

On the stability of the Ginzburg-Landau vortex

Eliot Pacherie (joint work with Philippe Gravejat and Didier Smets)

NYUAD Research Institute, Abu Dhabi

20/01/2022 - Abu Dhabi

The Gross-Pitaevskii equation

$$(GP) : \begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u \\ |u| \rightarrow 1 \text{ when } |x| \rightarrow \infty. \end{cases}$$

$$u : \mathbb{R}_x^2 \times \mathbb{R}_t \rightarrow \mathbb{C}.$$

Connected to Euler by the Madelung transform. Energy of a solution:

$$E_{GP}(u) := \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (|u|^2 - 1)^2.$$

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Cauchy problem: Global well posedness in

- $1 + H^1$ (Bethuel-Saut)
- The energy space (Gerard)
- More complex spaces (Gallo, Bethuel-Smets)

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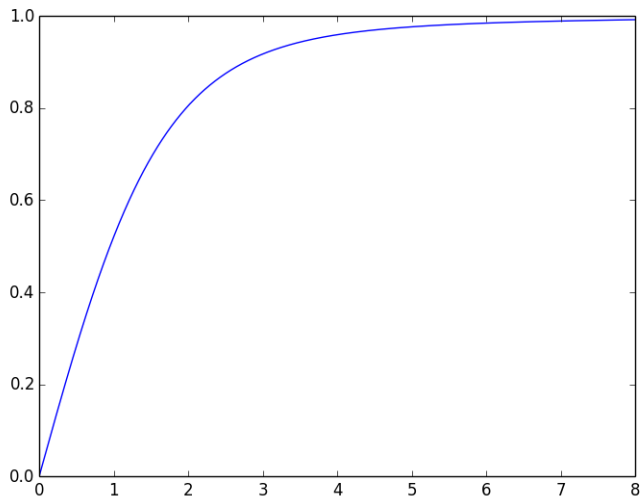
Theorem (Hervé, Hervé)

For all $n \in \mathbb{Z}^*$, there exists $V_n(x) = \rho_n(r)e^{in\theta}$ such that

$$\Delta V_n = (|V_n|^2 - 1)V_n$$

with $\rho_n(0) = 0, \rho_n(+\infty) = 1$.

Graph of ρ_1



About vortices of degrees ± 1

For $X \in \mathbb{R}^2, \gamma \in \mathbb{R}$,

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Question : Orbital stability of V_1 .

Coercivity around the vortex

$(GP)(V_1 + \varepsilon) = L(\varepsilon) + NL(\varepsilon)$ with

Linearized operator:

$$L_{V_1}(\varepsilon) := -\Delta\varepsilon - (1 - |V_1|^2)\varepsilon + 2\Re(\overline{V_1}\varepsilon)V_1$$

Quadratic form:

$$B(\varepsilon) := \int_{\mathbb{R}^2} |\nabla\varepsilon|^2 - (1 - |V_1|^2)|\varepsilon|^2 + 2\Re^2(\overline{V_1}\varepsilon)$$

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Theorem (Del Pino, Felmer, Kowalczyk)

For $\varepsilon \in H$ with $\|\varepsilon\|_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\varepsilon)|^2 + \frac{|\nabla\varepsilon|^2}{1+r^2}$,

$$B(\varepsilon) \geq \kappa(\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \Re^2(\overline{V_1}\varepsilon)).$$

$$\|\varepsilon\|_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1\varepsilon})|^2 + \frac{|\nabla\varepsilon|^2}{1+r^2}$$

controls

$$\|\chi\varepsilon\|_{L^2} \leq K\|\chi\overline{\nabla V_1\varepsilon}\|_{L^2} \leq K(\|\chi\nabla(\overline{V_1\varepsilon})\|_{L^2} + \|\chi\nabla\varepsilon\|_{L^2})$$

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controls

$$\|\chi\varepsilon\|_{L^2} \leq K\|\chi\nabla\overline{V_1}\varepsilon\|_{L^2} \leq K(\|\chi\nabla(\overline{V_1}\varepsilon)\|_{L^2} + \|\chi\nabla\varepsilon\|_{L^2})$$

and by Hardy inequality,

$$\int_{\mathbb{R}^2} \frac{|V_1|^2|\varepsilon|^2}{(1+r^2)\ln(2+r)^2} \leq K(\|\chi\varepsilon\|_{L^2} + \|\nabla(\overline{V_1}\varepsilon)\|_{L^2}).$$

Sketch of the proof of the coercivity

- Decompose $\varepsilon = e^{i\theta} \sum_{j \in \mathbb{Z}} \phi_j(r, \theta)$ in well chosen harmonics

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- $B_{\pm 1}(\phi_{\pm 1}) = \int_{\mathbb{R}^2} |\partial_r \phi_{\pm 1} - A(r)\phi_{\pm 1}|^2$ which cancels only if $\varepsilon = \partial_{x_{1,2}} V_1$

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- B_j for $|j| \geq 2$ is always coercive

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- 3) $B(\varepsilon) \geq \kappa \|\varepsilon\|^2$ under three orthogonality conditions
- 4) $NL(\varepsilon) \geq -K \|\varepsilon\|^3$
- 5) $1 \gg E_{GP}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) - E_{GP}(V_1) \geq \kappa \|\varepsilon\|^2(1 - K \|\varepsilon\|)$

The energy

$$E_{GP}(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (|u|^2 - 1)^2 = \int_{\mathbb{R}^2} e_{GP}(u).$$

We have $E_{GP}(V_1) = +\infty$ because $\nabla V_1 \sim \frac{e_\theta}{r} \notin L^2(\mathbb{R}^2)$.

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We define the renormalized energy by

$$\mathcal{E}(u) := \lim_{r \rightarrow \infty} \int_{B_r} e_{GP}(u) - e_{GP}(V_1)$$

and the space

$$E := \{\psi \in H_{loc}^1(\mathbb{R}^2), \|\psi\|_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}$$

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Proposition

\mathcal{E} is well defined on E and is invariant by translation and shift of phase. Furthermore, it is conserved by the flow.

Minimizing the energy

We have

$$E_{GP}(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (|u|^2 - 1)^2 = \int_{\mathbb{R}^2} e_{GP}(u),$$

$$\mathcal{E}(u) := \lim_{r \rightarrow \infty} \int_{B_r} e_{GP}(u) - e_{GP}(V_1)$$

and

$$E := \{\psi \in H_{loc}^1(\mathbb{R}^2), \|\psi\|_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}.$$

Theorem (Mironescu)

The functional \mathcal{E} satisfies $\mathcal{E}(u) \geq 0$ for $u \in E$. Furthermore, if $\mathcal{E}(u) = 0$, then there exists $a \in \mathbb{R}^2, \gamma \in \mathbb{R}$ such that

$$u = V_1(\cdot - a)e^{i\gamma}.$$

- 1) The renormalized energy \mathcal{E} is conserved by the flow
- 2) $\mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) = B(\varepsilon) + NL(\varepsilon)$
- 3) $B(\varepsilon) \geq \kappa \|\varepsilon\|^2$ under three orthogonality conditions
- 4) $NL(\varepsilon) \geq -K \|\varepsilon\|^3$
- 5) $1 \gg \mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) \geq \kappa \|\varepsilon\|^2 (1 - K \|\varepsilon\|)$

Decomposition of the renormalized energy

We have $\mathcal{E}(V_1 + \varepsilon) = B(\varepsilon) + NL(\varepsilon)$ with

$$B(\varepsilon) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^2 - (1 - |V_1|^2) |\varepsilon|^2 + 2\Re \varepsilon^2(\overline{V_1} \varepsilon)$$

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$$\|\varepsilon\|^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1\varepsilon})|^2 + \frac{|\nabla\varepsilon|^2}{1+r^2} + \Re e^2(\overline{V_1\varepsilon}) + \frac{|\varepsilon|^2}{(1+r^2)\ln(2+r)^2}$$

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$$\int_{\mathbb{R}^2} \Re(\overline{V_1}\varepsilon)|\varepsilon|^2 \geq -K\|\varepsilon\| \times \|\varepsilon\|_{L^4(\mathbb{R}^2)}^2$$

But $\|\varepsilon\|_{L^4(\mathbb{R}^2)}$ can not be controlled by $\|\varepsilon\|$

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Remark on the nonlinear part

We have

$$NL(\varepsilon) = \int_{\mathbb{R}^2} \Re(\overline{V_1\varepsilon})|\varepsilon|^2 + \frac{1}{4}|\varepsilon|^4 = \frac{1}{4} \int_{\mathbb{R}^2} (2\Re(\overline{V_1\varepsilon}) + |\varepsilon|^2)^2 - \int_{\mathbb{R}^2} \Re^2(\overline{V_1\varepsilon}).$$

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We try the decomposition

$\mathcal{E}(V_1 + \varepsilon) = \tilde{B}(\varepsilon) + \tilde{N}L(\varepsilon)$ with

$$\tilde{N}L(\varepsilon) := \frac{1}{4} \int_{\mathbb{R}^2} (2\Re(\overline{V_1}\varepsilon) + |\varepsilon|^2)^2.$$

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- 3) $\tilde{B}(\varepsilon) \geq \kappa \|\varepsilon\|^2$ under three orthogonality conditions
- 4) $\tilde{N}L(\varepsilon) \geq 0$
- 5) $1 \gg \mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) \geq \kappa \|\varepsilon\|^2$

The new quadratic form

The quadratic form

$$\tilde{B}(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^2 - (1 - |V_1|^2) |\varepsilon|^2$$

has infinitely many negative directions.

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Remark: $\|\varepsilon\|_H$ does not control $\|\varepsilon\|_{L^4(\mathbb{R}^2)}$ but it controls $\|\varepsilon\|_{L^4_{loc}(\mathbb{R}^2)}$.

Decomposition with a cutoff

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For $R > 0$ we define χ_R a cutoff with $\chi_R = 1$ in $B(0, R)$ and 0 outside of $B(0, 2R)$. We decompose

$$\mathcal{E}(V_1 + \varepsilon) = B_R(\varepsilon) + N_R(\varepsilon)$$

with

$$B_R(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^2 - (1 - |V_1|^2) |\varepsilon|^2 + 2\chi_R^2 \Re^2(\overline{V_1} \varepsilon)$$

and

$$N_R(\varepsilon) := \frac{1}{4} \int_{\mathbb{R}^2} (1 - \chi_R^2) (2\Re(\overline{V_1} \varepsilon) + |\varepsilon|^2)^2 + \chi_R^2 (4\Re(\overline{V_1} \varepsilon) |\varepsilon|^2 + |\varepsilon|^4).$$

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The quadratic form B_R has infinitely many negative eigenvalues. This is because of the term $\mathcal{P}_R(\varepsilon) := P_R(V_1 + \varepsilon)$ where

$$P_R(\psi) := \int_{\mathbb{R}^2} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \Re(i\bar{\psi} V_1 \nabla(\psi \bar{V}_1)).$$

We decompose

$$\mathcal{E}(V_1 + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_R(\varepsilon) + \mathcal{N}_R(\varepsilon).$$

The final decomposition

- 1) The renormalized energy \mathcal{E} is conserved by the flow
- 2) $\mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_R(\varepsilon) + \mathcal{N}_R(\varepsilon)$
- 3) $\mathcal{B}_R(\varepsilon) \geq \kappa \|\varepsilon\|_?^2$ under three orthogonality conditions
- 4) $\mathcal{N}_R(\varepsilon) + \mathcal{P}_R(\varepsilon) \geq -K \|\varepsilon\|_?^3$
- 5) $1 \gg \mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) \geq \kappa \|\varepsilon\|_?^2 (1 - K \|\varepsilon\|_?)$

Proposition

There exist universal constants $\kappa_0 > 0$ and $N_0 > 0$ such that, given any $\varepsilon \in H$ verifying the orthogonality conditions

$$\int_{\mathbb{R}^2} \chi \Re(\bar{\varepsilon} \partial_{x_1} V_1) = \int_{\mathbb{R}^2} \chi \Re(\bar{\varepsilon} \partial_{x_2} V_1) = \int_{\mathbb{R}^2} \chi \Re(\bar{\varepsilon} i V_1) = 0,$$

and for any $R_0 \geq 1$, there exists $R_0 \leq R \leq 2^{N_0} R_0$ such that

$$\mathcal{B}_R(\varepsilon) \geq \kappa_0 (\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \Re^2(\bar{\varepsilon} V_1)).$$

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Sketch of the proof

- We decompose ε in harmonics
- Locally the analysis is similar to Del Pino, Felmer, Kowalczyk (working with $\chi_R \varepsilon$)
- We have removed the non controllable part $\mathcal{P}_R(\varepsilon)$ at infinity
- The cutoff location R is chosen to glue these two estimates together

The key difficulty is to have

$$\int_{\mathbb{R}^2} |\nabla(\chi_R \overline{V_1} \varepsilon)|^2 + |\nabla((1 - \chi_R) \overline{V_1} \varepsilon)|^2 \geq \kappa \int_{\mathbb{R}^2} |\nabla(\overline{V_1} \varepsilon)|.$$

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- 2) $\mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_R(\varepsilon) + N_R(\varepsilon)$
- 3) $\mathcal{B}_R(\varepsilon) \geq \kappa_0(\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \Re^2(\bar{\varepsilon} V_1))$ under three orthogonality conditions
- 4) $N_R(\varepsilon) + \mathcal{P}_R(\varepsilon) \geq -K\|\varepsilon\|_?^3$
- 5) $1 \gg \mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) \geq \kappa\|\varepsilon\|_?^2(1 - K\|\varepsilon\|_?)$

The two "nonlinear" terms

We estimate

$$N_R(\varepsilon) \geq \frac{\kappa_0}{8} \|\eta_\varepsilon\|_{L^2(\mathbb{R}^2)}^2 - \|\varepsilon\|_{L^3(B_{2R})}^3 - \frac{\kappa_0}{2} \int_{\mathbb{R}^2} \chi_R^2 \Re^2(\bar{\varepsilon} V_1)$$

where $\eta_\varepsilon = 2\Re(\bar{\varepsilon} V_1) + |\varepsilon|^2$

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where $\eta_\varepsilon = 2\Re e(\bar{\varepsilon} V_1) + |\varepsilon|^2$

Lemma

There exist universal constant $\delta > 0$, $\Lambda > 0$ and $K \geq 1$ such that, for $V_1 + \varepsilon \in E$, if $R \geq \Lambda$ and $\|\varepsilon\|_H + \|\eta_\varepsilon\|_{L^2} \leq \delta$, then

$$|\mathcal{P}_R(\varepsilon)| \leq \frac{K}{R} (\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2).$$

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Lemma

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$$|\mathcal{P}_R(\varepsilon)| \leq \frac{K}{R} (\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2).$$

This uses a decomposition in frequencies of ε first introduced by Patrick Gérard.

We choose R_0 large enough such that $\frac{K}{R} \leq \frac{\kappa_0}{16}$

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- 2) $\mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_R(\varepsilon) + N_R(\varepsilon)$
- 3) $\mathcal{B}_R(\varepsilon) \geq \kappa_0(\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \Re^2(\bar{\varepsilon} V_1))$ under three orthogonality conditions
- 4) $N_R(\varepsilon) + \mathcal{P}_R(\varepsilon) \geq \frac{\kappa_0}{16}(\|\eta_\varepsilon\|_{L^2}^2 - \|\varepsilon\|_H^2) - K\|\varepsilon\|_H^3$
- 5) $1 \gg \mathcal{E}(V_1(\cdot - X(t))e^{i\gamma(t)} + \varepsilon) \geq \frac{\kappa_0}{16}(\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2) - K\|\varepsilon\|_H^3$

Main theorem

We recall

$$E = \{\psi \in H_{loc}^1(\mathbb{R}^2), \|\psi\|_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}$$

for $\|\psi\|_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\psi)|^2 + \frac{|\nabla\psi|^2}{1+r^2}$. We consider the distance

$$d_E(\psi_1, \psi_2) := \|\psi_1 - \psi_2\|_H + \| |\psi_1|^2 - |\psi_2|^2 \|_{L^2}.$$

Main theorem

We recall

$$E = \{\psi \in H_{loc}^1(\mathbb{R}^2), \|\psi\|_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}$$

for $\|\psi\|_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\psi)|^2 + \frac{|\nabla\psi|^2}{1+r^2}$. We consider the distance

$$d_E(\psi_1, \psi_2) := \|\psi_1 - \psi_2\|_H + \| |\psi_1|^2 - |\psi_2|^2 \|_{L^2}.$$

Theorem

There exist universal constant $\delta > 0$, $C > 0$ such that, if $\psi_0 \in E$ and $d_E(V_1, \psi_0) \leq \delta$, then the solution ψ_t with initial data ψ_0 of the Gross-Pitaevskii equation satisfies

$$d_E(V_1, \text{Orb}(\psi_t)) \leq C d_E(V_1, \psi_0)$$

for any $t \in \mathbb{R}$, where $\text{Orb}(\psi) := \{e^{i\gamma}\psi(\cdot + X), \gamma \in \mathbb{R}, X \in \mathbb{R}^2\}$.

Proposition

There exist universal constants $\tau > 0$, $C > 0$ such that, if $\psi_0 \in E$ and $d_E(V_1, \psi_0) \leq \tau$, then there exists two functions $X \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^2)$ and $\gamma \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$d_E(V_1, e^{i\gamma(t)}\psi_t(\cdot + X(t))) \leq Cd_E(V_1, \psi_0)$$

and

$$|X'(t)| + |\gamma'(t)| \leq Cd_E(V_1, \psi_0).$$

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- Orbital stability for a pair of vortices

Theorem (Chiron, P.)

There exists $c_0 > 0$ a small constant such that, for any $0 < c \leq c_0$, there exists a solution of $(TW_c)(u) = ic\partial_{x_2}u + \Delta u - (|u|^2 - 1)u = 0$ of the form

$$Q_c = V_1(\cdot - d_c \vec{e}_1) V_{-1}(\cdot + d_c \vec{e}_1) + \Gamma_{c,d_c},$$

where $d_c = \frac{1+o_{c \rightarrow 0}(1)}{c}$ is a continuous function of c and $\|\Gamma_{c,d_c}\|_{C^1(\mathbb{R}^2, \mathbb{C})} = o_{c \rightarrow 0}(1)$.

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Theorem (Chiron, P.)

The function Q_c is, for c small enough, the unique (up to translation and shift of phase) minimizer of the energy at fixed momentum. As such, it is orbitally stable for the semi distance

$$D_0(u, v) = \|\nabla u - \nabla v\|_{L^2} + \||u| - |v|\|_{L^2}.$$

Thank you for your attention !