Parabolic gluing and construction of blow-up solutions

Yannick Sire

Johns Hopkins University

Introduction

GOAL: Construct blow-up solutions in parabolic problems in finite and/or infinite time

Dichotomy: blow-up in finite time *vs* Global well-posedness

- Either CONVERGENCE in infinite time (Example: Yamabe flow on closed manifolds)
- Or blow-up in infinite time (Examples: Harmonic map flow (for some targets), Fujita equation, etc...)

TOOL: inner/outer parabolic gluing

Inner–outer gluing method for parabolic equations

- good approximation $\Longrightarrow$ small error
- $u = \text{approximation} + \overbrace{\underbrace{\eta_R \phi(y,t)}_{\text{inner}} + \underbrace{\psi(x,t)}_{\text{outer}}}^{\text{perturbation}}, \quad y = \frac{x-\xi(t)}{\lambda(t)}$
- Inner problem: $\lambda^2 \phi_t = L[\phi] + \underbrace{\text{coupling}(\psi) + \text{error}}_{\mathcal{H}}$
- Outer problem (maximum principle):
  $\psi_t = \Delta_x \psi + \underbrace{(\phi \Delta \eta_R + 2\nabla \eta_R \cdot \nabla \phi)}_{\text{coupling}} + \text{nonlinear terms} + \text{error}$
- Orthogonality conditions
  - $\int \mathcal{H} Z_j dy = 0 \Longrightarrow$ good inner solution
  - $\int \mathcal{H} Z_j dy = 0 \Longrightarrow$ reduced equations for λ, ξ
- Fixed point argument: ϕ, ψ, λ, ξ

Fast diffusion equations and Yamabe flow

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, we consider the fast diffusion equation

$$\begin{cases} \frac{\partial w}{\partial \tau} = \Delta w^m \text{ in } \Omega \times (0, \infty), \\ w = 0 \text{ on } \partial\Omega \times (0, \infty), \\ w(\cdot, 0) = w_0 \text{ in } \overline{\Omega} \end{cases} \tag{1}$$

with $m \in (0, 1)$. The equation is a singular but non-degenerate parabolic one. There exists a unique positive classical solution w which is local in time for the Dirichlet problem (1), the solution vanishes at finite time as $\tau \to T^- < \infty$, $w > 0$ in $\Omega \times (0, T)$ and $w(x, T) = 0$.

The asymptotic behaviour of solutions for (1) near the extinction time T has attracted much attention in the past two decades. Suppose $\Omega = B_1(0) := \{x \in \mathbb{R}^n ||x| < 1\} \subset \mathbb{R}^n$, when $m \in (m_s, 1)$ and $m_s := \frac{n-2}{n+2}$, from the classical work of Berryman and Holland, the solution near the extinction time has a separated self-similar form

$$w(x,\tau) = (T-\tau)^{\frac{1}{1-m}} S(x),$$

where $S(x)$ is the positive solution of the following nonlinear elliptic problem

$$\Delta S^m + (1-m)^{-1} S = 0 \text{ in } \Omega, \quad S = 0 \text{ on } \partial\Omega.$$

Under the transformation

$$u(x,t) = (T-\tau)^{-m/(1-m)} w(x,\tau)^m|_{\tau=T(1-e^{-t})}, \tag{2}$$

Problem (1) changes into the Yamabe flow equation on the bounded domain Ω as follows,

$$\begin{cases} \frac{\partial u^p}{\partial t} = \Delta u + u^p \text{ in } \Omega \times (0,\infty), \\ u = 0 \text{ on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \text{ in } \overline{\Omega}, \end{cases} \tag{3}$$

for a function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ and positive initial datum u_0 satisfying $u_0|_{\partial\Omega} = 0$, $p = \frac{n+2}{n-2}$.

The role of the Green function

Let $H(x,y)$ be the regular part of the Green's function on Ω with Dirichlet boundary condition, i.e., for fixed $y \in \Omega$, $H(x,y)$ satisfies $\Delta_x H(x,y) = 0$ in Ω, $H(x,y) = \frac{(n(n-2))^{\frac{n-2}{4}}}{|x-y|^{n-2}}$ for $x \in \partial\Omega$.

Let $q_1, \cdots, q_k$ to be k different but fixed points in Ω, define the following matrix,

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1,q_1) & -G(q_1,q_2) & \cdots & -G(q_1,q_k) \\ -G(q_1,q_2) & H(q_2,q_2) & -G(q_2,q_3)\cdots & -G(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1,q_k) & \cdots & -G(q_{k-1},q_k) & H(q_k,q_k) \end{bmatrix}. \quad (4)$$

Theorem (S.-Wei-Zheng)

Suppose $n \geq 3$, k is a positive integer and $q_1, \cdots, q_k$ are k different but fixed points in Ω such that the matrix defined in (4) is positive definite, then there exist an initial data u_0 and smooth functions $\mu_j(t)$, $\xi_j(t)$ such that the solution of problem (3) has the following asymptotic form when $t \to +\infty$,

$$u(x,t) = \sum_{j=1}^{k} \left(\alpha_n \left(\frac{\mu_j(t)}{\mu_j^2(t) + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}} - \mu_j^{\frac{n-2}{2}}(t) H(x, q_j) \right) + \varphi(x,t),$$

where $\mu_j = \beta_j t^{-\frac{1}{n-2}}(1 + o(1))$ for some $\beta_j > 0$, $\xi_j - q_j = o(t^{-\frac{1}{n-2}})$, $\alpha_n = (n(n-2))^{\frac{n-2}{4}}$ and $\varphi(x,t) \to 0$ uniformly away from the points $q_1, \cdots, q_k$ as $t \to +\infty$.

Yamabe flow (Hamilton)

In the case of $\mathbb{S}^n$ with its standard Riemannian metric $g_{\mathbb{S}^n}$, the Yamabe flow evolving a conformal metric $g = v^{\frac{4}{n-2}}(\cdot, t) g_{\mathbb{S}^n}$ takes the following form

$$(v^{\frac{n+2}{n-2}})_t = \Delta_{\mathbb{S}^n} v - c_n v, \quad c_n = \frac{n(n-2)}{4}, \tag{5}$$

which is equivalent to the problem via the stereographic projection and cylindrical changes of variables

$$\begin{cases} \frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \Delta u + u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 \text{ in } \mathbb{R}^n \end{cases} \tag{6}$$

Global existence and convergence on closed manifolds: B. Chow, R. Ye, S. Brendle

Study near the extinction time: Del Pino-Saez

Bubble towers

- Daskalopoulos, del Pino and Sesum constructed a new class of type II ancient solutions to the the Yamabe flow, these solutions are rotationally symmetry and converge to a tower of spheres when $t \to -\infty$.
- del Pino-Musso-Wei constructed bubble tower solutions for the energy critical heat equation.
- We conjecture that bubble-tower solutions occur for the Yamabe flow in bounded domains

Harmonic map heat flows

Harmonic maps into spheres

Critical points of the conformally invariant Dirichlet energy ($u : \Omega \subset \mathbb{R}^2 \to \mathbb{S}^\ell$)

$$\mathcal{L}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx.$$

Euler-Langrange equation:

$$-\Delta u = |\nabla u|^2 u.$$

Major Issue for this equation: REGULARITY (Full in 2D and partial in higher dimension) Hélein, Schoen-Uhlenbeck, Evans, Rivière, Struwe-Rivière, etc..

Harmonic map heat flow

Harmonic map heat flow or *how to construct harmonic maps*

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0,\infty), \\ u = \varphi \quad \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \quad \text{in } \Omega \end{cases} \tag{7}$$

for a function $u : \Omega \times [0,\infty) \to \mathbb{S}^2$ and $u_0 : \Omega \to \mathbb{S}^2$ is a given smooth map satisfying $\varphi = u_0|_{\partial\Omega}$.

Eells-Sampson, Chen, Chen-Struwe, etc...

Singularity Analysis

If $T > 0$ is the first time where the continuity breaks, one must have

$$\|\nabla u(\cdot, t)\|_\infty \to +\infty \text{ as } t \uparrow T.$$

From the results of Ding-Tian, Lin-Wang, Qing, Qing-Tian, Struwe, Topping and C.-Y. Wang:

There exists a sequence $t_n \to T$ such that $u(z, t_n)$ blows-up at given k points $q_1, \dots, q_k \in \Omega$ by bubbling. Precisely, there holds

$$u(z, t_n) - u_*(z) - \sum_{i=1}^{k} [U_i \left(\frac{z - q_i^n}{\lambda_i^n} \right) - U_i(\infty)] \to 0 \text{ in } H^1(\Omega), \tag{8}$$

where $u_* \in H^1(\Omega)$, $q_i^n \to q_i$, $0 < \lambda_i^n \to 0$ satisfying for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \to +\infty.$$

Singularity Analysis

Here U_i's are finite energy entire harmonic maps, i.e., they are solutions $U : \mathbb{R}^2 \to \mathbb{S}^2$ of the harmonic map equation on $\mathbb{R}^2$

$$\Delta U + |\nabla U|^2 U = 0 \text{ in } \mathbb{R}^2. \tag{9}$$

It is well known that U corresponds to a complex rational function or its conjugate after stereographic projection, whose energy is given by

$$\int_{\mathbb{R}^2} |\nabla U|^2 d\varrho = 4\pi m, \quad m \in \mathbb{N}.$$

Singularity Analysis

Therefore, $u(\cdot, t_n) \rightharpoonup u_*$ in $H^1(\Omega)$ and, for some $m_i \in \mathbb{N}^+$, there holds

$$|\nabla u(\cdot, t_n)|^2 \ \rightharpoonup^* \ |\nabla u_*|^2 + \sum_{i=1}^{k} 4\pi m_i \, \delta_{q_i}$$

in the measure sense, where δ_q is the Dirac mass at q.

Topping found the blow-up rates as $\lambda_i^n = o(T - t_n)^{\frac{1}{2}}$, which tells us that the blow-up is of "type II", i.e., it does not occur in a self-similar rate.

If the blow-up occurs in infinite time, i.e., $T = +\infty$, decomposition formula also holds.

Constructions of blow up solutions

- Chang, Ding and Ye, J. Differential Geom., 1992 constructed a first example of blow-up solutions

- Raphael and Schweyer, Comm. Pure Appl. Math., 2013 constructed a 1-corrotational blow-up solution rigorously.

- Davila, del Pino and Wei, Inventiones Mathematicae, 2020, the general, nonsymmetric case was considered, which asserts that for any given finite set of points in Ω, a solution blowing-up at those points simultaneously exists under suitable initial and boundary conditions.

Harmonic maps with free boundary

Let (M, g) be an m-dimensional Riemannian manifold with boundary ∂M and N be an l-dimensional manifold without boundary. Suppose Σ is a k-dimensional submanifold in N without boundary.

Any continuous map $u_0 : M \to N$ satisfying $u_0(\partial M) \subset \Sigma$ defines a relative homotopy class in maps from $(M, \partial M)$ to (N, Σ).

A map $u : M \to N$ with $u(\partial M) \subset \Sigma$ is called homotopic to u_0 if there exist a continuous homotopy $h : [0, 1] \times M \to N$ satisfying $h([0, 1] \times \partial M) \subset \Sigma$, $h(0) = u_0$ and $h(1) = u$.

Harmonic maps with free boundary

An interesting problem is that whether or not each relative homotopy class of maps has a representation by harmonic maps, which is equivalent to the following problem,

$$\begin{cases} -\Delta u = \Gamma(u)(\nabla u, \nabla u), \\ u(\partial M) \subset \Sigma, \\ \frac{\partial}{\partial \nu} u \perp T_u \Sigma. \end{cases} \tag{10}$$

Harmonic map flow with free boundary

A classical method for (10) is to study the following parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \Gamma(u)(\nabla u, \nabla u) \text{ on } M \times [0, \infty), \\ u(x,t) \in \Sigma \text{ on } x \in \partial M, t \geq 0, \\ \frac{\partial}{\partial \nu} u(x,t) \perp T_{u(x,t)}\Sigma \text{ for } x \in \partial M, t \geq 0, \\ u(\cdot, 0) = u_0 \text{ on } M. \end{cases} \tag{11}$$

This is the so-called harmonic map flow with free boundary.

Geometric applications

A. Fraser and R. Schoen, Sharp eigenvalue bounds and minimal surfaces in the ball. Invent. Math. (2016)

J. Jost, L. Liu and M. Zhu, The qualitative behavior at the free boundary for approximate harmonic maps from surfaces. Math. Ann. (2019)

Paul Laurain and Romain Petrides, Regularity and quantification for harmonic maps with free boundary, Advances in Calculus of Variations (2017).

Vincent Millot and Yannick Sire, On a fractional Ginzburg-Landau equation and 1/2-harmonic maps into spheres, Arch. Ration. Mech. Anal. (2015).

Harmonic map flow with free boundary

Y. Chen and F. Lin (JGEA) raised the following question:

"When M is a smooth domain in $\mathbb{R}^2$, $N = \mathbb{R}^n$ and Σ a smooth compact submanifold of $\mathbb{R}^n$, is there a smooth initial datum u_0 such that (11) has no global smooth solutions ?"

Blow-up solutions

We consider the problem (11) when $M = \mathbb{R}^2_+$ and $\Sigma = \mathbb{S}^1 \subset \mathbb{R}^2$, i.e. the following parabolic equation

$$\begin{cases} u_t = \Delta u \text{ in } \mathbb{R}^2_+ \times (0,T), \\ u(x,0,t) \in \mathbb{S}^1 \text{ for all } (x,0,t) \in \partial\mathbb{R}^2_+ \times (0,T), \\ -\frac{du}{dy}(x,0,t) \perp T_{u(x,0,t)}\mathbb{S}^1 \text{ for all } (x,0,t) \in \partial\mathbb{R}^2_+ \times (0,T), \\ u(\cdot,0) = u_0 \text{ in } \mathbb{R}^2_+ \end{cases} \tag{12}$$

for a function $u : \mathbb{R}^2_+ \times [0,T) \to \mathbb{R}^2$. Here $u_0 : \mathbb{R}^2_+ \to \mathbb{R}^2$ is a given smooth map and $\perp$ stands for orthogonality.

Harmonic map flow with free boundary

Theorem (S.-Wei-Zheng)

Given points $q = (q_1, \cdots, q_k) \in \left(\partial\mathbb{R}^2_+\right)^k := (\mathbb{R} \times \{0\})^k$ *and any sufficiently small* $T > 0$*, there exists* u_0 *such that the solution* $u_q(x,t)$ *of Problem (12) blows-up at exactly those* k *points as* $t \nearrow T$*. More precisely, there exist numbers* $k_i^* > 0$ *and a function* $u_* \in H^1(\mathbb{R}^2_+) \cap C(\mathbb{R}^2_+)$ *such that*

$$u_q(x,y,t) - u_*(x,y) - \sum_{j=1}^{k} \left[\omega\left(\frac{x-q_i}{\lambda_i}, \frac{y}{\lambda_i}\right) - \omega(\infty)\right] \to 0 \text{ as } t \nearrow T,$$

in the H^1 *and uniform senses in* $\mathbb{R}^2_+$ *where*

$$\lambda_i(t) = k_i^* \frac{T-t}{|\log(T-t)|^2}(1+o(1)) \text{ as } t \nearrow T.$$

$$|\nabla u_q(\cdot,\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 2\pi \sum_{j=1}^{k} \delta_{q_j} \text{ as } t \nearrow T.$$

Comments

Compared to *Davila, del Pino and Wei*, the main difficulty is the *nonlocality* of the problem. Problem (12) is equivalent to:

$$\sqrt{\partial_t - \Delta} u = \frac{1}{8\pi}$$
$$\left[\int_0^{+\infty} \int_{\mathbb{R}} |u(x,0,t) - u(x-z,0,t-\tau)|^2 \frac{e^{-\frac{|z|^2}{4\tau}}}{\tau^2} dz d\tau \right] u(x,0,t).$$

The problem under consideration interpolates between the two-dimensional harmonic map flow and the half-harmonic map flow. It inherits characteristics from both problems.

Partially regular flows

We will try to solve the following version of the heat flow:

$$\begin{cases} \partial_t U - \Delta U = 0 & \text{in } \mathbb{R}^{n+1}_+ \times \mathbb{R}_+, \\ U(x,0,t) \in \Sigma & x \in \mathbb{R}^n, t > 0, \\ -\lim_{y \to 0^+} \dfrac{\partial U}{\partial y}(x,y,t) \perp T_{U(x,0,t)}\Sigma & x \in \mathbb{R}^n, t > 0, \\ U(x,y,0) = U_0(x,y) & (x,y) \in \mathbb{R}^{n+1}_+. \end{cases} \tag{13}$$

I will focus on the study of (13) for

$$\boxed{\Sigma = \mathbb{S}^{L-1}}$$

and solve (13) through convergence analysis of its Ginburg-Landau approximation scheme.

Given $U_0 \in \dot{H}^1(\mathbb{R}^{n+1}_+, \mathbb{S}^{L-1})$ and $\varepsilon > 0$, consider

$$\begin{cases} (\partial_t - \Delta) U_\varepsilon(x,y,t) = 0 & \text{in } \mathbb{R}^{n+1}_+ \times (0,\infty), \\ U_\varepsilon(x,y,0) = U_0(x,y) & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial U_\varepsilon}{\partial y} = -\dfrac{1}{\varepsilon^2}(1-|U_\varepsilon|^2)U_\varepsilon & \text{on } \partial\mathbb{R}^{n+1}_+ \times (0,\infty). \end{cases} \tag{14}$$

For fixed $\varepsilon > 0$, (14) is the gradient flow of

$$E_\varepsilon(U) = \int_{\mathbb{R}^{n+1}_+} \frac{1}{2}|\nabla U|^2 \, dxdy + \int_{\partial\mathbb{R}^{n+1}_+} \frac{(1-|U|^2)^2}{4\varepsilon^2} \, dx.$$

There exist smooth solutions $U_\varepsilon : \mathbb{R}^{n+1}_+ \times (0,\infty) \to \mathbb{R}^L$ of (14):

$$\begin{aligned} &E_\varepsilon(U_\varepsilon)(t) + \int_0^t \int_{\mathbb{R}^{n+1}_+} |\partial_t U_\varepsilon|^2 \, dxdydt \\ &\le E_\varepsilon(U_0) = \int_{\mathbb{R}^{n+1}_+} \frac{1}{2}|\nabla U_0|^2 \, dxdy. \end{aligned} \tag{15}$$

For $U_0 \in H^1(\mathbb{R}^{n+1}_+, \mathbb{S}^{L-1})$, let $u_0 = U_0|_{\partial\mathbb{R}^{n+1}_+}$. Let $\mathcal{P}^k$ denote the k-dimensional Hausdorff measure on $\mathbb{R}^{n+1} \times \mathbb{R}$ with respect to

$$\delta((X,t),(Y,s)) = \max\left\{|X-Y|, \sqrt{|t-s|}\right\}.$$

Theorem (Hyder, Segatti, S., Wang)

1) $\exists\, U_* \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^{n+1}_+, \mathbb{S}^{L-1}))$ with $\partial_t U_* \in L^2(\mathbb{R}^{n+1}_+ \times \mathbb{R}_+)$ solving

$$\begin{cases} (\partial_t - \Delta) U_* = 0 & \text{in } \mathbb{R}^{n+1}_+ \times (0,\infty), \\ U_*\big|_{t=0} = U_0 & \text{on } \mathbb{R}^{n+1}_+, \\ U_*(x,0,t) \in N; \quad \frac{\partial U_*}{\partial y}(x,0,t) \perp T_{U_*(x,0,t)} N & \text{on } \mathbb{R}^n \times (0,\infty). \end{cases}$$

such that $U_\varepsilon \rightharpoonup U_*$ in $H^1(\mathbb{R}^{n+1}_+ \times \mathbb{R}_+)$.
2) $\exists\, \Sigma \subset \partial\mathbb{R}^{n+1}_+ \times (0,\infty)$, with $\mathcal{P}^{n+1}(\Sigma) < \infty$, such that

$$U_\varepsilon \to U_* \in C^2_{loc}\big(\overline{\mathbb{R}^{n+1}_+} \times (0,\infty) \setminus \Sigma\big).$$

Theorem

3) Set $u_* = U_*\big|_{\partial\mathbb{R}^{n+1}_+\times[0,\infty)}$. *Then* $u_* \in C^\infty(\mathbb{R}^n \times (0,\infty)\setminus\Sigma)$ *solves the* $\frac{1}{2}$*-harmonic heat flow:*

$$\begin{cases}(\partial_t - \Delta)^{\frac{1}{2}} u_* \perp T_{u_*}\mathbb{S}^{L-1} & \text{in} \quad \mathbb{R}^n \times (0,\infty),\\ u_*(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^n.\end{cases} \tag{16}$$

4) For any $C_0 > 0$, $\exists \epsilon_0 > 0$ *such that if*

$$\big\|\nabla U_0\big\|_{L^\infty(\mathbb{R}^{n+1}_+)} \le C_0, \quad E(U_0) \le \epsilon_0,$$

$U_* \in C^\infty\big(\overline{\mathbb{R}^{n+1}_+} \times (0,\infty)\big)$ $(\Rightarrow u_* = U_*\big|_{\partial\mathbb{R}^{n+1}_+\times[0,\infty)} \in C^\infty)$.

Lemma 1 *For* $Z_0 = (X_0, t_0)$ *with* $X_0 \in \partial\mathbb{R}^{n+1}_+$ *and* $t_0 > 0$, *the renormalized energy*

$$E(U_\varepsilon, Z_0, R) = \frac{1}{2}\int_{T_R^+(t_0)} G_{X_0,t_0}(X,t)|\nabla U_\varepsilon|^2\, dX dt + \frac{1}{4\varepsilon^2}\int_{\partial T_R^+(t_0)} G_{X_0,t_0}(X,t)(1-|u_\varepsilon|^2)^2\, dx dt \qquad (17)$$

is monotone nondecreasing in R. Here

$$G_{X_0,t_0}(X,t) = |t-t_0|^{-(\frac{n}{2}+\frac{1}{2})} e^{-\frac{|X-X_0|^2}{4|t-t_0|}},\ t < t_0;$$
$$T_R^+(t_0) = \left\{X \in \mathbb{R}^{n+1}_+,\ t_0 - 4R^2 < t < t_0 - R^2\right\};$$
$$\partial T_R^+(t_0) = \left\{x \in \mathbb{R}^n,\ t_0 - 4R^2 < t < t_0 - R^2\right\}.$$

Lemma 2 *Let* $U_\varepsilon : \mathbb{R}^{n+1}_+ \times [0,\infty) \to \mathbb{R}^L$ *solve* (14). *Then* $|U_\varepsilon| \leq 1$.

Proof. By maximum principle and Hopf boundary Lemma.

Lemma 3 (cleaning-out) $\exists \eta_0 > 0$ *and* $\delta_0 > 0$ *such that if*

$$E(U_\varepsilon, (X_0, t_0), 1) \leq \eta_0,$$

for $X_0 \in \partial\mathbb{R}^{n+1}_+$ *and* $t_0 > 0$, *then*

$$\big|U_\varepsilon\big| \geq \frac{1}{2} \quad \text{on} \quad P^+_{\delta_0}(X_0, t_0).$$

Here

$$P^+_{\delta_0}(X_0, t_0) = \big\{X \in B_{\delta_0}(X_0) \cap \mathbb{R}^{n+1}_+,\ |t - t_0| \leq \delta_0^2\big\}.$$

A key ingredient is an η_0-uniform gradient estimate (for the stationnary case, see Millot-S. ARMA).

Lemma 3 *For* $\alpha \in (0,1)$, $\exists \eta_0 > 0$ *such that if*

$$E(U_\varepsilon, (X_0, t_0), 1) \leq \eta_0$$

for $X_0 \in \partial \mathbb{R}^{n+1}_+$ *and* $t_0 > 0$, *then*

$$\left\| U_\varepsilon \right\|_{C^{1+\alpha}(P^+_{\frac{1}{4}}(X_0,t_0))} \leq C(\eta_0, \alpha), \ \forall \varepsilon > 0. \tag{18}$$

Proof. Assume $(X_0, t_0) = (0, 2)$. It suffices to show (18) under additional assumptions:

$$\frac{1}{2} \leq |U_\varepsilon| \leq 1, \ |\partial_t U_\varepsilon| \leq 4, \ |\nabla U_\varepsilon| \leq 4 \ \text{ in } \ P^+_1(0,2).$$

Polar decomposition:

$$U_\varepsilon = \rho_\varepsilon \omega_\varepsilon,\ with\ \rho_\varepsilon = |U_\varepsilon|\ and\ \omega_\varepsilon = \frac{U_\varepsilon}{|U_\varepsilon|}.$$

Then

$$\frac{1}{2} \le \rho_\varepsilon \le 1, \quad |\nabla \rho_\varepsilon| + |\nabla \omega_\varepsilon| \le 4,$$

and

$$\begin{cases} \partial_t \rho_\varepsilon - \Delta \rho_\varepsilon = -|\nabla \omega_\varepsilon|^2 \rho_\varepsilon & \text{in} \quad P_1^+ \\ \partial_\nu \rho_\varepsilon = \dfrac{1}{\varepsilon^2}(1 - |\rho_\varepsilon|^2)\rho_\varepsilon & \text{on} \quad \Gamma_1, \end{cases} \tag{19}$$

$$\begin{cases} \partial_t \omega_\varepsilon - \Delta \omega_\varepsilon = f_\varepsilon = 2\dfrac{\nabla \rho_\varepsilon}{\rho_\varepsilon} \cdot \nabla \omega_\varepsilon + |\nabla \omega_\varepsilon|^2 \omega_\varepsilon & \text{in } P_1^+ \\ \partial_\nu \omega_\varepsilon = 0 & \text{on } \Gamma_1, \end{cases} \tag{20}$$

$$|f_\varepsilon| \le 160 \Rightarrow \omega_\varepsilon \in W_q^{2,1}, \forall q < \infty \Rightarrow \|\nabla \omega\|_{C^\alpha} \le C(\alpha).$$

Some comments

- For the intrinsic version, i.e embedding the target N isometrically into an Euclidean space and $u(\partial M) \subset N$, the heat flow has been studied by Struwe and Chen-Lin.

- Our approach is a singular perturbation on the boundary

Sketch of the proof

Step 1. Construction of approximation

Let $t_0 > 0$ be a large number to be chosen later and consider the following problem

$$\begin{cases} (u^p)_t = \Delta u + u^p \text{ in } \Omega \times (t_0, \infty), \\ u = 0 \text{ on } \partial\Omega \times (t_0, \infty), \end{cases} \tag{21}$$

for $p = \frac{n+2}{n-2}$. Let $q_1, \cdots, q_k \in \mathbb{R}^n$ be k fixed points, we are going to find a positive solution to (21) of form

$$u(x,t) \approx \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x)$$

with $\xi_j(t) \to q_j$, $\mu_j(t) \to 0$ as $t \to \infty$ for all $j = 1, \cdots, k$ and $U_{\mu_j(t),\xi_j(t)}(x) = \mu_j(t)^{-\frac{n-2}{2}} U\left(\frac{x-\xi_j(t)}{\mu_j(t)}\right)$, $U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}$.

Denote the error operator as follows

$$S(u) := -(u^p)_t + \Delta u + u^p.$$

Then we have

$$S(U_{\mu_j(t),\xi_j(t)}) = \mu_j^{-\frac{n+2}{2}-1} U(y_j)^{p-1} \left(\dot{\mu}_j Z_{n+1}(y_j) + \dot{\xi}_j \cdot \nabla U(y_j)\right)$$

for $y_j = \frac{x-\xi_j(t)}{\mu_j(t)}$. Since $u=0$ on $\partial\Omega$, a natural better approximation than $\sum_{j=1}^k U_{\mu_j(t),\xi_j(t)}(x)$ should be

$$\tilde{z}(x,t) = \sum_{j=1}^{k} \tilde{z}_j(x,t) \text{ with } \tilde{z}_j(x,t) := U_{\mu_j,\xi_j}(x) - \mu_j^{\frac{n-2}{2}} H_{\mu_j}(x,q_j). \quad (22)$$

Here for fixed $y \in \Omega$, $H_{\mu_j}(x,y)$ satisfies $\Delta_x H_{\mu_j}(x,y) = 0$ in Ω, $H_{\mu_j}(x,y) = \frac{(n(n-2))^{\frac{n-2}{4}}}{(\mu_j^2+|x-y|^2)^{\frac{n-2}{2}}}$ for $x \in \partial\Omega$.

Then from the equation satisfied by $U_{\mu_j(t),\xi_j(t)}(x)$ and the fact that $H_{\mu_j}(x,q)$ is a harmonic function, the error of $\tilde{z}$ is

$$S(\tilde{z}) = -\sum_{i=1}^{k} \partial_t \tilde{z}_i^p + \left(\sum_{i=1}^{k} \tilde{z}_i\right)^p - \sum_{i=1}^{k} U_{\mu_i,\xi_i}^p. \tag{23}$$

Moreover, for a fixed index j, in the region $|x - q_j| \leq \frac{1}{2}\min_{i\neq l}|q_i - q_l|$, set $x = \xi_j + \mu_j y_j$, there holds

$$S[\tilde{z}] = \mu_j^{-\frac{n+2}{2}}(\mu_j E_{0j} + \mu_j E_{1j} + \mathcal{R}_j)$$

with

$$E_{0j} = pU(y_j)^{p-1}\left[-\mu_j^{n-3}H(q_j,q_j) + \sum_{i\neq j}\mu_j^{\frac{n-2}{2}-1}\mu_i^{\frac{n-2}{2}}G(q_j,q_i)\right] + \mu_j^{-2}\dot{\mu}_j pU(y_j)^{p-1}Z_{n+1}(y_j),$$

$$E_{1j} = pU(y_j)^{p-1}\left[-\mu_j^{n-2}\nabla H(q_j,q_j) + \sum_{i\neq j}\mu_j^{\frac{n-2}{2}}\mu_i^{\frac{n-2}{2}}\nabla G(q_j,q_i)\right]\cdot y_j + \mu_j^{-2}pU(y_j)^{p-1}\dot{\xi}_j\cdot\nabla U(y_j)$$

Suppose $u = \tilde{z} + \tilde{\phi}$ is the exact solution of (21) and write $\tilde{\phi}(x,t)$ in self-similar form around the point q_j,

$$\tilde{\phi}(x,t) = \mu_j^{-\frac{n-2}{2}} \phi\left(\frac{x-\xi_j}{\mu_j}, t\right). \tag{24}$$

Then we have

$$\begin{aligned} 0 &= \mu_j^{\frac{n+2}{2}} S[\tilde{z} + \tilde{\phi}] \\ &= -pU^{p-1}(y)\partial_t \phi + \Delta_y \phi + pU(y)^{p-1}\phi + \mu_j^{\frac{n+2}{2}} S[\tilde{z}] + A[\phi] \end{aligned} \tag{25}$$

with $A[\phi]$ being a high order term. To improve the approximation error, we require $\phi(y,t)$ equals (at main order) to the solution $\phi_{0j}(y,t)$ of the following equation

$$-pU^{p-1}(y)\partial_t \phi_{0j} + \Delta_y \phi_{0j} + pU(y)^{p-1}\phi_{0j} = -\mu_j^{\frac{n+2}{2}} S[\tilde{z}] \text{ in } \mathbb{R}^n. \tag{26}$$

Near the blow-up point q_j, equation (26) is mainly an elliptic problem of form

$$L_0[\phi] := \frac{1}{U^{p-1}} \left(\Delta_y \phi + pU(y)^{p-1}\phi\right) = h(y) \text{ in } \mathbb{R}^n. \tag{27}$$

Consider the eigenvalue problem $L_0[\phi] + \lambda\phi = 0$ on the weighted space $L^2(U^{p-1}dx)$, which has an infinite sequence of eigenvalues

$$\lambda_0 < \lambda_1 = \cdots = \lambda_n = \lambda_{n+1} = 0 < \lambda_{n+2} < \lambda_{n+3} < \cdots,$$

the associated eigenfunctions Z_j, $j = 0, 1, \cdots$ constitute an orthonormal basis of $L^2(U^{p-1}dx)$. It is well known that λ_0 is simple and $Z_0(y) = U(y)$. Therefore every bounded solution of $L_0[\phi] = 0$ in $\mathbb{R}^n$ is the linear combination of the functions

$$Z_1, \cdots, Z_{n+1},$$

where

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \cdots, n, \quad Z_{n+1}(y) := \frac{n-2}{2}U(y) + y \cdot \nabla U(y).$$

Furthermore, problem (27) is solvable if the following conditions

$$\int_{\mathbb{R}^n} h(y)Z_i(y)U^{p-1}(y)dy = 0 \quad \text{for all} \quad i = 1, \cdots, n+1$$

hold.
Now we consider the solvability condition for equation (26) with $i = n+1$,

$$\int_{\mathbb{R}^n} \mu_j^{\frac{n+2}{2}} S[\tilde{z}](y,t)Z_{n+1}(y)dy = 0. \tag{28}$$

We claim that if one choose $\mu_{0j} = b_j\mu_0(t)$ for some positive constants b_j, $j = 1, \cdots, k$ to be determined later, $\mu_0(t) = \gamma_n t^{-\frac{1}{n-2}}$ and γ_n is a positive constant depending only on n, identity (28) holds at main order.

Observe that the main contribution term to the integral on the left hand side of (28) is

$$E_{0j} = pU(y_j)^{p-1}\left[-\mu_j^{n-3}H(q_j,q_j) + \sum_{i\neq j}\mu_j^{\frac{n-2}{2}-1}\mu_i^{\frac{n-2}{2}}G(q_j,q_i)\right] + \mu_j^{-2}\dot{\mu}_j U(y_j)^{p-1}Z_{n+1}(y_j).$$

Then direct computations yield the following

$$\int_{\mathbb{R}^n}\mu_j^2(t)E_{0j}(y,t)Z_{n+1}(y)dy \approx c_1\left[\mu_j^{n-1}H(q_j,q_j) - \sum_{i\neq j}\mu_j^{\frac{n-2}{2}+1}\mu_i^{\frac{n-2}{2}}G(q_j,q_i)\right] + c_2\dot{\mu}_j$$

with

$$c_1 = -p\int_{\mathbb{R}^n}U(y)^{p-1}Z_{n+1}(y)dy,$$

$$c_2 = \int_{\mathbb{R}^n}U(y)^{p-1}\left|Z_{n+1}(y)\right|^2 dy.$$

Note that c_1, c_2 are finite positive numbers since we assume that $n \geq 3$. Set

$$\mu_j(t) = b_j \mu_0(t).$$

Then (28) holds at main order if we have the following identities,

$$b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} (b_i b_j)^{\frac{n-2}{2}} G(q_j, q_i) + c_2 c_1^{-1} \mu_0^{1-n} \dot{\mu}_0 = 0 \text{ for all } j = 1, \cdots, k. \tag{29}$$

Set $c_2 c_1^{-1} \mu_0^{1-n} \dot{\mu}_0 = -\frac{2}{n-2}$, we then have

$$\dot{\mu}_0(t) = -\frac{2 c_1 c_2^{-1}}{n-2} \mu_0^{n-1}(t), \tag{30}$$

with the solution $\mu_0(t) = \left(\frac{c_1^{-1} c_2}{2}\right)^{\frac{1}{n-2}} t^{-\frac{1}{n-2}}$. Furthermore, from the identities (29) and (30), the constants b_j must satisfy the following system

$$b_j^{n-3} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2}{2}-1} b_i^{\frac{n-2}{2}} G(q_j, q_i) = \frac{2}{n-2} \frac{1}{b_j} \text{ for all } j = 1, \cdots, k. \tag{31}$$

System (31) can be viewed as the Euler-Lagrange equation $\nabla_b I(b) = 0$ for the functional

$$I(b) := \frac{1}{n-2}\left[\sum_{j=1}^{k} b_j^{n-2} H(q_j, q_j) - \sum_{i\neq j} b_j^{\frac{n-2}{2}} b_i^{\frac{n-2}{2}} G(q_j, q_i) - \sum_{j=1}^{k} \ln b_j^2\right].$$

Set $\Lambda_j = b_j^{\frac{n-2}{2}}$, then we have

$$(n-2)I(b) = \tilde{I}(\Lambda) = \left[\sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i\neq j} G(q_j, q_i)\Lambda_i\Lambda_j - \sum_{j=1}^{k} \ln \Lambda_j^{\frac{4}{n-2}}\right].$$

System (31) possesses a unique solution with all its components be positive if and only if the matrix

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_2, q_1) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -G(q_k, q_1) & -G(q_k, q_2) & \cdots & H(q_k, q_k) \end{bmatrix}$$

is positive definite. For the following solvability conditions of (26),

$$\int_{\mathbb{R}^n} \mu_j^{\frac{n+2}{2}} S(\tilde{z})(y,t) Z_i(y) dy = 0, \quad i = 1, \cdots, n,$$

choose $\xi_{0j} = q_j$, then these identities can be satisfied at main order. Now we denote

$$\bar{\mu}_0 = (\mu_{01}, \cdots, \mu_{0k}) = (b_1 \mu_0, \cdots, b_k \mu_0)$$

and let Φ_j be the unique solution of (26) for $\mu = \bar{\mu}_0$. Then

$\Delta_y \Phi_j + pU(y)^{p-1} \Phi_j = -\mu_{0j} E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}]$ in $\mathbb{R}^n$, $\Phi_j(y,t) \to 0$ as $|y| \to \infty$.

From the definitions of μ_0 and b_j as above, there holds

$$\mu_{0j} E_{0j} = -\tilde{\gamma}_j \mu_0^{n-2} q_0(y),$$

where $\tilde{\gamma}_j$ is a positive constant and

$$q_0(y) := pU(y)^{p-1} c_2 + c_1 U(y)^{p-1} Z_{n+1}(y).$$

Let $p_0 = p_0(|y|)$ be the unique solution of $\Delta_y \Phi + pU(y)^{p-1}\Phi = q_0$, then $p_0(|y|) = O(|y|^{-2})$ as $|y| \to \infty$ and

$$\Phi_j(y,t) = \tilde{\gamma}_j \mu_0^{n-2} p_0(y).$$

Now we define the improved approximation as follows

$$z(x,t) = \tilde{z}(x,t) + \tilde{\Phi}(x,t)$$

with

$$\tilde{\Phi}(x,t) = \sum_{j=1}^{k} \mu_j^{-\frac{n-2}{2}} \eta_0(x-q_j)\Phi_j\left(\frac{x-\xi_j}{\mu_j}, t\right)$$

and $\eta_0(x)$ is a smooth function defined on $\mathbb{R}^n$ which equals to 0 for $x \in \mathbb{R}^n \setminus B_\epsilon(0)$ and equals to 1 for $x \in B_{\frac{\epsilon}{2}}(0)$, $\epsilon > 0$ is a small but fixed positive number satisfying
$0 < \epsilon < \frac{1}{2}\min\{\min_{i\neq l, i,l=1,\cdots,k} |q_i - q_l|, \min_{i=1,\cdots,k} dist(q_i, \partial\Omega)\}$. Here $dist(x, \partial\Omega)$ means the distance of x to the boundary $\partial\Omega$ of Ω. Finally we set

$$\mu(t) = \bar{\mu}_0 + \lambda(t) \text{ with } \lambda(t) = (\lambda_1(t), \cdots, \lambda_k(t)).$$

Step 2. The inner-outer gluing scheme

Now we use the ansatz

$$u(x,t) = \sum_{j=1}^{k} z_j(x,t) + \tilde{\phi}(x,t)$$

for $z_j(x,t) = U_{\mu_j,\xi_j}(x) - \mu_j^{\frac{n-2}{2}} H(x,q_j) + \mu_j^{-\frac{n-2}{2}} \Phi_j\left(\frac{x-\xi_j}{\mu_j}, t\right)$. Problem (21) becomes

$$-\left(\left(z+\tilde{\phi}\right)^p\right)_t + \Delta\left(z+\tilde{\phi}\right) + \left(z+\tilde{\phi}\right)^p = 0,$$

which can be linearized as

$$-pz^{p-1}\tilde{\phi}_t + \Delta\tilde{\phi} + pz^{p-1}\tilde{\phi} + S[z] + N[\tilde{\phi}] - \left(N[\tilde{\phi}]\right)_t - \left(pz^{p-1}\right)_t \tilde{\phi} = 0. \tag{32}$$

Here we denote

$$N[\tilde{\phi}] = \left(z+\tilde{\phi}\right)^p - z^p - pz^{p-1}\tilde{\phi}.$$

Using the inner outer gluing method, we write

$$\tilde{\phi}(x,t) = \psi(x,t) + \phi^{in}(x,t)$$

with

$$\phi^{in}(x,t) := \sum_{j=1}^{k} \eta_{j,R}(x,t)\tilde{\phi}_j(x,t)$$

$$\tilde{\phi}_j(x,t) = \mu_{0j}^{-\frac{n-2}{2}} \phi\left(\frac{x-\xi_j}{\mu_{0j}}, t\right)$$

and

$$\eta_{j,R} = \eta\left(\frac{x-\xi_j}{R\mu_{0j}}\right).$$

Here $\eta(s)$ is a cut-off function satisfying $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$. The positive number R is independent of t but sufficiently large, for convenience, we choose it as

$$R = t_0^{\varepsilon}, \text{ with } 0 < \varepsilon \ll 1. \tag{33}$$

Then $\tilde{\phi}$ solves equation (32) if ψ and $\tilde{\phi}^{in}$ satisfies the following system of two equations respectively

$$\begin{cases} pz^{p-1}\psi_t \\ \quad = \Delta\psi + V_{\mu,\xi}\psi + \sum_{j=1}^{k}\left[2\nabla\eta_{j,R}\nabla_x\tilde{\phi}_j + \tilde{\phi}_j\left(\Delta_x - pU_j^{p-1}\partial_t\right)\eta_{j,R}\right] \\ \qquad + S^{*,out}_{\mu,\xi} + N[\tilde{\phi}] - \left(N[\tilde{\phi}]\right)_t - \left(pz^{p-1}\right)_t\tilde{\phi} \\ \qquad - pz^{p-1}\partial_t\sum_{j=1}^{k}\eta_{j,R}\tilde{\phi}_j + \sum_{j=1}^{k}pU_j^{p-1}\partial_t\left(\eta_{j,R}\tilde{\phi}_j\right) \ \text{ in } \Omega\times[t_0,+\infty), \\ \psi = 0 \quad \text{on} \quad \partial\Omega\times[t_0,+\infty) \end{cases} \tag{34}$$

and

$$\begin{aligned} &pU_j^{p-1}\partial_t\tilde{\phi}_j \\ &\quad = \Delta\tilde{\phi}_j + pU_0^{p-1}\tilde{\phi}_j + pU_0^{p-1}\psi + S^{*,in}_{\mu,\xi,j} \quad \text{ in } B_{2R\mu_0}(\xi)\times[t_0,+\infty). \end{aligned} \tag{35}$$

Here

$$V_{\mu,\xi} = \sum_{j=1}^{k} p \left(z^{p-1} - \left(\mu_j^{-\frac{n-2}{2}} U\left(\frac{x-\xi_j}{\mu_j} \right) \right)^{p-1} \right) \eta_{j,R} + p \left(1 - \sum_{j=1}^{k} \eta_{j,R} \right) z^{p-1}, \quad U_j := \mu_j^{-\frac{n-2}{2}} U\left(\frac{x-\xi_j}{\mu_j} \right),$$

and

$$S^{*,out}_{\mu,\xi} = \left(S[z] - \sum_{j=1}^{k} S^{*,in}_{\mu,\xi,j} \right) + \sum_{j=1}^{k} (1-\eta_{j,R}) S^{*,in}_{\mu,\xi,j}.$$

$$
\begin{aligned}
&S^{*,in}_{\mu,\xi,j}(y,t)\\
&= \mu_j^{-\frac{n+2}{2}}\Big\{\mu_{0j}^{-1}\dot{\lambda}_j pU(y)^{p-1}Z_{n+1}(y) - 2\mu_{0j}^{-2}b_j\dot{\mu}_0\lambda_j pU(y)^{p-1}Z_{n+1}(y)\\
&\quad - \mu_{0j}\mu_0^{n-4}pU(y)^{p-1}\sum_{i=1}^{k}\mathcal{M}_{ij}\lambda_i + \mu_j^{-2}pU(y)^{p-1}\dot{\xi}_j\cdot\nabla U(y)\\
&\quad + \mu_j pU(y)^{p-1}\Big[-\mu_j^{n-2}\nabla H(q_j,q_j) + \sum_{i\neq j}\mu_j^{\frac{n-2}{2}}\mu_i^{\frac{n-2}{2}}\nabla G(q_j,q_i)\Big]\cdot y\Big\}\\
&\quad + \mu_j^{-\frac{n+2}{2}}\lambda_j b_j\Big[b_j^{-2}\mu_0^{-2}\dot{\mu}_0 pU(y)^{p-1}Z_{n+1}(y)\\
&\quad + pU(y)^{p-1}\mu_0^{n-3}\Big(-b_j^{n-4}H(q_j,q_j) + \sum_{i\neq j}b_j^{\frac{n-6}{2}}b_i^{\frac{n-2}{2}}G(q_j,q_i)\Big)\Big]
\end{aligned}
$$

- (34) is the so-called outer problem,
- (35) is the so-called inner problem
- Solve the outer problem as a function of λ, ξ and ϕ. Then solve the inner problem (35) based on a linear theory and suitable choice of the parameter functions λ, ξ.

Step 3. Linear theory for the outer problem

We consider the linear equation of the outer problem

$$\begin{cases} pz^{p-1}\psi_t = \Delta\psi + V_{\mu,\xi}\psi + f(x,t) \text{ in } \Omega \times [t_0, +\infty), \\ \psi(x,t) = 0 \text{ on } \partial\Omega \times [t_0, +\infty), \\ \psi(x,t_0) = h(x) \text{ on } \Omega, \end{cases} \tag{36}$$

First we consider the H^2-estimate of (36). We have

Lemma

Suppose $\|g\|_{L^2_{t_0},\nu} < +\infty$ *and* $\|h\|_{L^2(\Omega)} < +\infty$, *there exists a solution* $\psi = \psi(x,t)$ *of the following problem*

$$\begin{cases} -pz^{p-1}\psi_t + \Delta\psi + V_{\mu,\xi}\psi + z^{p-1}g = 0 \ \textit{in } \Omega \times [t_0, +\infty), \\ \psi = 0 \ \textit{on } \partial\Omega \times [t_0, +\infty), \\ \psi(\cdot, t_0) = h(x) \ \textit{on } \Omega, \end{cases} \tag{37}$$

furthermore, there exists a positive constant C *such that*

$$\|\psi\|_{H^2_{t_0},\nu} \leq C\left(\|h\|_{L^2(\Omega)} + \|g\|_{L^2_{t_0},\nu}\right) \tag{38}$$

holds for t_0 *sufficiently large and* $\nu > 0$.

Notations: For $\Lambda_\tau := \Omega \times [\tau, \tau+1]$ and $\nu > 0$, we define

$$\|\psi(\cdot,\tau)\|_{L^2} = \left(\int_\Omega |\psi(\cdot,\tau)|^2 z^{p-1} dx\right)^{\frac{1}{2}},$$

$$\|\psi\|_{L^2(\Lambda_\tau)} = \left(\int\int_{\Lambda_\tau} |\psi|^2 z^{p-1} dx dt\right)^{\frac{1}{2}},$$

$$\|\psi\|_{H^1(\Lambda_\tau)} = \|\psi\|_{L^2(\Lambda_\tau)} + \|z^{-\frac{p-1}{2}} \nabla\psi\|_{L^2(\Lambda_\tau)},$$

$$\|\psi\|_{H^2(\Lambda_\tau)} = \|\psi_t\|_{L^2(\Lambda_\tau)} + \|z^{-\frac{p-1}{2}} \Delta\psi\|_{L^2(\Lambda_\tau)} + \|\psi\|_{H^1(\Lambda_\tau)},$$

$$\|\psi\|_{L^2_{t_0},\nu} = \sup_{\tau > t_0} \mu_0^{-\nu} \|\psi\|_{L^2(\Lambda_\tau)},$$

$$\|\psi\|_{H^1_{t_0},\nu} = \sup_{\tau > t_0} \mu_0^{-\nu} \|\psi\|_{H^1(\Lambda_\tau)},$$

$$\|\psi\|_{H^2_{t_0},\nu} = \sup_{\tau > t_0} \mu_0^{-\nu} \|\psi\|_{H^2(\Lambda_\tau)}.$$

For $s > t_0$, we also define

$$\|\psi\|_{L^2_{t_0,s},\nu} = \sup_{t_0 < \tau < s} \mu_0^{-\nu} \|\psi\|_{L^2(\Lambda_\tau)},$$

$$\|\psi\|_{H^1_{t_0,s},\nu} = \sup_{t_0 < \tau < s} \mu_0^{-\nu} \|\psi\|_{H^1(\Lambda_\tau)},$$

$$\|\psi\|_{H^2_{t_0,s},\nu} = \sup_{t_0 < \tau < s} \mu_0^{-\nu} \|\psi\|_{H^2(\Lambda_\tau)},$$

In the region $\cup_{j=1}^{k} B_{2\mu_j R}(\xi_j)$, we consider the following model problem of (34),

$$\begin{cases} pz^{p-1}\psi_t = \Delta\psi + V_{\mu,\xi}\psi + f_j(x,t) \text{ in } B_{2\mu_j R}(\xi_j) \times [t_0, +\infty), \\ \psi(\cdot, t_0) = h_j(x) \text{ on } B_{2\mu_j R}(\xi_j), \end{cases} \tag{39}$$

$j = 1, \cdots, k$. For $\alpha, \beta > 0$, we assume $f_j(x,t)$ satisfies

$$|f_j(x,t)| \le M \frac{\mu_0^{-2}\mu_0^{\beta}}{1 + |y|^{2+\alpha}} \tag{40}$$

and denote by $\|f_j\|_{*,\beta,2+\alpha}$ the least M such that (40) holds. It is convenient to lift (39) onto the standard sphere $\mathbb{S}^n$.

Conformal Laplacian on $\mathbb{S}^n$. Let $\pi : \mathbb{R}^n \to \mathbb{S}^n$ be the stereographic projection given by

$$\pi(y_1, \cdots, y_n) = \left(\frac{2y}{1+|y|^2}, \frac{|y|^2-1}{|y|^2+1} \right).$$

For a function $\phi : \mathbb{R}^n \to \mathbb{R}$, we define the lifted function $\tilde{\phi}$ of ϕ on $\mathbb{S}^n$ by the relation

$$\phi(y) = \tilde{\phi}(\pi(y)) \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n. \tag{41}$$

The conformal Laplacian on $\mathbb{S}^n$ can be defined as

$$P = \Delta_{\mathbb{S}^n} - \frac{1}{4} n(n-2),$$

here $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on $\mathbb{S}^n$. Then the following well known property holds,

$$\left(\frac{2}{1+|y|^2} \right)^{\frac{n+2}{2}} P(\tilde{\phi}) \circ \pi = \Delta_{\mathbb{R}^n} \phi$$

for ϕ and $\tilde{\phi}$ satisfying the relation (41).

We have the following result.

Lemma

Suppose $\|f_j\|_{*,\beta,2+\alpha} < +\infty$ *for some* $\alpha > 0$ *and* $\beta > 0$*. Then there exists a solution* $\psi = \psi[f_j, h_j]$ *of (39) satisfies the following estimates*

$$|\psi(x,t)| \lesssim \|f_j\|_{*,\beta,2+\alpha} \sum_{j=1}^{k} \frac{\mu_0^{\beta}(t)}{1+|y_j|^{\alpha}} + \sum_{j=1}^{k} e^{-\delta(t-t_0)} \|h_j(x)\|_{L^\infty(B_{\mu_j R}(\xi_j))},$$

$$|\partial_t \psi(x,t)| \lesssim \|f_j\|_{*,\beta,2+\alpha} \sum_{j=1}^{k} \frac{\mu_0^{\beta}(t)}{1+|y_j|^{\alpha-2}}$$

here $y_j := \frac{x-\xi_j}{\mu_j}$.

Combine the above discussions, we have the following linear theory for the outer problem. Define the norm $\|\psi\|_{**,\beta,\alpha,\nu}$ of ψ as the least positive number such that

$$
\begin{aligned}
(1+|y|)^{-1}\mu_0|\nabla\psi(x,t)|\chi_{\cup_{j=1}^k B_{2R\mu_j}(\xi_j)} \\
&+ (1+|y|)^{-2}|\partial_t\psi(x,t)|\chi_{\cup_{j=1}^k B_{2R\mu_j}(\xi_j)} \\
&+ |\psi(x,t)|\chi_{\cup_{j=1}^k B_{2R\mu_j}(\xi_j)} \lesssim M\sum_{j=1}^k \frac{\mu_0^\beta(t)}{1+|y_j|^\alpha}
\end{aligned}
$$

and

$$
\|\psi\|_{H^2_{t_0},\nu} \lesssim M.
$$

Also we define $\|f\|_{*,\beta,2+\alpha,\nu} = \|f\chi_{\cup_{j=1}^k B_{2R\mu_j}(\xi_j)}\|_{*,\beta,2+\alpha} + \|z^{1-p}f\|_{L^2_{t_0},\nu}$.

We have the following result.

Lemma

There exists a bounded linear operator which maps functions $f : \Omega \times (t_0, +\infty) \to \mathbb{R}$, $h : \Omega \to \mathbb{R}$ *with* $\|f\|_{*,\beta,2+\alpha,\nu} < \infty$, $\|h\|_{L^2_{t_0},\nu} < +\infty$ *into a solution* ψ *of(36), furthermore, the following estimate holds*

$$
\begin{aligned}
&\|\psi\|_{**,\beta,\alpha,\nu} \\
&\leq C\left(\|f\|_{*,\beta,2+\alpha,\nu} + \|h\|_{L^2(\Omega)} + e^{-\delta(t-t_0)}\|h\chi_{\cup_{j=1}^{k} B_{2R\mu_j}(\xi_j)}\|_{L^\infty(\Omega)}\right)
\end{aligned}
$$

for a small constant $\delta > 0$.

Step 4. Linear theory for the inner problem

we consider the following linear equation

$$-pU^{p-1}\phi_t + \Delta\phi + pU^{p-1}\phi + U^{p-1}h = 0 \text{ on } \mathbb{R}^n, \tag{42}$$

with $h = h(y,t)$ being supported on the ball $B_{2R}(0)$ and under the orthogonality conditions

$$\int_{B_{2R}} h(y,t)Z_j(y)U^{p-1}(y)dy = 0 \text{ for } j = 0, 1, \cdots, n+1. \tag{43}$$

Equation (42) is a degenerate parabolic equation, therefore a natural way is to lift it to the standard sphere $\mathbb{S}^n$, which becomes a classical (non-degenerate) parabolic problem on $\mathbb{S}^n$. Similarly to (41), we define $\tilde{g}$ on $\mathbb{S}^n$ to be

$$h(y) = \tilde{h}(\pi(y))\left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n.$$

Then standard computation shows that (42) is equivalent to the following linear heat problem on $\mathbb{S}^n$

$$\partial_t \tilde{\phi} = \left(\Delta_{\mathbb{S}^n} + \lambda_1\right) \tilde{\phi} + \tilde{h} \quad \text{on} \quad \mathbb{S}^n. \tag{44}$$

Here $\lambda_1 = n$ is the second eigenvalue of $\Delta_{\mathbb{S}^n}$ with eigenfunctions $\tilde{Z}_j$, $j = 1, \cdots, n+1$, given by the functions

$$Z_i(y) = \tilde{Z}_i(\pi(y)) \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad y \in \mathbb{R}^n.$$

Recall that the space $L^2(\mathbb{S}^n)$ has an orthonormal basis Θ_m, $m = 0, 1, \cdots$, which are eigenfunctions of the problem

$$\Delta_{\mathbb{S}^n} \Theta_m + \lambda_m \Theta_m = 0 \quad \text{in} \quad \mathbb{S}^n \tag{45}$$

so that

$$0 = \lambda_0 < \lambda_1 = \cdots = \lambda_{n+1} = n < \lambda_{n+2} \leq \cdots .$$

One has $\Theta_0(y) = \alpha_0$ and $\Theta_j(y) = \alpha_1 y_j$, $j = 1, \cdots, n+1$, for constant numbers α_0 and α_1.

Lemma

Suppose $a \in (-n, -2)$, $\nu > 0$, $\|\tilde{h}\|_{a,\nu} < +\infty$ *and*

$$\int_{\mathbb{S}^n} h(\tilde{y}, t) Z_j(\tilde{y}) d\tilde{y} = 0 \quad \textit{for all} \quad t \in (t_0, \infty), \;\; j = 1, \cdots, n+1,$$

then there exists a function $\tilde{\phi} = \tilde{\phi}[\tilde{h}](\tilde{y}, t)$ *satisfying (44) and the estimate*

$$(\pi - |\tilde{y}|)|\nabla \tilde{\phi}(\tilde{y}, t)| + |\tilde{\phi}(\tilde{y}, t)| \lesssim t^{-\nu}(\pi - |\tilde{y}|)^{2+a}\|\tilde{h}\|_{a,\nu}.$$

Step 5. Solving the inner-outer gluing system

We solve the inner-outer gluing system based on the linear theory and the Contraction Mapping Theorem.

Thank you for your attention!