Source-type solutions for the stable thin-film equation

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Consider the stable thin-film equation

$$\text{(STF)} \qquad h_t + (h^n h_{zzz})_z = (h^m)_{zz}, \quad t > 0,\, z \in (Z_-(t), Z_+(t))$$

$$\text{(FBC)} \qquad h = h_z = 0, \quad t > 0,\ z = Z_\pm(t)$$

$$\text{(CLV)} \qquad \frac{dZ_\pm}{dt}(t) = \lim_{z \to Z_\pm(t)} \left(h^{n-1}\, h_{zzz}\right), \quad t > 0$$

$$\text{(IC)} \qquad h(0,z) = \omega\, \delta(z)$$

- $h = h(t,z) \geq 0$ is the film height.
- $n > 0$ represents the mobility exponent.
- The term $+(h^m)_{zz}$, $m > 0$, represents the effect of the gravity.
- The functions $Z_\pm(t)$ define the boundary of the droplet.
- $\omega > 0$ and δ is the Dirac mass at the origin.
- (STF) is a forth order degenerate parabolic equation.
- (FBC) are two homogeneous free boundary conditions.

- (CLV) is of kinematic character: the (vertically averaged) velocity of the film $h^{n-1}h_{zzz}$ at the contact lines equals the contact line velocities.
- (STF) is relevant to surface tension dominated motion of thin viscous films and spreading droplets. The second-order term in the equation, $(h^m)_{zz}$, arises as a cut off of van der Waals interactions.[1]
- In terms of (vertically averaged horizontal) **velocity** $v = h^{n-1}h_{zzz}$:

$$h_t + (vh)_z = (h^m)_{zz}.$$

- Using (FBC) and (CLV), we obtain the conservation of mass:

$$\frac{d}{dt}\int_{Z_-(t)}^{Z_+(t)} h(t,z)\,dz = 0.$$

[1] H. P. Greenspan, *On the motion of a small viscous droplet that wets a surface*,J. Fluid Mech., **84** (1978), 125–143.

Self-similar / Source-type solutions

- In the case $m = n + 3$, $(\texttt{STF})$ enjoy a mass invariant scaling:

$$h \longmapsto h_\lambda(t,z) = \lambda h\left(\lambda^{n+4} t, \lambda z\right), \ \lambda > 0.$$

- Self-similar means that $h_\lambda = h$, $\forall\, \lambda > 0$. This yields

$$h(t,z) = t^{-\frac{1}{n+4}} \mathcal{H}\left(t^{-\frac{1}{n+4}} z\right) = t^{-\frac{1}{n+4}} h\left(1, t^{-\frac{1}{n+4}} z\right)$$

- $\mathcal{H} = h(1,\cdot)$ is the profile.
- Source-type means that

$$h(t,z) \longrightarrow \omega\, \delta(z) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}) \quad \text{as} \quad t \searrow 0.$$

This yields $\displaystyle\int \mathcal{H}(y)dy = \omega = \int h(t,z)dz \quad (t > 0).$

Equation satisfied by the profile

- The profile $\mathcal{H} = \mathcal{H}(y)$ satisfies
$$\mathcal{H}^n \mathcal{H}''' = \frac{1}{n+4} y \mathcal{H} + (n+3)\mathcal{H}^{n+2}\mathcal{H}'.$$
- We look for $\mathcal{H} : \mathbb{R} \longrightarrow [0, \infty)$ regular, even and with compact support $[-a, a]$, $a > 0$, $\mathcal{H} > 0$ on $(-a, a)$ implying $Z_{\pm}(t) = \pm a t^{\frac{1}{n+4}}$.
- Define
$$H(x) = (n+4)^{1/n} a^{-4/n} \mathcal{H}(a(-1+x)).$$
The problem reduces to finding $H \in C^1\big([0,1]\big) \cap C^\infty\big((0,1]\big)$ s.t.

$$\text{(1a)} \qquad H^{n-1}H''' = -1 + x + \mu H^{n+1} H', \qquad x \in (0,1]$$

$$\text{(1b)} \qquad H(0) = H'(0) = 0$$

$$\text{(1c)} \qquad H'(1) = 0$$

$$\text{(1d)} \qquad \int_0^1 H(x)dx = \kappa(\mu) := \frac{\sqrt{n+3}}{2\sqrt{\mu}}\omega$$

Here $\mu = (n+3)(n+4)^{-\frac{2}{n}} a^{2+\frac{8}{n}} > 0$.

Overview

In the case of standard thin-film equation

(TF) $$h_t + (h^n h_{zzz})_z = 0,$$

(1a) reads

(1a') $$H^{n-1} H''' = -1 + x.$$

Theorem [Bernis-Peletier-Williams, 1992]

Let $n \in (3/2, 3)$. There exists a unique non-negative solution $H \in C^1([0,1]) \cap C^\infty((0,1])$ to (1a')-(1b)-(1c). Moreover

$$H(x) = A^{-\frac{\nu}{3}} x^\nu \ (1 + \circ(1)) \ \text{ a } \ x \searrow 0,$$

where $\nu := \frac{3}{n} \in (1,2)$ and $A := \nu(\nu-1)(2-\nu) > 0$.

Remarks

1. The leading order term

$$H_{TW}(x) := A^{-\frac{\nu}{3}} x^{\nu}$$

corresponds to the traveling-wave solution:

$$H_{TW}^{n-1} H_{TW}''' = -1, \quad x > 0$$

$$H_{TW}(0) = H_{TW}'(0) = 0$$

2. The same result holds for $0 < n < 3/2$ and $n = 3/2$ but for other leading order terms.

3. If $n \geq 3$, then there exist no solutions.

4. By scaling argument we can reach any mass $\omega > 0$ from any given solution of (1a')-(1b)-(1c) that is, (1a) without gravity.

Theorem [Giacomelli-Gnann-Otto, 2013]

Let $n \in (3/2, 3)$. The solution H satisfies:

$$H(x) = A^{-\frac{\nu}{3}} x^{\nu} \left(1 + v(x, x^{\beta}) \right) \text{ as } x \searrow 0,$$

for some $\beta \in (0, 1)$ and an analytic function $v = v(x, y)$ in a neighborhood of $(0, 0)$ with $v(0, 0) = 0$, $\partial_y v(0, 0) < 0$.

Remarks

1. For $3/2 < n < 3$, the droplet can only spread.
2. The fraction power x^{β} may be understood by writing $H = A^{-\frac{\nu}{3}} x^{\nu}(1 + u)$ and linearizing around $u = 0$:

$$p\left(x \frac{d}{dx}\right) u := p(D)u = Ax, \quad p(\zeta) := (\zeta + 1)(\zeta - \alpha)(\zeta - \beta), \quad \alpha < 0.$$

Theorem [Beretta, 1997]

Let $n \in (3/2, 3)$. There exists a non-negative solution H to (1a)-(1b)-(1c)-(1d). Moreover

$$H(x) \sim H_{TW}(x) \quad \text{as} \quad x \searrow 0.$$

- The uniqueness is an open problem.
- Our aim, which is stated in the next theorem, is to go beyond the regularity given by Beretta.
- In particular, the effect of the stabilization term $(h^{n+3})_{zz}$ on the regularity near the boundary will be highlighted.
- We expect that our analysis will be helpful in the well-posedness theory.

Main Result

Theorem [M.-Tayachi, 2021]

Let $3/2 < n < 3$. Then we have the following:
(i) There exists $\varepsilon > 0$ such that for any $\mu > 0$ there exists a solution $H_\mu \in C^1([0,1]) \cap C^\infty((0,1])$ of (1a)-(1b)-(1c) satisfying

$$H_\mu(x) = A^{-\frac{\nu}{3}} x^\nu \left(1 + \bar{u}\left(x, b(\mu)x^\beta, \mu x^\gamma\right)\right), \quad x \searrow 0,$$

for some $b(\mu) > 0$, where $\bar{u}(x_1, x_2, x_3) : [0, \varepsilon^2] \times [0, \varepsilon] \times [0, \varepsilon^2] \to \mathbb{R}$ is analytic with

$$\bar{u}(0,0,0) = 0,\ \partial_1 \bar{u}(0,0,0) > 0,\ \partial_2 \bar{u}(0,0,0) < 0,\ \partial_3 \bar{u}(0,0,0) > 0$$

and

$$0 < \beta := \frac{\sqrt{-3\nu^2 + 12\nu - 8} - 3\nu + 4}{2} < 1 < \gamma := 2 + \frac{6}{n}.$$

(ii) There exists $\bar{\mu} > 0$ such that the solution $H_{\bar{\mu}}$ satisfies also (1d).

Comments

- We have

$$\bar{u}(x_1, x_2, x_3) = \frac{A}{p(1)}\big(1+O(\varepsilon)\big)x_1 - \big(1+O(\varepsilon)\big)x_2 + \frac{\nu A^{-\frac{2}{3}\nu}}{p(\gamma)}\big(1+O(\varepsilon)\big)x_3.$$

- Our result shows that solutions depend analytically on two non-smooth variables (unless $n = 2$), whereas in the absence of gravity, there is only one such variable.
- This shows the effect of the gravity on the expansion of source type solutions.
- Since no uniqueness result is known yet, we are not necessarily expanding the solution obtained by Beretta.
- We construct and give refined asymptotic of the solution than that given by Beretta.
- We cannot reach any mass $\omega > 0$ by scaling as in the standard thin-film equation. This justify why we need Part (ii) in the previous theorem.

Linearization around H_{TW}

By writing $H(x) = A^{-\frac{\nu}{3}} x^{\nu}\Big(1 + u(x)\Big) = H_{TW}(x)\Big(1 + u(x)\Big)$, the problem becomes

$$\begin{aligned} p(D)u &= Ax + A\big[(1+u)^{n-1} - 1 - (n-1)u\big] \\ &\quad -\big[(1+u)^{n-1} - 1\big]q(D)u \\ &\quad +\mu A^{-\frac{2\nu}{3}} x^{\gamma}(1+u)^{n+1}(D+\nu)(1+u),\ x \in (0,1] \\ u(0) &= 0 \end{aligned}$$

Here $D = x\partial_x$ and $p(\xi) = (\xi+1)(\xi-\alpha)(\xi-\beta)$.

$$\boxed{n \in (3/2, 3) \Longrightarrow \alpha \in (-2, 0),\ \beta \in (0, 1)}$$

Equation in non-smooth variables

- x^{-1} and x^{α} are ruled out by the boundary condition.
- We introduce a second and third variable $y := bx^{\beta},\ z := \mu x^{\gamma}$ for some $b, \mu > 0$ to be fixed later.
- We look for $u(x) = \bar{u}(x, bx^{\beta}, \mu x^{\gamma})$. Hence $Du(x) = \bar{\mathbf{D}}\bar{u}(x, bx^{\beta}, \mu x^{\gamma})$, where $\bar{\mathbf{D}} := x\partial_x + \beta y\partial_y + \gamma z\partial_z$.

- In the new variables

$$\begin{aligned} p(\bar{\mathbf{D}})\bar{u} &= Ax + A\left[(1+\bar{u})^{n-1} - 1 - (n-1)\bar{u}\right] \\ &\quad - \left[(1+\bar{u})^{n-1} - 1\right] q(\bar{\mathbf{D}})\bar{u} \\ &\quad + A^{-\frac{2}{3}\nu} z (1+\bar{u})^{n+1}(\bar{\mathbf{D}} + \nu)(1+\bar{u}) \end{aligned}$$

with $\big(\bar{u}, \partial_y \bar{u}\big)(0,0,0) = \big(0, -1\big)$.

Strategy of the Proof

- Solve the corresponding linear problem: $p(\bar{\mathbf{D}})\bar{u} = \bar{f}$.
- Solve the nonlinear problem using a fixed point argument.
- We obtain a solution $H_{b,\mu}(x) = A^{-\frac{\nu}{3}} x^{\nu}\Big(1 + u_{b,\mu}(x)\Big)$ of (1a)-(1b).
- To fulfill condition (1c) we shoot with the parameter b. Thus, we obtain a solution $H_\mu := H_{\overline{b}(\mu),\mu}$ of (1a) which satisfies (1b) and (1c).
- We conclude by a shooting argument with μ to fulfill condition (1d).

Linear Problem

Notation: $\|\cdot\|$ is the sup norm on $Q_\varepsilon := [0,\varepsilon^2]\times[0,\varepsilon]\times[0,\varepsilon^2]$.
$\mathbb{N}_0 = \mathbb{N}\cup\{0\} = \{0,1,2,\cdots\}$. $(x,y,z) := (x_1,x_2,x_3)$.

Lemma

Let $\Lambda \le \beta$ and $\bar{f}(x_1,x_2,x_3)$ be a smooth function with $\big(\bar{f},\partial_2\bar{f}\big)(0,0,0) = \big(0,0\big)$. Then

$$\bar{v}(x_1,x_2,x_3) = \big(T_\Lambda\,\bar{f}\big)\,(x_1,x_2,x_3) := \int_0^1 r^{-\Lambda}\,\bar{f}(rx_1, r^\beta x_2, r^\gamma x_3)\,\frac{dr}{r},$$

solves

$$\big(\bar{\mathbf{D}}-\Lambda\big)\,\bar{v} = \bar{f},\quad \big(\bar{v},\partial_2\bar{v}\big)(0,0,0) = \big(0,0\big).$$

Moreover $T_\Lambda\bar{\mathbf{D}} = \bar{\mathbf{D}}\,T_\Lambda$ and

$$\sum_{j=0}^{1}\left\|\partial_1^k\partial_2^\ell\partial_3^m\bar{\mathbf{D}}^j\,\bar{v}\right\| \lesssim \left\|\partial_1^k\partial_2^\ell\partial_3^m\bar{f}\right\|,\quad (k,\ell,m)\in\mathbb{N}_0^3,$$

with $(k,\ell,m)\notin\{\,(0,0,0),\ (0,1,0)\}$ if $\Lambda=\beta$.

- $(\bar{f}, \partial_2 \bar{f})(0,0,0) = (0,0) \to \bar{f}(x,y,z) = O(x + y^2 + z)$.[2] Hence $x \sim \varepsilon^2, y \sim \varepsilon, z \sim \varepsilon^2$.
- As a consequence, with $T := T_\beta \, T_{-1} \, T_\alpha$, the function $\bar{u}_0 := T\bar{f}$ is well defined, smooth and solves the linear problem with homogeneous boundary conditions

$$p(\bar{\mathbf{D}})\bar{u}_0 = \bar{f}, \quad \big(\bar{u}, \partial_2 \bar{u}\big)(0,0,0) = \big(0,0\big).$$

Moreover

$$|\bar{u}_0|_1 = |T\bar{f}|_1 \lesssim |\bar{f}|_0,$$

where, for fixed $K, L, M \in \mathbb{N}$,

$$\begin{aligned} |\bar{h}|_0 &:= \sum_{k=0}^{K} \sum_{\ell=0}^{L} \sum_{m=0}^{M} \frac{\varepsilon^{2k+\ell+2m}}{k!\ell!m!} \left\| \partial_1^k \partial_2^\ell \partial_3^m \bar{h} \right\| \\ |\bar{h}|_1 &:= \sum_{j=0}^{3} |\bar{\mathbf{D}}^j \, \bar{h}|_0. \end{aligned}$$

[2] $\bar{f}(x,y,z) = \int_0^y \int_0^s \partial_2^2 \bar{f}(0,\tau,0) d\tau ds + \int_0^x \partial_1 \bar{f}(s,y,0) ds + \int_0^z \partial_3 \bar{f}(x,y,s) ds$.

Local Existence

The problem

$$\begin{aligned} p(\bar{\mathbf{D}})\bar{u} = \bar{f}_{\bar{u}}, \quad \text{for } x_1, x_2, x_3 > 0, \\ \big(\bar{u}, \partial_2 \bar{u}\big)(0,0,0) = \big(0,-1\big), \end{aligned}$$

where

$$\begin{aligned} \bar{f}_{\bar{u}} &= Ax_1 - \left((1+\bar{u})^{n-1} - 1\right) q(\bar{\mathbf{D}})\bar{u} + A\left[(1+\bar{u})^{n-1} - 1 - (n-1)\bar{u}\right] \\ &+ A^{-\frac{2}{3}\nu} x_3 (1+\bar{u})^{n+1}\left(\bar{\mathbf{D}} + \nu\right)(1+\bar{u}), \end{aligned}$$

can be written as a fixed point equation

$$\boxed{\bar{u} = -x_2 + T\bar{f}_{\bar{u}} := \mathcal{T}(\bar{u})}$$

For K, L, M fixed integers, let

$$\mathbf{S}_{K,L,M} := \left\{ \bar{v} \in C^{K+L+M+3}(Q_\varepsilon);\ (\bar{v}, \partial_2 \bar{v})(0,0,0) = (0,0) \text{ and } |\tilde{\bar{v}}|_1 \leq \varepsilon \right\},$$

where

$$|\tilde{\bar{v}}|_1 := |\bar{v}|_1 + \sum_{\substack{\alpha=(\alpha_1,\alpha_2,\alpha_3) \\ |\alpha|=K+L+M+3}} \frac{\varepsilon^{2\alpha_1+\alpha_2+2\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!} \|\partial^\alpha \bar{v}\|.$$

Note that $\left(C^{K+L+M+3}(Q_\varepsilon), |\cdot\tilde{|}_1 \right)$ is a Banach space and

$$\bar{u} \in \mathbf{S}_{K,L,M} \Longrightarrow \left(\bar{f}_{\bar{u}}, \partial_2 \bar{f}_{\bar{u}} \right)(0,0,0) = (0,0).$$

Goal: $\mathcal{T} : \mathbf{S}_{K,L,M} \longrightarrow \mathbf{S}_{K,L,M}$ is a contraction for ε sufficiently small.

Claim: We have

$$\begin{aligned} |\mathcal{T}(\bar{u}) - \mathcal{T}(\bar{v})\tilde{|}_1 &\lesssim \varepsilon|\bar{u} - \bar{v}\tilde{|}_1 \quad \text{for all } \bar{u}, \bar{v} \in \mathbf{S}_{K,L,M} \\ |\mathcal{T}(\bar{u})\tilde{|}_1 &\lesssim \varepsilon^2 \quad \text{for all } \bar{u} \in \mathbf{S}_{K,L,M} \end{aligned}$$

The proof uses the following estimates:

Lower order norms

$$\left(\bar{f}, \partial_2 \bar{f}\right)(0,0,0) = \left(0,0\right) \Longrightarrow \|\bar{f}\| + \varepsilon\|\partial_2 \bar{f}\| \lesssim \varepsilon^2 \left(\|\partial_1 \bar{f}\| + \|\partial_2^2 \bar{f}\| + \|\partial_3 \bar{f}\|\right)$$

Algebra property

$$|\bar{f}\bar{g}|_0 \leq |\bar{f}|_0 |\bar{g}|_0$$

Nonlinear estimates

$$\begin{aligned} \left|G(\bar{f})\right|_0 &\leq G_1\left(\left|\bar{f}\right|_0\right), \\ \left|G(\bar{f}) - G(\bar{g})\right|_0 &\leq G_2\left(\max\{|\bar{f}|_0, |\bar{g}|_0\}\right) |\bar{f} - \bar{g}|_0, \end{aligned}$$

where

$$G(z) = \sum_{j=0}^{\infty} a_j z^j, \quad G_1(z) := \sum_{j=0}^{\infty} |a_j| z^j \ \text{ and } \ G_2(z) := \sum_{j=0}^{\infty} (j+1)|a_{j+1}| z^j.$$

Claim: There exist $\varepsilon \in (0,1)$ and $\bar{u}$ analytic in Q_ε such that $\bar{u} = \mathcal{T}(\bar{u})$.
Proof:

- All the estimates are in terms of constants independent of K, L, M and ε does not depend on K, L, M.
- The complete metric spaces $\mathbf{S}_{K,L,M}$ are nested as K, L, M increase.
- They all share the same unique fixed point which is C^∞ and the Taylor series

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{\partial_1^k\partial_2^\ell\partial_3^m\bar{u}(0,0,0)}{k!\ell!m!}x_1^k x_2^\ell x_3^m$$

  converges absolutely in Q_ε.
- The corresponding error terms converge uniformly to zero.
- The Taylor series also represents the solution.
- The solution is analytic.

How to reach $[0, 1]$?

Until now, we have constructed a local solution of (1a)-(1b)

$$H_{b,\mu}(x) = A^{-\frac{\nu}{3}} x^{\nu}(1 + u_{b,\mu}(x)) = H_{TW}(x)\left(1 + \bar{u}\left(x, bx^{\beta}, \mu x^{\gamma}\right)\right)$$

where $\bar{u}(x_1, x_2, x_3)$ is analytic in $Q_\varepsilon = [0, \varepsilon^2] \times [0, \varepsilon] \times [0, \varepsilon^2]$.
Note that $H_{b,\mu}$ is defined for

$$0 \le x \le \hat{x}_{b,\mu}(\varepsilon) := \min\left\{\varepsilon^2, \left(\frac{\varepsilon}{b}\right)^{\frac{1}{\beta}}, \left(\frac{\varepsilon^2}{\mu}\right)^{\frac{1}{\gamma}}\right\}.$$

By ODE arguments, we can extend $H_{b,\mu}$ to a smooth solution of (1a)-(1b) on a maximal interval $(0, x^*_{b,\mu})$ with

$$H_{b,\mu} > 0 \text{ in } (0, x^*_{b,\mu}) \text{ and } x^*_{b,\mu} \le \infty.$$

Claim: Let $\mu > 0$ be fixed. Then there exists $\bar{b}(\mu) > 0$ such that the function $H_{\bar{b}(\mu),\mu}$ satisfies (1a)-(1b)-(1c). Moreover, $H'_{\bar{b}(\mu),\mu} > 0$ on $(0,1)$.

Proof:

- We compare $H_{b,\mu}$ to H_{TW} by nonlinear ODE arguments to obtain

$$H'_{b,\mu}(x) > 0 \quad \text{for} \quad x \searrow 0.$$

- This also gives by using monotonicity in b

$$x^*_{b,\mu} \longrightarrow 0 \quad \text{as} \quad b \to \infty.$$

- It follows that $H'_{b,\mu}$ must vanish somewhere.
- Setting

$$\begin{aligned} \bar{b}(\mu) &= \inf \Big\{ b \geq 0; H'_{b,\mu}(x) = 0 \ \text{for some}\, x \in (0,1] \cap (0, x^*_{b,\mu}) \Big\} \\ &:= \inf \mathcal{B}, \end{aligned}$$

we conclude the proof as follows.

- By continuous dependence and monotonicity $0 < \bar{b}(\mu) \in \mathcal{B}$.
- By contradiction: $\bar{\bar{x}}_{\bar{b}(\mu)} = 1$, where for $b \in \mathcal{B}$, $\bar{\bar{x}}_b$ stands for the first zero of $H'_{b,\mu}$.
- If $\bar{\bar{x}}_{\bar{b}(\mu)} < 1$, then $H'_{\bar{b}(\mu),\mu} < 0$ in a right neighborhood of $\bar{\bar{x}}_{\bar{b}(\mu)}$.
- Since $H'_{\bar{b}(\mu),\mu} > 0$ on $(0, \bar{\bar{x}}_{\bar{b}(\mu)})$ and by continuous dependence, we deduce that this property is satisfied by $H_{b,\mu}$ for b near $\bar{b}(\mu)$.
- This yields the existence of $0 < b < \bar{b}(\mu)$ such that $H'_{b,\mu}$ vanish somewhere.
- Clearly this contradicts the definition of $\bar{b}(\mu)$.

Remark: $H_\mu := H_{\bar{b}(\mu),\mu}$ solves (1a)-(1b)-(1c).

How to reach the mass?

Claim: There exists $\bar{\mu} > 0$ such that $H_{\bar{\mu}}$ satisfies (1d), namely

$$\int_0^1 H_{\bar{\mu}}(x)dx = \frac{\sqrt{n+3}}{2\sqrt{\bar{\mu}}}\omega.$$

Proof: Define

$$\mathcal{M}(\mu) := \frac{2\sqrt{\mu}}{\sqrt{n+3}} \int_0^1 H_\mu(x)dx.$$

It suffices to show that $\mathcal{M}([0,\infty)) = [0,\infty)$. For this end, we will show that

$$\mathcal{M}(\mu_j) \to \infty \quad \text{as} \quad j \to \infty,$$

for some sequence (μ_j).
This will be done in three steps.

Step 1. Define

$$\mathcal{H}_\mu(y) = (n+4)^{-1/n}\, a^{4/n}\, H_\mu(1 - \frac{y}{a}),$$

and using some estimates in[3], we obtain

$$-H_\mu''(1) = |H_\mu''(1)| \leq \; C_n \sqrt{n+4}\, H_\mu(1)^{1-n/2},$$

$$\max\left(H_\mu(1)^{n/4}, \sqrt{\mu} H_\mu(1)^{1+n/2}\right) \geq D_n,$$

for some positive constants C_n, D_n depending only on n.

[3] E. Beretta, *Selfsimilar source solutions of a fourth order degenerate parabolic equation*, Nonlinear Anal., **29** (1997), 741–760.

Step 2. Define

$$\alpha(\mu) = (n+4)^{-\frac{1}{n+4}} (n+3)^{-\frac{2}{n+4}} \mu^{\frac{2}{n+4}} H_\mu(1).$$

Then

$$\sup_{\mu>0} \alpha(\mu) = \infty.$$

Indeed, if not, by using the above estimates , we should have $1 \lesssim \mu^{-\frac{n}{2(n+4)}}$ for all $\mu > 0$. **Contradiction!**
In particular, there exists a sequence (μ_j) such that

$$\alpha_j := \alpha(\mu_j) \to \infty \quad \text{as} \quad j \to \infty.$$

Step 3. Let $\mathbf{v}_j = \mathbf{v}_j(y)$ be the solution of

$$\begin{cases} \mathbf{v}_j'' & = -2C_n\,\alpha_j^{1-n/2} + \alpha_j^2(n+3)\,(\mathbf{v}_j - \alpha_j) \;\; \text{for} \;\; y > 0, \\ \mathbf{v}_j(0) & = \alpha_j, \quad \mathbf{v}_j'(0) = 0, \end{cases}$$

The solution $\mathbf{v}_j$ is given explicitly by

$$\mathbf{v}_j(y) = \frac{1}{(n+3)\alpha_j^{1+n/2}}\left[2C_n + (n+3)\alpha_j^{2+n/2} - 2C_n\cosh\left(\alpha_j\sqrt{n+3}\,y\right)\right].$$

By comparison argument, we deduce that

$$\mathcal{M}(\mu_j) = 2\int_0^{a_j}\mathcal{H}_{\mu_j}(y)\,dy \;\;\geq\;\; \frac{1}{\sqrt{n+3}}\operatorname{arg\,cosh}\left(1 + \frac{n+3}{4C_n}\alpha_j^{2+n/2}\right)$$
$$\to\infty \quad \text{as} \quad j\to\infty.$$

A continuity argument finishes the proof.

Concluding Remarks

- No analogous result for $0 < n \leq 3/2$. Clearly $A = 0$ for $n = 3/2$.
- Non existence for $n \geq 3$.
- Our arguments doesn't work for the unstable regime

$$h_t + (h^n h_{zzz})_z = -\left(h^{n+3}\right)_{zz}.$$

- More precisely, with our approach we can construct a solution satisfying (1a)-(1b).
- To satisfy (1c)-(1d) we use some comparison arguments.
- This justify why we need the sign $+$ or equivalently $+\mu > 0$.
- To construct regular solutions satisfying Problem (1) in the unstable regime, the mass ω should be less then the critical mass $\omega_c = 2\pi\sqrt{2/3}$.[4]

[4] T. P. Witelski, A. J. Bernoff and A. L. Bertozzi, *Blowup and dissipation in a critical-case unstable thin film equation*, European J. Appl. Math., **15** (2004), 223–256.

Figure 1: *

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THANK YOU
FOR YOUR
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