Smooth Imploding Solutions for 3D Compressible Fluids

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Compressible Euler Equations

Non-Isentropic Form

Full non-isentropic Euler equations:

$$\partial_t(\rho\, u) + \mathrm{div}\,(\rho u \otimes u + p\,\mathrm{Id}\,) = 0 \qquad \text{(Conservation of momentum)}$$

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \qquad \text{(Conservation of mass)}$$

$$\partial_t E + \mathrm{div}\,((p + E)u) = 0 \qquad \text{(Conservation of energy)}$$

where u is the velocity, ρ, the density, p, the pressure and E, the energy. Conservation of energy can be replaced by transport of specific entropy $\partial_t S + u \cdot \nabla S = 0$. The pressure is

$$p = (\gamma - 1)(E - \tfrac{1}{2}\rho|u|^2) = \tfrac{1}{\gamma}\rho^\gamma e^S$$

for adiabatic exponent $\gamma > 1$. The sound speed is given by

$$c = \sqrt{\frac{\gamma P}{\rho}}\,.$$

Shock waves and imploding solutions

- Shock waves: The prototypical singularity for the Euler equations is a shock wave, which occurs when the speed of a disturbance exceeds the local speed of sound. Mathematically, one is interested in both the formation of the shock and the development of the shock.
- Implosions: Implosions involve spherically symmetric solutions that collapse at a point in finite time. Classically, one considers imploding shock waves. Recently, Merle-Raphael-Rodnianski-Szeftel showed there exist smooth imploding solutions.

Shock formation results

- Christodoulou '07, Christodoulou-Miao '14: 3D isentropic, irrotational.
- Luk-Speck '18: 2D isentropic, non-trivial vorticity.
- B-Shkoller-Vicol '19: 2D isentropic, azimuthal, non-trivial vorticity + description of self-similar profile.
- B-Shkoller-Vicol '19: 3D isentropic, non-trivial vorticity + description of self-similar profile.
- B-Shkoller-Vicol '20: Full 3D Euler + description of self-similar profile.
- Luk-Speck '21: Full 3D Euler, allow non-generic shocks.

Related works: John-Klainerman '84, Klainerman, John '87, Hörmander '87, John '81, Sideris '85, Alinhac '99

Shock development results

- Lebaud '94: 1D, 2x2 p-system, existence of discontinuous shock (uniqueness follows by T.P. Liu) (cf. Chen-Dong '01, Kong '02 for generalizations)
- Yin '04: Spherically symmetric Euler, existence of weak solution past formation, no uniqueness and no description of weak discontinuities.
- Christodoulou-Lisbach '16: Spherically symmetric isentropic Euler in formation, restricted problem thereafter, not a weak solution to Euler.
- Christodoulou '19: Multi-D, irrotational, isentropic shock development for the restricted problem, not a weak solution to Euler.
- B-Drivas-Shkoller-Vicol '21: Development for full Euler under azimuthal symmetry satisfying the Rankine-Hugoniot jump conditions, uniqueness, full description of weak discontinuities.

Implosion results

- Guderley '42: Self-similar imploding shock waves solutions to Euler.
- Merle-Raphael-Rodnianski-Szeftel '19: Smooth imploding self-similar solutions exist from a.e. adiabatic exponent $\gamma > 1$.
- Biasi' 21: Detailed numerical description of smooth self-similar imploding solutions.

  Related work:
  Navier-Stokes (Merle et al. '19), NLS via Madelung transform (Merle et al. '19), Euler Poisson (Guo-Hadzic-Jang-Schrecker '21)

Setup

- Isentropic, spherically symmetric Euler

$$\partial_t u + u\partial_R u + \frac{1}{\gamma\rho}\partial_R \rho^\gamma = 0 \quad \text{and} \quad \partial_t \rho + \frac{1}{R^2}\partial_R(R^2\rho u) = 0\,,$$

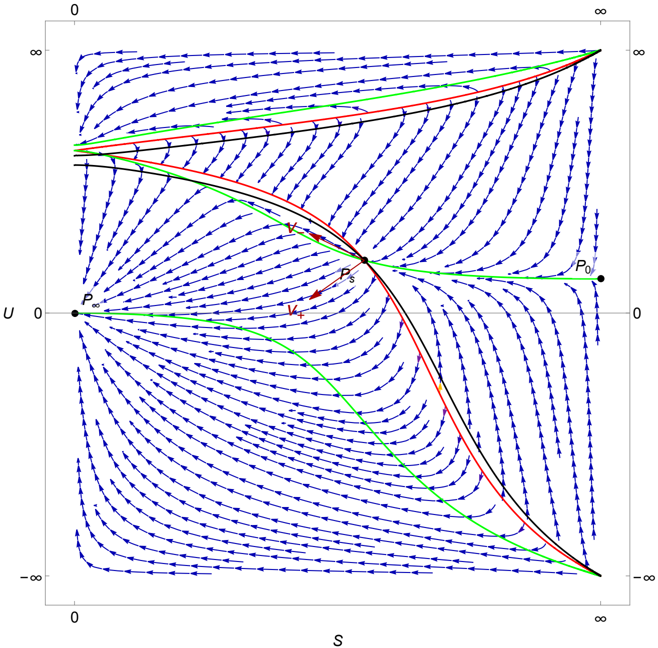
- The self-similar ansatz

$$u(r,t) = r^{-1}\tfrac{R}{T-t}U(\log(\tfrac{R}{(T-t)^{\frac{1}{r}}})) \text{ and } \sigma(r,t) = \alpha^{-\frac{1}{2}}r^{-1}\tfrac{R}{T-t}S(\log(\tfrac{R}{(T-t)^{\frac{1}{r}}}))\,,$$

where $\sigma = \frac{1}{\alpha}\rho^\alpha$ is the rescaled sound speed.

- Setting $\xi = \log(\frac{R}{(T-t)^{\frac{1}{r}}})$ leads to the autonomous ODE

$$\frac{dU}{d\xi} = \frac{N_U(U,S)}{D(U,S)}, \quad \text{and} \quad \frac{dS}{d\xi} = \frac{N_S(U,S)}{D(U,S)}\,.$$



0
∞
∞
∞
V_-
P_0
P_s
P_∞
V_+
U
0
0
$-\infty$
$-\infty$
0
∞
S

Result of Merle et al. '19

For a.e. $\gamma > 1$, there exists a countably infinite sequence of self-similar solutions to isentropic Euler. The velocity and density blow up at the origin.

The condition on γ relates to the non-vanishing of an analytic function. The condition is not proven for any specific γ, but may be checked numerically. The case $\gamma = 5/3$ (monatomic gases) is specifically ruled out.

Compressible Navier-Stokes

Isentropic 3D compressible Navier-Stokes with constant viscosity:

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) - \mu_1 \Delta u - (\mu_1 + \mu_2)\nabla \mathrm{div}\, u = 0\,,$$
$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0\,,$$

for $\mu_1 \geqslant 0$ and $2\mu_1 + \mu_2 \geqslant 0$.

Merle et al. '19: there exists imploding solutions to NS for a.e. $1 < \gamma < \frac{2+\sqrt{3}}{\sqrt{3}}$ with decaying density.

Previously, Xin '98: blow up for initial data with compact density and Rozanova '08: blow up for rapidly decaying density.

Problems left open

1. Do imploding solutions for Euler exist for all $\gamma > 1$?
2. Can one construct imploding solutions to the Navier-Stokes equation with initial density constant at infinity?

Main result

1. There exists smooth self-similar imploding solutions for all $\gamma > 1$.
2. For the case $\gamma = \frac{7}{5}$ (diatomic gas, e.g. oxygen, hydrogen, nitrogen) there exists a countably infinite sequence of imploding solutions.
3. Simplified proofs of linear stability and non-linear stability.
4. Asymptotically self-similar imploding solutions to NS for $\gamma = \frac{7}{5}$.
5. First example of initial data with density constant at infinity leading to blow up for NS.

Riemann invariants

- Riemann invariants: $w = u + \sigma$ and $z = u - \sigma$.
- Self-similar anzatz

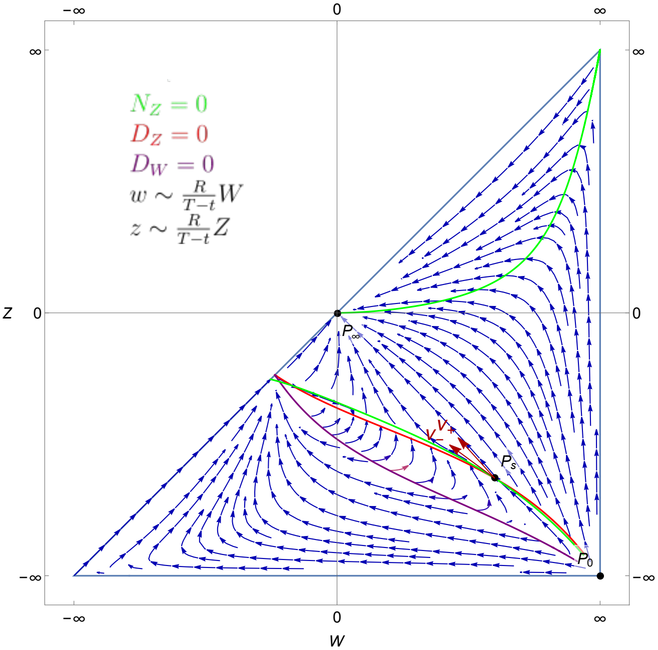
$$w(R,t) = \frac{1}{r} \cdot \frac{R}{T-t} W(\xi) \quad \text{and} \quad z(R,t) = \frac{1}{r} \cdot \frac{R}{T-t} Z(\xi)$$

- Setting $\xi = \log(\frac{R}{(T-t)^{\frac{1}{r}}})$ yields

$$(r + \frac{1}{2}((1+2\alpha)W + (1-\alpha)Z))W + (1 + \frac{1}{2}(W + Z + \alpha(W-Z)))\partial_\xi W - \frac{\alpha}{2}Z^2 = 0$$
$$(r + \frac{1}{2}((1-\alpha)W + (1+2\alpha)Z))Z + (1 + \frac{1}{2}(W + Z - \alpha(W-Z)))\partial_\xi Z - \frac{\alpha}{2}W^2 = 0$$

- Rearranging,

$$\frac{dW}{d\xi} = \frac{-(r+\frac{1}{2}((1+2\alpha)W+(1-\alpha)Z))W + \frac{\alpha}{2}Z^2}{1+\frac{1}{2}(W+Z+\alpha(W-Z))} = \frac{N_W}{D_W},$$
$$\frac{dZ}{d\xi} = \frac{-(r+\frac{1}{2}((1-\alpha)W+(1+2\alpha)Z))Z + \frac{\alpha}{2}W^2}{1+\frac{1}{2}(W+Z-\alpha(W-Z))} = \frac{N_Z}{D_Z}.$$



$N_Z = 0$
$D_Z = 0$
$D_W = 0$
$w \sim \frac{R}{T-t} W$
$z \sim \frac{R}{T-t} Z$
$-\infty$
0
∞
Z
W
P_∞
P_s
P_0
V_+
V_-

Analysis of the point P_s

Under the change of variables $\xi \mapsto \psi$ where $\partial_\psi = -D_W D_Z \partial_\xi$:

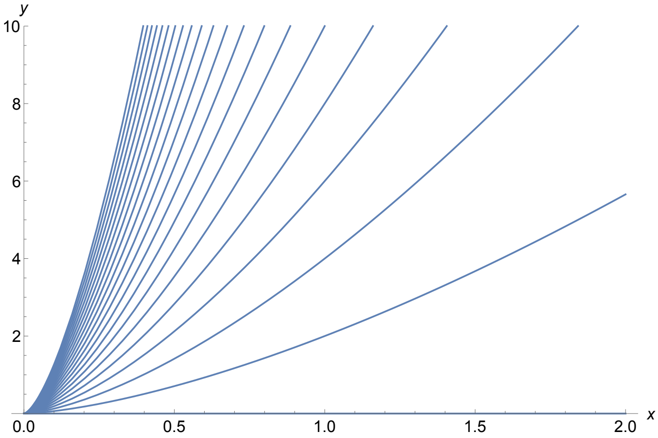
$$\partial_\psi W = -N_W D_Z \quad \text{and} \quad \partial_\psi Z = -N_Z D_W \,,$$

P_s becomes a stable stationary point. Consider the simple ODE:

$$\dot{x} = \lambda_+ x, \quad \dot{y} = \lambda_- y$$

for $\lambda_- < \lambda_+ < 0$. If $k = \frac{\lambda_-}{\lambda_+} \notin \mathbb{N}$, $x = 0$ and $y = 0$ are the sole smooth solutions. Non-smooth, C^k solutions exist of the form $y = Cx^k$ whose series agrees with the solution $y = 0$ up to order $\lfloor k \rfloor$.

Case $\lambda_- = -\frac{3}{2}$ and $\lambda_+ = -1$



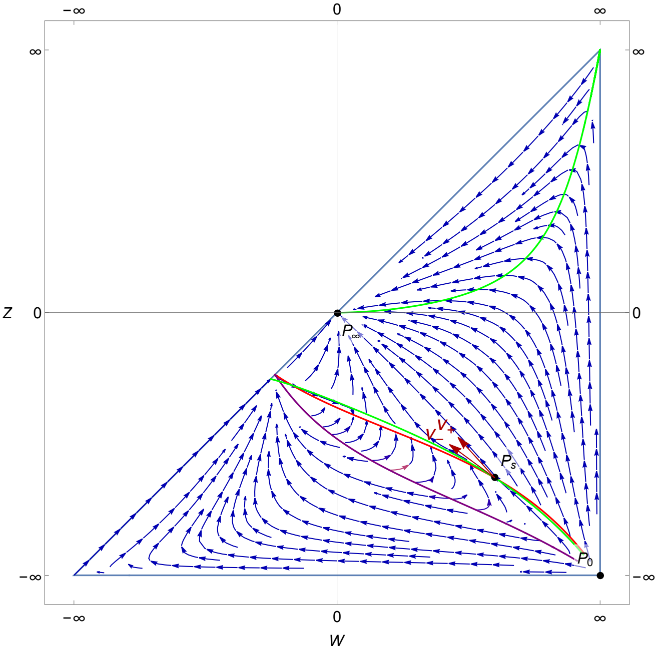
Returning to our ODE

$$\partial_\psi W = -N_W D_Z \quad \text{and} \quad \partial_\psi Z = -N_Z D_W \, ,$$

Let $\lambda_- < \lambda_+ < 0$ be the eigenvalues of the Jacobian at P_s, and define

$$k = \frac{\lambda_-}{\lambda_+} \, .$$

If ν_-, ν_+ are the corresponding eigenvectors, we consider smooth solutions tangent to ν_-. The smooth solutions tangent to ν_+ correspond to the Guderley solutions.



−∞
0
∞
∞
∞
Z
0
0
P∞
V+
V−
Ps
P0
−∞
−∞
−∞
0
∞
W

Taylor Expansion around P_s ($\xi = 0$)

- Write the solution crossing P_s as a series

$$(W(\xi), Z(\xi)) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}(W_n, Z_n)$$

- If $D_{\circ,n} = \nabla D_\circ \cdot (W_n, Z_n)$ for $\circ \in \{W, Z\}$, $n \geqslant 1$, then

$$D_{W,0} W_n = N_{W,n-1} - \sum_{j=0}^{n-2} \binom{n-1}{j} D_{W,n-1-j} W_{j+1},$$

$$Z_n D_{Z,1}(n-k) = -\sum_{j=1}^{n-2} \binom{n}{j} D_{Z,n-j} Z_{j+1}$$
$$+ \left(N_{Z,n} - (\partial_Z N_Z(P_2)) Z_n\right) + Z_1 \left(-D_{Z,n} + Z_n \partial_Z D_Z(P_2)\right).$$

The wiggle with r

- Restrict $1 < r < r^*$ where

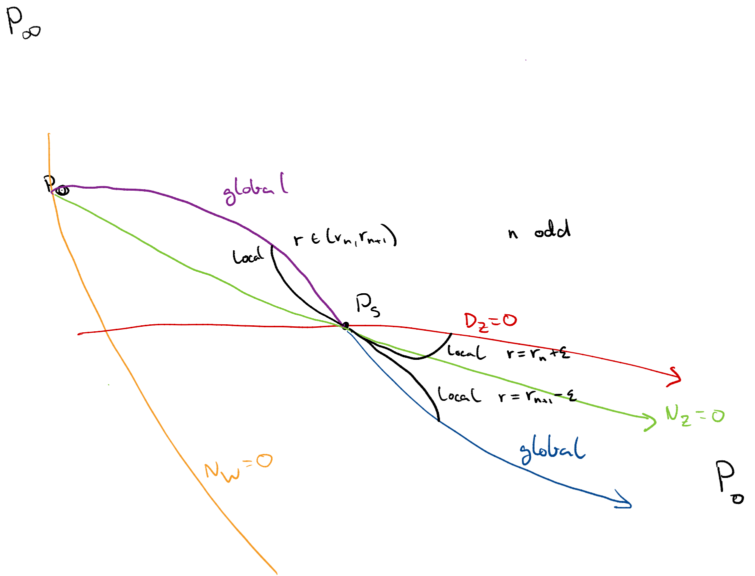
$$r^*(\gamma) = \begin{cases} \frac{2}{\left(\sqrt{2}\sqrt{\frac{1}{\gamma-1}}+1\right)^2} + 1 & 1 < \gamma < \frac{5}{3}. \\ \frac{3\gamma-1}{2+\sqrt{3(\gamma-1)}} & \gamma \geqslant \frac{5}{3}. \end{cases}$$

$k(r) : (1, r^*) \to \mathbb{R}$ is increasing with r and $k \to \infty$ as $r \to r^*$.

- For $j \in \mathbb{N}$, define r_j such that $j = k(r_j)$. At k_j the denominator of Z_n is singular and switches sign.
- To connect P_0 to P_∞, we show that for n odd
  1. For $r \in (r_n, r_{n+1})$ the solution to the left of P_c converges to P_∞ as $\xi \to \infty$.
  2. For $r = r_n + \varepsilon$ the solution to the right of P_c intersects the line $D_W = 0$.
  3. For $r = r_{n+1} - \varepsilon$ the solution to the right of P_c intersects the line $D_Z = 0$.

  For $\gamma > 1$, this is shown for $n = 3$, for $\gamma = \frac{7}{5}$, this is shown for odd n sufficiently large.

Barriers



P_∞
P_∞
global
$r \in (r_n, r_{n+1})$
n odd
Local
P_s
$D_z = 0$
Local $r = r_n + \varepsilon$
Local $r = r_{n+1} - \varepsilon$
$N_z = 0$
global
$N_w = 0$
P_0

The $\gamma = \frac{7}{5}$ case

- For $\gamma < \frac{5}{3}$ and fixed n, (W_n, Z_n) converge as $r \to r^*$ in a non-trivial manner.
- The *wiggle* can be determined from the sign and a lower bound on the coefficients of order $O(k)$.
- Via a computer assisted proof, compute the first 10000 coefficients with effective error bounds at r^* ($Z_{10000} \sim 6 \times 10^{46770}$).
- The proof works for general $\gamma < \frac{5}{3}$; however, the computation degenerates as $\gamma \to \frac{5}{3}$.

Strategy for stability for Euler and Navier-Stokes

The basic strategy is:

1. Linearize Euler/Navier-Stokes around self-similar profiles.
2. Show the linearized operator as finitely many unstable modes.
3. A Brouwer fix point argument shows there exists a manifold of initial data of finite co-dimension leading to imploding asymptotically self-similar blow-up.

Write

$$w(R,t) = r^{-1}(T-t)^{r^{-1}-1}\mathcal{W}\left(\frac{R}{(T-t)^{\frac{1}{r}}}, -\frac{\log(T-t)}{r}\right)$$

$$z(R,t) = r^{-1}(T-t)^{r^{-1}-1}\mathcal{Z}\left(\frac{R}{(T-t)^{\frac{1}{r}}}, -\frac{\log(T-t)}{r}\right).$$

and define the self-similar variables

$$s = -\frac{\log(T-t)}{r}, \quad \zeta = \frac{R}{(T-t)^{\frac{1}{r}}} = e^s R = \exp(\xi).$$

The Navier-Stokes equations ($\mu_1 = 1$ and $\mu_2 = -1$) become

$$(\partial_s + r - 1)\mathcal{W} + (\zeta + \tfrac{1}{2}(\mathcal{W} + \mathcal{Z} + \alpha(\mathcal{W} - \mathcal{Z})))\partial_\zeta \mathcal{W} + \frac{\alpha}{2\zeta}(\mathcal{W}^2 - \mathcal{Z}^2)$$

$$= \frac{r^{1+\frac{1}{\alpha}}2^{1/\alpha}}{\alpha^{1/\alpha}\zeta^2((\mathcal{W}-\mathcal{Z}))^{\frac{1}{\alpha}}} e^{(2-r+\frac{1}{\alpha}(1-r))s}\left(\partial_\zeta(\zeta^2\partial_\zeta(\mathcal{W}+\mathcal{Z})) - 2(\mathcal{W}+\mathcal{Z})\right)$$

$$(\partial_s + r - 1)\mathcal{Z} + (\zeta + \tfrac{1}{2}(\mathcal{W} + \mathcal{Z} - \alpha(\mathcal{W} - \mathcal{Z})))\partial_\zeta \mathcal{Z} - \frac{\alpha}{2\zeta}(\mathcal{W}^2 - \mathcal{Z}^2)$$

$$= \frac{r^{1+\frac{1}{\alpha}}2^{1/\alpha}}{\alpha^{1/\alpha}\zeta^2((\mathcal{W}-\mathcal{Z}))^{\frac{1}{\alpha}}} e^{(2-r+\frac{1}{\alpha}(1-r))s}\left(\partial_\zeta(\zeta^2\partial_\zeta(\mathcal{W}+\mathcal{Z})) - 2(\mathcal{W}+\mathcal{Z})\right)$$

If $2 - r + \frac{1}{\alpha}(1 - r) < 0$ then the dissipation term is an exponentially decaying error term, i.e.

$$r > \frac{2\gamma}{\gamma + 1},.$$

Recall $1 < r < r*$. Then, the adiabatic exponents are restricted to

$$1 < \gamma < \frac{2+\sqrt{3}}{\sqrt{3}}$$

.

Linearization

Let $(\overline{W}, \bar{Z})$ be an exact, self-similar Euler profile and define $(\widetilde{W}, \tilde{Z}) = (\mathcal{W} - \overline{W}, \mathcal{Z} - \bar{Z})$. Rewrite Euler/NS in linearized form

$$\partial_s \widetilde{W} = \mathcal{L}_W(\widetilde{W}, \tilde{Z}) + \mathcal{F}_W \quad \text{and} \quad \partial_s \tilde{Z} = \mathcal{L}_W(\widetilde{W}, \tilde{Z}) + \mathcal{F}_Z$$

where $(\mathcal{F}_W, \mathcal{F}_Z)$ include nonlinear and dissipative terms.

- We study the linearized operator $\mathcal{L} = (\mathcal{L}_W, \mathcal{L}_Z)$
- It is also sometimes useful to write the operator in terms of self-similar velocity $\mathcal{U}$ and sound speed $\mathcal{S}$, so that $\mathcal{L} = (\mathcal{L}_U, \mathcal{L}_S)$

Strategy for linear stability

1. Causality is used to cut-off the linearized equation on a neighborhood of the backwards acoustic cone ($|\zeta| \leqslant 1$). Redefine $\mathcal{L}$ using cut-offs.
2. Show that for large m and small $\delta_g > 0$, $\mathcal{L}$ decomposes as

$$\mathcal{L} = A = A_0 - \delta_g + K$$

   for some A_0 maximally dissipative on H^m_{Rad} and K is some compact.
3. Use (U, S) variables for high derivative energy estimates (dissipativity) and (W, Z) for low derivative arguments (maximality).
4. Lumer-Phillips theorem ensures exponential decay modulo a finite dimensional unstable space.

Theorem

Let $\delta > 0$, and T the (strongly continuous) semigroup generated by $A = A_0 - \delta + K$ where $A_0 : H \to H$ is a maximally dissipative operator and $K : H \to H$ is compact. Then, there are finitely many eigenvalues λ_i with $Re(\lambda_i) \geqslant 0$.

Let $\psi_i \in H$ be the corresponding eigenfunctions, let V be the finite dimensional space $V = span(\psi_i)$. There exists $U \subset H$ such that U, V are invariant spaces of A and $H = U \oplus V$. Moreover

$$\|T(t)X\| \leqslant Ce^{-\delta t/2}\|X\|$$

for all $X \in U$

Dissipativity

Aim is to show that for (U, S) in a finite co-dim subspace of $H^m_{\mathrm{Rad}}[0, 2]$

$$\langle \mathcal{L}(U, S),\ (U, S) \rangle_{H^m} \leqslant -\|(U, S)\|^2_{H^m}$$

- Highest order terms cancel.
- Left over with *m*th order terms have a good sign, if m is large.
- Projecting away low frequencies controls lower order terms.

Nonlinear stability argument

1. Control on unstable modes of truncated equations $\implies$ global control on low order derivatives.
2. Global control on low order derivatives $\implies$ global control on high order derivatives.
3. Global control on high order derivatives $\implies$ dissipation can be treated as an decaying forcing term for the linearized problem.
4. Topological argument closes the argument, leading to a global solution of the self-similar equation.

Topological argument

Let $\{\psi_i\}_{i=1,\ldots,N}$ be a basis for the unstable manifold and

$$\kappa_i = \langle (\widetilde{U}, \widetilde{S}), \psi_i \rangle$$

be the unstable modes of a solution. Let $\mathcal{R}(s) = B_{\mathbb{R}^N}(0, e^{-\delta s})$. If $\kappa \in \partial\mathcal{R}$, we can show that κ leaves $\mathcal{R}$ immediately.

Consider solutions with initial unstable modes $\kappa(s_0) \in \mathcal{R}(s_0)$. Suppose all such initial data leave $\mathcal{R}$ in finite time. This would imply (after rescaling) the existence of a continuous map from $B_{\mathbb{R}^N}(0, 1)$ to $\mathcal{S}^{N-1}$, which leads to a contradiction.

Computer assisted arguments

Computer assisted interval arithmetic is used to prove the positivity of certain quantities: e.g. positivity of a polynomial over finite interval (barrier arguments) or the sign of a Taylor coefficient.

We define an arithmetic such that for intervals X, Y and $x \in X$, $y \in Y$

$$x \star y \in X \star Y\,.$$

for a given operator $\star$. E.g.

$$[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$
$$[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}].$$

Thank you!