$\begin{cases} v_\varepsilon(x, y, z, t) = v(x, y, \varepsilon z, t), \\ w_\varepsilon(x, y, z, t) = \frac{1}{\varepsilon} w(x, y, \varepsilon z, t), \\ p_\varepsilon(x, y, z, t) = p(x, y, \varepsilon z, t), \end{cases}$$

for $(x, y, z) \in M \times (0, 1)$.

Formal small aspect ratio limit (continue)

Then u_ε and p_ε satisfy the scaled Navier-Stokes equations

$$\begin{cases} \partial_t v_\varepsilon + (u_\varepsilon \cdot \nabla) v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = 0, \\ \nabla_H \cdot v_\varepsilon + \partial_z w_\varepsilon = 0, \\ \varepsilon^2 (\partial_t w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon - \Delta w_\varepsilon) + \partial_z p_\varepsilon = 0, \end{cases} \quad \text{in } M \times (0,1).$$

Formally, if $(v_\varepsilon, w_\varepsilon, p_\varepsilon) \to (V, W, P)$, then $\varepsilon \to 0$ yields

$$\begin{cases} \partial_t V + (U \cdot \nabla) V - \Delta V + \nabla_H P = 0, \\ \nabla_H \cdot V + \partial_z W = 0, \\ \boxed{\partial_z P = 0}, \text{(Hydrostatic Approximation)}, \end{cases} \quad \text{in } M \times (0,1).$$

where $U = (V, W)$.

Formal convergence

Formally

$$(SNS)\begin{cases}\partial_t v_\varepsilon + (v_\varepsilon\cdot\nabla_H)v_\varepsilon + w_\varepsilon\partial_z v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = 0,\\ \nabla_H\cdot v_\varepsilon + \partial_z w_\varepsilon = 0,\\ \varepsilon^2(\partial_t w_\varepsilon + v_\varepsilon\cdot\nabla_H w_\varepsilon + w_\varepsilon\partial_z w_\varepsilon - \Delta w_\varepsilon) + \partial_z p_\varepsilon = 0,\end{cases}$$

$$\downarrow \varepsilon\to 0^+$$

$$(PEs)\begin{cases}\partial_t v + (v\cdot\nabla_H)v + w\partial_z v - \Delta v + \nabla_H p = 0,\\ \nabla_H\cdot v + \partial_z w = 0,\\ \partial_z p = 0.\end{cases}$$

Weak to strong (H^1 initial data)

Theorem (Li–Titi 2015)

Given a periodic function $v_0 \in H^1$, such that $\nabla_H \cdot \left(\int_{-1}^{1} v_0 dz \right) = 0$ and $\int_\Omega v_0 dxdydz = 0$. Let $(v_\varepsilon, w_\varepsilon)$ and (v, w), respectively, be an arbitrary weak solution to (SNS) and the unique global strong solution to (PEs), with initial data v_0.

Then, we have the following strong convergences

$$(v_\varepsilon, \varepsilon w_\varepsilon) \to (v, 0), \text{ in } L^\infty(0, \infty; L^2(\Omega)),$$
$$(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, w_\varepsilon) \to (\nabla v, 0, w), \text{ in } L^2(0, \infty; L^2(\Omega)),$$

and the convergence rate is of the order $O(\varepsilon)$.

Strong to strong (H^2 initial data)

Theorem (Li–Titi 2015)

Given a periodic function $v_0 \in H^2$, such that $\nabla_H \cdot \left(\int_{-1}^{1} v dz \right) = 0$ and $\int_\Omega v_0 dxdydz = 0$. Let $(v_\varepsilon, w_\varepsilon)$ and (v, w), respectively, be the unique **local** strong solution to (SNS) and the unique global strong solution to (PEs), with initial data v_0.

(i) There is a positive constant ε_0 depending only on norm $\|v_0\|_{H^2}$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, the strong solution $(v_\varepsilon, w_\varepsilon)$ exists **globally in time**.

(ii) The following strong convergences hold

$$(v_\varepsilon, \varepsilon w_\varepsilon) \to (v, 0), \text{ in } L^\infty(0,\infty; H^1(\Omega)),$$
$$(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, w_\varepsilon) \to (\nabla v, 0, w), \text{ in } L^2(0,\infty; H^1(\Omega)),$$
$$w_\varepsilon \to w, \text{ in } L^\infty(0,\infty; L^2(\Omega)),$$

and the convergence rate is of the order $O(\varepsilon)$.

Global Regularity of the Primitive Equations with Horizontal Viscosity & Diffusion Mixing

Cao-Li-Titi [CPAM 2015]

- Global Regularity of PE with Horizontal Viscosity and Horizontal Diffusion

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \frac{1}{\rho_0} \nabla_H p + f \vec{k} \times v_H$$

$$= A_h \Delta_H v_H$$

$$\partial_z p = gT$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = \kappa_h \Delta_H T$$

Rigorous justification of PE with horizontal viscosity

- J. Li, E.S.T. and G. Yuan (2021)

When the vertical viscosity scales like $\varepsilon^{\alpha-2}$, for $\alpha > 2$. Then the solutions of the scaled three-dimensional Navier-Stokes converge to the PE equations with horizontal viscosity, and the convergence rate is $O(\varepsilon^{\beta})$ with $\beta = \min(2, \alpha - 2)$.

- For the case of $\alpha = 2$ J. Li and E.S.T.

Tropical atmosphere

- In the tropics, the wind in the lower troposphere is of equal magnitude but with opposite sign to that in the upper troposphere, in other words, the primary effect is captured in the first baroclinic mode $(cos(\pi z/H))$.
- For the study of the tropical-extratropical interactions, where the transport of momentum between the barotropic and baroclinic modes plays an important role, it is necessary to retain both the barotropic and baroclinic modes.

Anasatz

Because of the above two, we impose an ansatz of the form

$$\begin{pmatrix} V \\ \Phi \end{pmatrix}(x, y, z, t) = \begin{pmatrix} u \\ p \end{pmatrix}(x, y, t) + \sqrt{2}\cos(\pi z/H)\begin{pmatrix} v \\ p_1 \end{pmatrix}(x, y, t)$$

and correspondingly

$$\begin{pmatrix} W \\ \Theta \end{pmatrix}(x, y, z, t) = \sqrt{2}\sin(\pi z/H)\begin{pmatrix} w \\ \theta \end{pmatrix}(x, y, t),$$

which carry the barotropic and first baroclinic modes of the unknowns.

Barotropic-baroclinic interaction system

Using the above ansatz, and performing the Galerkin projection to PEs in the vertical direction (Frierson–Majda–Pauluis (Commum. Math. Sci. 2004)), one has

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p + \nabla \cdot (v \otimes v) = 0, \\ \nabla \cdot u = 0, \\ \partial_t v + (u \cdot \nabla) v - \Delta v + (v \cdot \nabla) u = \nabla \theta, \\ \partial_t \theta + u \cdot \nabla \theta - \nabla \cdot v = S_\theta, \end{cases} \tag{1}$$

in $\mathbb{R}^2$, where $u = (u_1, u_2)$ is the barotropic velocity, and $v = (v_1, v_2)$, p and θ, respectively, are the first baroclinic modes of the velocity, pressure and the temperature, S_θ is the temperature source.

Remarks

Remark

- The subsystem of (1) for (u, v) "looks like" the Magnetohydrodynamics (MHD), but $\nabla \cdot v = 0$ is not preserved by the system, i.e. v is compressible;
- The subsystem of (1) for (v, θ) is the linearize compressible Navier-Stokes equations around $(0, 1)$;
- The corresponding inviscid system of (1) has one derivative loss for (u, v), and thus one can not close the energy estimate at finite order derivatives (need appeal to infinite order derivatives).

The moisture equation (for the tropical atmosphere)

For the tropical atmosphere, following Frierson–Majda–Pauluis (Commum. Math. Sci. 2004), we use the following large-scale moisture equation:

$$\partial_t q + u \cdot \nabla q + \bar{Q} \nabla \cdot v = -P,$$

where $\bar{Q}$ is the described gross moisture stratification, and P **is the precipitation**, given by

$$P = \frac{1}{\varepsilon}(q - \alpha\theta - \hat{q})^+,$$

where $0 < \varepsilon << 1$ is the convective adjustment time scale parameter, α and $\hat{q}$ are constants, with $\hat{q} > 0$.

A Tropical Atmosphere Model with Moisture

- Frierson-Majda-Pauluis(2004)

$$
\begin{aligned}
&\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p + \nabla \cdot (v \otimes v) = 0, \\
&\nabla \cdot u = 0, \\
&\partial_t v + (u \cdot \nabla) v - \Delta v + (v \cdot \nabla) u = \frac{1}{1+\alpha} \nabla (T_e - q_e), \\
&\partial_t T_e + u \cdot \nabla T_e - (1 - \bar{Q}) \nabla \cdot v = 0, \\
&\partial_t q_e + u \cdot \nabla q_e + (\bar{Q} + \alpha) \nabla \cdot v = -\frac{1+\alpha}{\varepsilon} q_e^{+},
\end{aligned}
$$

Relaxation Limit Model

$$
\begin{aligned}
&\partial_t u + (u \cdot \nabla) u - \mu \Delta u + \nabla p + \nabla \cdot (v \otimes v) = 0, \\
&\nabla \cdot u = 0, \\
&\partial_t v + (u \cdot \nabla) v - \mu \Delta v + (v \cdot \nabla) u = \frac{1}{1+\alpha} \nabla (T_e - q_e), \\
&\partial_t T_e + u \cdot \nabla T_e - (1 - \bar{Q}) \nabla \cdot v = 0, \\
&\partial_t q_e + u \cdot \nabla q_e + (\bar{Q} + \alpha) \nabla \cdot v \leq 0, \\
&q_e \leq 0, \\
&\partial_t q_e + u \cdot \nabla q_e + (\bar{Q} + \alpha) \nabla \cdot v = 0, \quad \text{a.e. on } \{q_e < 0\}.
\end{aligned}
$$

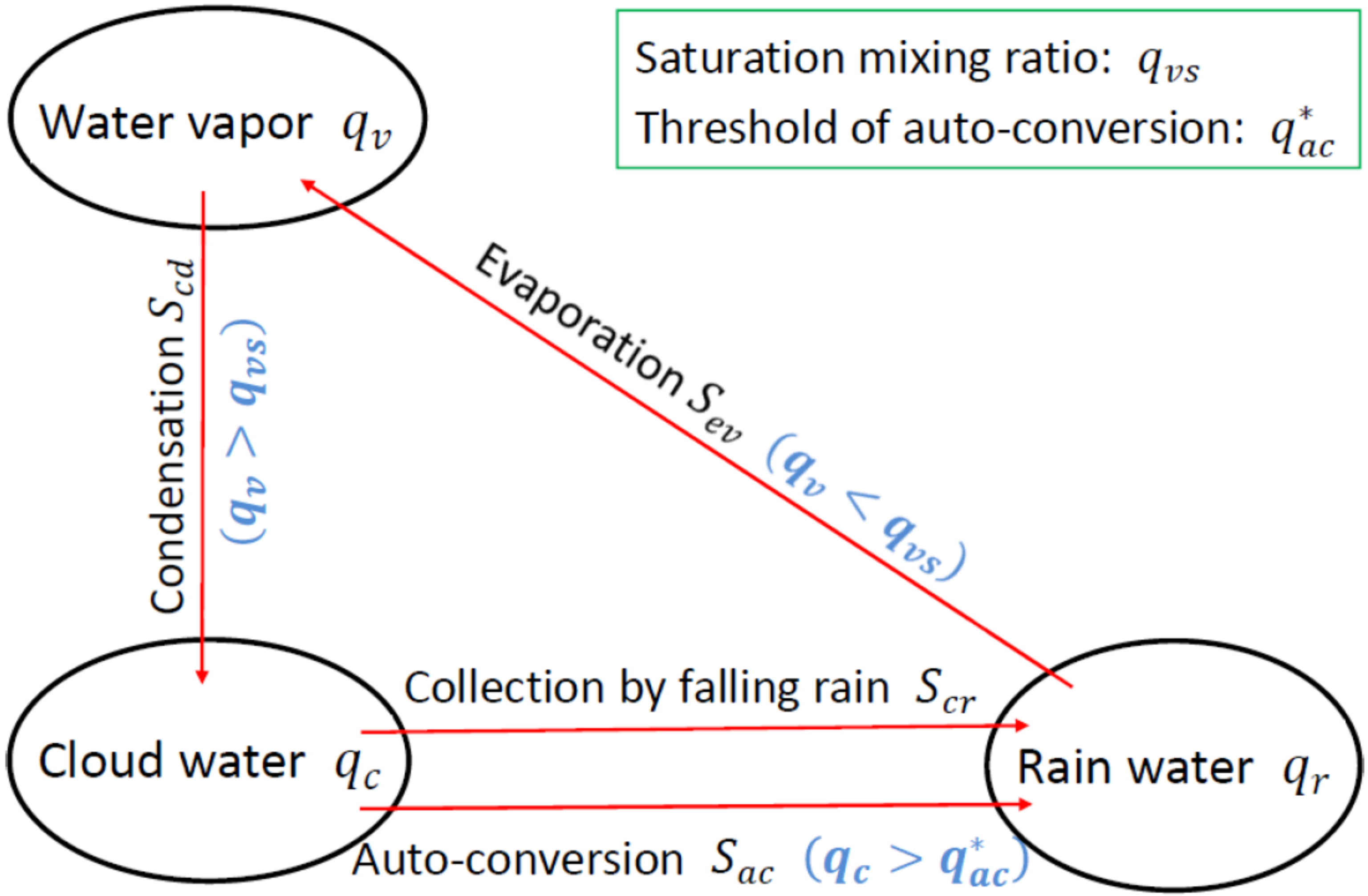
Resutls Concerning the Tropical Models with Moisture

- Global regularity and justification of the relaxation limit of the linearized version of the inviscid model [Majda-Souganidis (2010)].
- Global well-posedness and justification of the nonlinear relaxation limit of the viscous model. Explicit rate of convergence $\sqrt{\varepsilon}$ [Li-Titi (2015)]

A moisture model for warm clouds

Phase and phase changes

Mixing ratios: $q_j = \frac{\rho_j}{\rho_d}, j \in \{v, c, r\}$.



Assumptions on q_{vs} and source terms

The saturation mixing ratio $q_{vs} = q_{vs}(p, T)$ satisfies:

- nonnegative and bounded;
- Lipschitz in the temperature T, uniformly in the pressure p.

Source terms:

$$\begin{aligned}
S_{ev} &= C_{ev}\, T (q_r^+)^\beta (q_{vs} - q_v)^+ , \qquad \beta \in (0,1), \\
S_{cd} &= C_{cd}(q_v - q_{vs}) q_c + C_{cn}(q_v - q_{vs})^+ , \\
S_{cr} &= C_{cr}\, q_c\, q_r \\
S_{ac} &= C_{ac}(q_c - q_{ac}^*)^+ , \\
S_T &= \frac{L}{c_p}(S_{cd} - S_{ev}),
\end{aligned}$$

where C_{ev}, C_{cr}, C_{ac}, C_{cd}, C_{cn}, L, and c_p are positive constants.

The moisture system with temperature

Moisture balances and thermodynamic equation (Klein–Majda (Theor. Comput. Fluid Dyn. 2006)):

$$
\begin{aligned}
\frac{Dq_v}{Dt} &= S_{ev} - S_{cd} + \mathcal{D}^{q_v} q_v \,, \\
\frac{Dq_c}{Dt} &= S_{cd} - S_{ac} - S_{cr} + \mathcal{D}^{q_c} q_c \,, \\
\frac{Dq_r}{Dt} - \frac{V}{g\rho}\partial_z(\rho q_r) &= S_{ac} + S_{cr} - S_{ev} + \mathcal{D}^{q_r} q_r, \\
\frac{DT}{Dt} - \frac{RT}{c_p p}\omega &= \frac{L}{c_p}(S_{cd} - S_{ev}) + \mathcal{D}^T T,
\end{aligned}
$$

where, for $f \in \{q_v, q_c, q_r, T\}$,

$$
\frac{Df}{Dt} = \partial_t f + u\partial_x f + v\partial_y f + \omega\partial_p f,
$$

$$
\mathcal{D}^f f = \mu_f \Delta_h f + \nu_f \partial_p \left(\left(\frac{gp}{R\bar{T}} \right)^2 \partial_p f \right).
$$

Boundary conditions

Domain and boundaries:

$$p_0: \text{ pressure on the ground (bottom);} \quad p_1: \text{ pressure at the top,}$$
$$\mathcal{M} = \mathcal{M}' \times (p_1, p_0), \quad \mathcal{M}' \subseteq \mathbb{R}^2 \text{ bounded,}$$
$$\Gamma_0 = \mathcal{M}' \times \{p_0\}, \quad \text{(bottom)}, \quad \Gamma_1 = \mathcal{M}' \times \{p_1\}, \quad \text{(top)}$$
$$\Gamma_\ell = \partial\mathcal{M}' \times (p_1, p_0), \quad \text{(lateral boundary).}$$

Boundary conditions:

$$\begin{aligned}
\Gamma_0: &\quad \partial_p T = \alpha_{0T}(T_{b0} - T), &\quad \partial_p q_j = \alpha_{0j}(q_{b0j} - q_j), \\
\Gamma_1: &\quad \partial_p T = 0, &\quad \partial_p q_j = 0, \\
\Gamma_\ell: &\quad \partial_n T = \alpha_{\ell T}(T_{b\ell} - T), &\quad \partial_n q_j = \alpha_{\ell j}(q_{b\ell j} - q_j),
\end{aligned}$$

where $\alpha_{0j}, \alpha_{\ell j}, \alpha_{0T}, \alpha_{\ell T}$ and T_{b0}, $T_{b\ell}$, q_{b0j}, $q_{b\ell j}$ are nonnegative, sufficiently smooth, and uniformly bounded.

Global well-posedness

Theorem (Global well-posedness) (Hittmeir–Klein–Li–Titi Nonlinearity 2017)

Let $(T_0, q_{v0}, q_{c0}, q_{r0}) \in L^\infty \cap H^1$ be nonnegative and given velocity field $(\mathbf{v}_h, \omega)$

$$(\mathbf{v}_h, \omega) \in L_t^\infty(L_X^2) \cap L_t^2(H_X^1) \cap L_t^r(L_X^\sigma),$$
$$\nabla_h \cdot \mathbf{v}_h + \partial_p \omega = 0, \quad \mathbf{v}_h \cdot \mathbf{n}_h + \omega n_p = 0, \quad \text{on } \partial\mathcal{M},$$
$$\frac{2}{r} + \frac{3}{\sigma} < 1, \quad r \in [2, \infty], \quad \sigma \in [3, \infty].$$

$\Longrightarrow \exists!$ **global solution** (T, q_v, q_c, q_r), satisfying

$$(T, q_v, q_c, q_r) \in L_t^\infty(L_X^\infty) \cap L_t^2(H_X^2),$$
$$(T, q_v, q_c, q_r) \in C([0, \mathcal{T}]; H^1),$$
$$(\partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r) \in L_t^2(L_X^2).$$

Uniqueness

Recall the equation for q_v

$$\frac{Dq_v}{Dt} = S_{ev} - S_{cd} + \mathcal{D}^{q_v} q_v \ ,$$

where S_{ev} is given by

$$S_{ev} = C_{ev} T (q_r^+)^\beta (q_{vs} - q_v)^+, \quad \beta \in (0, 1].$$

$$\Downarrow$$

Energy estimate to the difference system for (q_v, q_r, q_c, T) does not lead to the uniqueness !!

Observations for the uniqueness

We will introduce

$$Q = q_v + q_r, \quad H = T - \frac{L}{c_p}(q_c + q_r),$$

and use (Q, q_c, q_r, H) as the new unknowns.

Remark: advantages of the system (Q, q_c, q_r, H)

- in the system for (Q, q_c, q_r, H), the fractional power nonlinearity S_{ev} **appears only in the equation for** q_r;
- in the equation for q_r, the monotonicity of S_{ev} w.r.t q_r guarantees the uniqueness $S_{ev} = C_{ev} T (q_r^+)^\beta (q_{vs} - q_v)^+$:

$$(q_{r1}^\beta - q_{r2}^\beta)(q_{r1} - q_{r2}) \geq 0.$$

Coupling microphysics to the primitive equations

Theorem (Global well-posedness) (Hittmeir–Klein–Li–Titi 2019)

Assume that $\mathbf{u}_0, T_0, q_{v0}, q_{c0}, q_{r0} \in H^1(\mathcal{M})$ and $T_0, q_{v0}, q_{c0}, q_{r0} \in L^\infty(\mathcal{M})$, with $T_0,\ q_{v0},\ q_{c0},\ q_{r0} \geq 0$ in $\mathcal{M}$ and $\int_{p_0}^{p_1} \nabla_h \cdot \mathbf{u}_0 dp = 0$ on $\mathcal{M}'$.

Then, the coupled microphysics dynamical system with the primitive equations has a unique global in time solution $(\mathbf{u}, T, q_v, q_c, q_r)$, satisfying:

$$T, q_v, q_c, q_r \geq 0 \quad \text{and} \quad T, q_v, q_c, q_r \in L^\infty(0, \mathcal{T}; L^\infty(\mathcal{M})),$$
$$\mathbf{u}, T, q_v, q_c, q_r \in C([0, \mathcal{T}]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^2(\mathcal{M})),$$
$$\partial_t \mathbf{u}, \partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r \in L^2(0, \mathcal{T}; L^2(\mathcal{M})),$$

for any $\mathcal{T} \in (0, \infty)$.

Primitive Equations with a Moisture Model

- Coti-Zelati-Huang-Kukavica-Temam-Ziane (2015)

Compressible Primitive Equations
[X. Liu and E.S.T.]

- Global existence and uniqueness of the compressible PE with small data.
- Convergence of the compressible Navier-Stokes to the compressible PE as the aspect ratio tends to zero.
- Convergence of the compressible PE to the incompressible PE as the Mach number tends to zero.

Thank You!