Remarks on local regularity of axisymmetric solutions to the 3D Navier-Stokes equations

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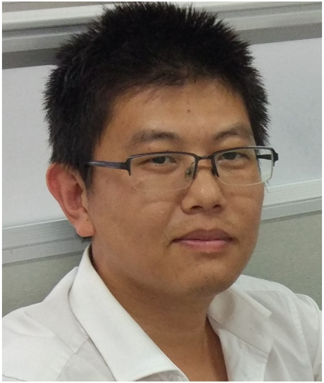
Collaborators



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Incompressible Navier-Stokes equations

Consider the *incompressible Navier-Stokes equations*

$$\left.\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0, \infty), \qquad \text{(NS)}$$

with initial condition

$$u(\cdot, 0) = u_0, \quad \operatorname{div} u_0 = 0.$$

Axisymmetric solutions

In cylindrical coordinates r, θ, z with $r = \sqrt{x_1^2 + x_2^2}$, and

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \; e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1),$$

a vector field u is *axisymmetric* if

$$u = u^r(r, z, t)e_r + u^\theta(r, z, t)e_\theta + u^z(r, z, t)e_z$$

with components u^r, u^z, u^θ independent of θ. This class is preserved by (NS).

We call the angular component u^θ the *swirl*. The subclass of axisymmetric vector fields with zero swirl is also preserved by (NS).

The regularity problem

An open question is whether for all smooth initial data u_0 with fast decay, there is a global in time regular solution of (NS).

We are interested in the restriction of the above problem to the class of axisymmetric solutions because:

1. The number of space variables is 2, the problem is between 2D and 3D, more chance of regularity, an intermediate step before the general problem
2. Many numerical work in this class seeking blow-up
3. Many interesting examples are constructed in this class

Known regularity results for axisymmetric solutions

1. Zero swirl solutions are globally regular
   (Ladyzhenskaya, Ukhovskii-Yudovich)
2. regularity criteria for general NS
   (Serrin-type, Scheffer, Caffarelli-Kohn-Nirenberg, etc)
3. regularity criteria of Neustupa–Pokorny, Chae-Lee, Jiu-Xin:
   finite integrals, or smallness of sup of scaled integrals

Criteria for Type 1 singularity

4. Chen-Strain-Tsai-Yau

$$|u(x,t)| \leq \frac{C}{|x - x_0| + \sqrt{t_0 - t}}$$

5. CSTY2 and Koch-Nadirashivili-Seregin-Sverak

$$|u(x,t)| \leq \frac{C}{r^{1-\varepsilon}|t|^{\varepsilon/2}}, \quad 0 \leq \varepsilon \leq 1$$

6. Seregin: bounded (large) $\limsup$ of some scaled integrals

Regularity criteria in swirl

Note a priori $r|u^\theta| \le C$. The following assume vanishing as $r \to 0$:

7. $r^\delta |u^\theta| \in L^q(0,T;L^p(\mathbb{R}^3)), \quad \frac{2}{q}+\frac{3}{p} \le 1-\delta, \quad 0 \le \delta < 1$
   Chen-Fang-Zhang
8. $r|u^\theta| \le C|\ln r|^{-2}, \quad 0 \le r < 1/2$ Lei-Q.Zhang
9. $r|u^\theta| \le C|\ln r|^{-3/2},$ Wei 2016

Slightly supercritical criteria

10. Pan [2016] revised CSTY2 and showed for $\alpha \le 0.028$

$$|u(x,t)| \le \frac{C}{r}\left(\ln\ln\frac{3}{r}\right)^{\alpha}, \quad (r<1)$$

11. Palasek [2022], using the quantitative estimate of Tao for Escauriaza, Seregin and Sverak $L^\infty L^3$ regularity

$$\left\| r^{1-\frac{3}{p}} u \right\|_{L^p(\mathbb{R}^3)} \le C(\ln\ln \tfrac{1}{T-t})^{\alpha(p)}, \quad 2<p<\infty$$

12. Seregin, arXiv:2109.09344 ,

$$R^{-\frac{1}{2}}\|u\|_{L^4L^3(Q(z_0,R))} + R^{-\frac{1}{2}}\|u\|_{L^{\frac{10}{3}}(Q(z_0,R))} \le N\left(\ln\ln\frac{100}{R}\right)^{\frac{1}{224}},$$

with an error corrected in arXiv:2201.00153 .

Our goal and notation

Our motivation is to improve the slightly supercritical result of Pan.

Denote

$$\omega(R) = \left(\ln \ln \frac{100}{R}\right)^{-1},$$

$$A(z_0, R) = \sup_{t_0 - R^2 < t < t_0} \frac{1}{R} \int_{B(x_0, R)} |u(x, t)|^2 \, dx$$

$$\Gamma = r u^{\theta}$$

Key oscillation estimate

Proposition. Assume that (u, p) is an axisymmetric suitable weak solution to (NS) in $Q(1)$ and there are $\beta \in (0, \frac{1}{8})$ and $K > 0$

$$A(z_0, R) \leq K\omega(R)^{-\beta}, \quad \forall R \leq \frac{1}{4}, \tag{1}$$

for some $z_0 = (0, x_{0,3}, t_0) \in Q(\frac{1}{8})$. Then for any $0 < \tau < 1$, there is a constant $c = c(K, \beta, \tau) > 0$ such that for $\Gamma = ru^\theta$

$$\operatorname*{osc}_{Q(z_0,\rho)} \Gamma \leq e^{-c\left(\left(\ln \frac{100}{\rho}\right)^\tau - \left(\ln \frac{100}{R}\right)^\tau - 2\right)} \operatorname*{osc}_{Q(z_0,R)} \Gamma \tag{2}$$

for $0 < \rho < R \leq \frac{1}{4}$. Assume, in addition, that (1) holds for *all* $z_0 = (0, x_{0,3}, t_0) \in Q(\frac{1}{8})$, then we have that for $(r, x_3, t) \in Q(\frac{1}{8})$

$$|\Gamma(r, x_3, t)| \leq Ne^{-c\,|\ln r|^\tau} \tag{3}$$

Comments:

1. (3) improves Pan's Theorem 1.2 with similar assumption, smaller β, better decay.
2. Seregin arXiv:2201.00153 has a similar and independent oscillation estimate. His assumption is the same as in arXiv:2109.09344, and his decay exponent τ is $1/4$.
3. Our τ is limited to $0 < \tau < 1$, weaker than the Hölder continuity case $\tau = 1$.
4. We need to assume (1) for all z_0 to get (3).

Corollaries

Let $b = u^r e_r + u^z e_z$, part of u.

Theorem 1. Let (u, p) be an axisymmetric suitable weak solution to (NS) in $Q(1)$. If for $\gamma = 2 - \frac{2}{q} - \frac{3}{p} \in (0, 1)$ and $0 < \alpha < \frac{\gamma}{48+16\gamma}$,

$$R^{\gamma-1}\|b\|_{L^{p,q}(Q(z_0,R))} \leq G\omega(R)^{-\alpha}, \tag{4}$$

for *all* $z_0 = (0, x_{0,3}, t_0) \in Q(\frac{1}{8})$ and $0 < R \leq \frac{1}{4}$, then the solution is regular at $(0, 0)$.

This is similar to Seregin, arXiv:2109.09344.

Theorem 2. If (4) is valid for $\alpha = 0$ at *one* z_0, then the solution is regular at z_0.

Theorem 3. Let (u, p) be a classical axisymmetric solution to (NS) in $\mathbb{R}^3 \times (-1, 0)$ which blows up at time $t = 0$. Then

$$\limsup_{t \to 0} \frac{\|b(\cdot, t)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}}{\left(\ln \ln \frac{100}{-t}\right)^a} = \infty, \quad \forall a < \frac{1}{48} \tag{5}$$

This is similar to the result of Palasek.

Sketch of the proof of the oscillation estimate

Assume $z_0 = (0, 0)$. Let $\Gamma = ru^\theta$ and $A(R) = \sup\limits_{-R^2<t<0} \frac{1}{R} \int_{B_R} |u|^2 dx$.

Γ satisfies

$$\partial_t \Gamma + (b \cdot \nabla)\,\Gamma - \Delta\Gamma + \frac{2}{r}\partial_r \Gamma = 0 \tag{6}$$

which allows maximal principle.

The sign of $\frac{2}{r}\partial_r$ is bad, but it can be treated using $\Gamma|_{r=0} = 0$.

Step 1. Local maximum estimate

For $0 < R < 1$,

$$\sup_{Q(\frac{1}{2}R)} |\Gamma| \leq N \left(\frac{1 + A(R)}{R} \right)^{\frac{5}{2}} \|\Gamma\|_{L^2(Q(R))}. \tag{7}$$

It is by Moser type energy estimate of exponents m_j, $j \in \mathbb{N}$, over parabolic cylinders of radius $(\frac{1}{2} + 2^{-1-j})R$.

Step 2. Initial lower bound for rescaled function

Let $m_R = \inf_{Q(R)} \Gamma$, $M_R = \sup_{Q(R)} \Gamma$, and

$$h(x,t) = \begin{cases} \dfrac{2\,(M_R - \Gamma)}{M_R - m_R}, & \text{if } M_R > -m_R, \\[2ex] \dfrac{2\,(\Gamma - m_R)}{M_R - m_R}, & \text{else.} \end{cases} \tag{8}$$

Then h satisfies the same equation of Γ, and

$$0 \le h \le 2, \quad h(0, x_3, t) = a \ge 1.$$

By testing the equation with a cut-off function, one can show

$$R^{-5} \|h\|_{L^1\left(B(\frac{1}{2}R) \times \left(-R^2, -\frac{1}{4}R^2\right)\right)} \ge \frac{N}{1 + A(R)}$$

Step 3. weak Harnack inequality

$$- \int_{\mathbb{R}^3} \ln h(x,t) \cdot \zeta_R^2(x)\, dx \leq N\left(1 + A(R)\right)^3, \tag{9}$$

for $-\frac{1}{4}R^2 \leq t < 0$ with $0 < R \leq \frac{1}{4}$.

It is by integrating the equation of $H = -\ln h$

$$\partial_t H + (b \cdot \nabla) H - \Delta H + \frac{2}{r}\partial_r H + |\nabla H|^2 = 0,$$

using Nash's inequality and weighted Poincare inequality, to get a nonlinear damped equation of $\overline{H}(t) = \int H(x,t)\zeta_R dx$. Step 2 gives a lower bound of the set of time t where $\int_{B(R/2)} h(x,t)dx$ is bounded from below.

Step 4. Strong Harnack inequality

Assuming $A(R)\omega(R)^{\beta} \leq K$, show that,

$$\inf_{Q(\frac{1}{4}R)} h \geq \frac{1}{2}\lambda(R), \tag{10}$$

where $\lambda(R) = \varepsilon \left(\ln \frac{100}{R}\right)^{\tau-1}$, $0 < \tau < 1$, and $0 < \varepsilon = \varepsilon(\tau) \ll 1$.

It is by bounding

$$\sup_{Q(R/4)} (\lambda(R) - h)_+ \lesssim \left(\frac{1 + A(R)}{R}\right)^{5/2} \|(\lambda(R) - h)_+\|_{L^2(Q(R/2))}$$

as in Step 1, and then using the upper bound of the set $\{h \leq \lambda(R)\}$ from Step 3.

Final Step: Proof of the oscillation estimate

The lower bound (10) of h translates to

$$\operatorname*{osc}_{Q(R/4)} \Gamma \leq \left(1 - \frac{1}{4}\lambda(R)\right) \operatorname*{osc}_{Q(R)} \Gamma$$

Iterating this estimate for $R = R_k$ and ensuring the convergence of the product

$$\Pi_{k=0}^{j-1}(1 - \frac{1}{4}\lambda(R_k))$$

gives the oscillation estimate.