Global regularity and Asmptotic behavior for the steady Prandtl equation

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Steady Prandtl equation

The 2-D steady Prandtl equation

$$\begin{cases} u\partial_x u + v\partial_y u - \partial_y^2 u + \dfrac{dp}{dx} = 0, \quad x \geq 0,\, y \geq 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y \to +\infty} u(x,y) = U(x), \end{cases}$$

where the outer flow $\big(U(x), p(x)\big)$ satisfies

$$U(x)U'(x) + p'(x) = 0.$$

This system was used to describe the behavior of the steady Navier-Stokes equations near the boundary in the case when a flow past a plate.

Oleinik classical result

Let $\mathcal{K}$ be a set consisting of functions $u \in C_b^{2,\alpha}\big([0,+\infty)\big)$ with

$$u(0) = 0,\ u_y(0) > 0, \quad u(y) > 0 \quad \text{for } y > 0,$$
$$\lim_{y\to+\infty} u(y) = U(0) > 0, \quad u_{yy}(y) - \partial_x p(0) = O(y^2).$$

Theorem. (*Oleinik 1960*)

Let $u_0 \in \mathcal{K}$ and $p(x)$ be smooth. There exists $X > 0$ so that the Prandtl equation admits a unique solution with

(1) **Regularity**: *$u \in C_b([0,X] \times \mathbb{R}_+)$ and $u_y, u_{yy} \in C([0,X) \times \mathbb{R}_+)$.*

(2) **Non-degeneracy**: *$u(x,y) > 0$ in $[0,X) \times (0,+\infty)$ and for $\bar{x} < X$, there exist $y_0 > 0, c > 0$ so that*

$$\partial_y u(x,y) \geq c \quad \text{in } [0,\bar{x}] \times [0,y_0].$$

(3) **Global regularity**: *if $p'(x) \leq 0$, then $X = +\infty$.*

Some basic questions

- **Problem 1.** Global C^∞ regularity of the Oleinik solution when $p'(x) \le 0$;
- **Problem 2.** Asymptotic behavior of the Oleinik solution when $p'(x) \le 0$;
- **Problem 3.** Boundary layer separation and local behavior near the separation point when $p'(x) > 0$;
- **Problem 4.** Prandtl boundary layer expansion:

$$u^{NS}(x,y) = u^E(x,y) + u^P(x, y/\sqrt{\nu}) + O(\sqrt{\nu}),$$
$$v^{NS}(x,y) = v^E(x,y) + \sqrt{\nu} v^P(x, y/\sqrt{\nu}) + o(\sqrt{\nu}).$$

von Mises transformation:

$$\psi(x,y) = \int_0^y u(x,z)dz, \quad w(x,\psi) = u(x,y)^2.$$

Then the new unknown $w(x,\psi)$ satisfies

$$\partial_x w - \sqrt{w}\partial_\psi^2 w = -2\frac{dp}{dx} \quad \text{in} \quad [0,X)\times\mathbb{R}_+,$$

together with the boundary conditions

$$w(x,0) = 0, \quad \lim_{\psi\to+\infty} w(x,\psi) = U(x)^2.$$

Main difficulty: degenerate parabolic equation.

Global C^∞ regularity

Guo and Iyer(*CMP 2021*) proved the higher regularity of the solution in a local time, i.e., $0 < x < X \ll 1$.

The following is our result.

Theorem 1.(*Wang-Zhang, AIHP 2021*)

Let u be a global Oleinik solution with $u_0 \in \mathcal{K}$ and $p'(x) \leq 0$ smooth. For any positive integers m, k and any positive constants ϵ, X with $\epsilon < X$, there exists a positive constant C depending only on $\epsilon, X, u_0, p, k, m$ so that

$$|\partial_x^k \partial_y^m u(x,y)| \leq C \quad \textit{for} \quad (x,y) \in [\epsilon, X] \times [0, +\infty).$$

Global C^∞ regularity

Step 1. There exists $\delta_1 > 0$ so that

$$|\partial_x w(x,\psi)| \le C\psi \quad \text{in} \quad [\epsilon, X] \times [0, \delta_1].$$

Proof. *Consider the equation of $\partial_x w$:*

$$\partial_x\Big[\partial_x w\Big] - \sqrt{w}\partial_\psi^2\Big[\partial_x w\Big] = \frac{(\partial_x w)^2}{2w} + 2\frac{\partial_x w \frac{dp}{dx}}{2w} - 2\frac{d^2 p}{dx^2} := I.$$

Thanks to $|\partial_x w| \le C\psi^{1/2+\alpha}$ for some $\alpha \in (0, 1/2)$ and $w \sim \psi$, we have $|I| \le C\psi^{\alpha-\frac{1}{2}}$. Construct the barrier function $\varphi = A_1\psi - A_2\psi^{1/2+\alpha}$. By taking A_1, A_2 suitably, we have

$$\partial_x\Big[\partial_x w - \varphi\Big] - \sqrt{w}\partial_\psi^2\Big[\partial_x w - \varphi\Big] < 0 \quad \textit{in} \quad (0, X] \times (0, \delta_1),$$

which implies $\partial_x w - \varphi \le 0$ by using the maximum principle.

Step 2. There exists $\delta_2 > 0$ so that in $[\epsilon, X] \times (0, \delta_2]$,

$$|\partial_\psi \partial_x w(x,\psi)| \le C, \ |\partial_\psi^2 \partial_x w(x,\psi)| \le C\psi^{-1}, \ |\partial_x^2 w(x,\psi)| \le C\psi^{-\frac{1}{2}}.$$

Proof. *For any* $(x_3, \psi_3) \in [\epsilon, X] \times (0, \delta_0]$, *we denote*

$$Q = \left\{x_3 - \psi_3^{\frac{3}{2}} \le x \le x_3\right\} \times \left\{\frac{1}{2}\psi_3 \le \psi \le \frac{3}{2}\psi_3\right\}.$$

We make a coordinate transformation:

$$T: \quad Q \longrightarrow [-1,0]_{\tilde{x}} \times \left[-\frac{1}{2}, \frac{1}{2}\right]_{\tilde{\psi}} := \widetilde{Q},$$
$$(x,\psi) \mapsto (\tilde{x}, \tilde{\psi}),$$

where

$$x - x_3 = \psi_3^{\frac{3}{2}} \tilde{x}, \quad \psi - \psi_3 = \psi_3 \tilde{\psi}.$$

Global C^∞ regularity

It holds that

$$\partial_{\tilde{x}}(\psi_3^{-1}w) - \frac{\sqrt{w}}{\psi_3^{\frac{1}{2}}}\partial^2_{\tilde{\psi}}(\psi_3^{-1}w) = -2\frac{d}{d\tilde{x}}p\,\psi_3^{-1} \quad in \quad \widetilde{Q}.$$

Thanks to $w \sim \psi$ *and* $|\partial_x w| \le C\psi$, *we have*

$$0 < c \le \frac{\sqrt{w}}{\psi_3^{\frac{1}{2}}} \le C, \quad |\psi_3^{-1}w| \le C \quad in \quad \widetilde{Q},$$

and for any $\alpha \in (0,1)$,

$$\left|\frac{\sqrt{w}}{\psi_3^{\frac{1}{2}}}\right|_{C^\alpha(\widetilde{Q})} \le C, \quad \left|\frac{d}{d\tilde{x}}p\,\psi_3^{-1}\right|_{C^{0,1}([-1,0]_{\tilde{x}})} \le C.$$

Then standard interior a priori estimates yield

$$|\partial^2_{\tilde{\psi}}w\psi_3^{-1}|_{C^\alpha\left([-\frac{1}{2},0]_{\tilde{x}}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}}\right)} \le C.$$

Global C^∞ regularity

Let $f = \partial_x w \psi_3^{-1}$*, which satisfies*

$$\partial_{\tilde{x}} f - \frac{\sqrt{w}}{\psi_3^{\frac{1}{2}}} \partial_{\tilde{\psi}}^2 f - \frac{\partial_{\tilde{\psi}}^2 w}{2\sqrt{w}\psi_3^{\frac{1}{2}}} f = -2 \frac{d}{d\tilde{x}} \frac{d}{dx} p \, \psi_3^{-1}.$$

It holds that

$$|f| \le C \quad in \quad \widetilde{Q}, \quad \Big| \frac{\psi_3^{\frac{1}{2}}}{\sqrt{w}} \Big|_{C^\alpha(\widetilde{Q})} \le C,$$

$$\Big| \frac{\partial_{\tilde{\psi}}^2 w}{2\sqrt{w}\psi_3^{\frac{1}{2}}} \Big|_{C^\alpha([-\frac{1}{2},0]_{\tilde{x}} \times [-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}})} \le C, \quad \Big| \frac{d}{d\tilde{x}} \frac{d}{dx} p \, \psi_3^{-1} \Big|_{C^{0,1}([-1,0]_{\tilde{x}})} \le C.$$

Then standard interior a priori estimates yield

$$|(\partial_{\tilde{x}} f, \partial_{\tilde{\psi}} f, \partial_{\tilde{\psi}}^2 f)|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}} \times [-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})} \le C,$$

which implies our results.

Step 3. Let $k \geq 2$. There exists $\delta_3 > 0$ so that in $[\epsilon, X] \times [0, \delta_3]$,

$$|\partial_x^k w| \leq C\psi, \quad |\partial_\psi \partial_x^k w| \leq C, \quad |\partial_\psi^2 \partial_x^k w| \leq C\psi^{-1}.$$

Proof. *Assume that for* $1 \leq j \leq k - 1$,

$$|\partial_\psi \partial_x^j w| \leq C, \ |\partial_\psi^2 \partial_x^j w| \leq C\psi^{-1}, \ |\partial_x^j w| \leq C\psi,$$
$$|\partial_x^j \sqrt{w}| \leq C\psi^{\frac{1}{2}}, \ |\partial_x^k w| \leq C\psi^{-\frac{1}{2}},$$

where the case of $j = 1$ *has been proved in Step 2. Next we need to show that*

$$|\partial_\psi \partial_x^k w| \leq C, \quad |\partial_\psi^2 \partial_x^k w| \leq C\psi^{-1}, \quad |\partial_x^k w| \leq C\psi,$$
$$|\partial_x^k \sqrt{w}| \leq C\psi^{\frac{1}{2}}, \quad |\partial_x^{k+1} w| \leq C\psi^{-\frac{1}{2}}.$$

Global C^∞ regularity

Step 3.1 $|\partial_x^k w| < M_1\psi^{1-\beta}$, $|\partial_x^k \sqrt{w}| \le M_1\psi^{\frac{1}{2}-\beta}$ *for* $0 < \beta \ll 1$. *Consider the equation of*

$$g = \begin{cases} \dfrac{\partial_x^{k-1} w(x-h,\psi) - \partial_x^{k-1} w(x,\psi)}{-h}\zeta + M\psi\ln\psi & \dfrac{5\epsilon}{8} \le x \le X, \\ M\psi\ln\psi & 0 \le x < \dfrac{5\epsilon}{8}. \end{cases}$$

Step 3.2 $|\partial_x^k w| \le C\psi$, $|\partial_x^k \sqrt{w}| \le C\psi^{\frac{1}{2}}$. *Consider the equation of*

$$g = \partial_x^k w - A_1\psi + A_2\psi^{\frac{3}{2}-\beta}.$$

Step 3.3 $|\partial_\psi\partial_x^k w| \le C$, $|\partial_\psi^2\partial_x^k w| \le C\psi^{-1}$, $|\partial_x^{k+1} w| \le C\psi^{-\frac{1}{2}}$.

Use the coordinate transform in step 2 and consider the equation of $f = \partial_x^k w\psi_3^{-1}$.

Step 4. For any $m, k \geq 0$, there exists $\delta > 0$ so that

$$|\partial_\psi^m \partial_x^k w| \leq C\psi^{1-m} \quad \text{in} \quad [\epsilon, X] \times [0, \delta].$$

Proof. *Use the equation and the estimates in Step 3.*

Step 5. For any $m, k \geq 0$, there exists $Y_1 > 0$ so that

$$|\partial_x^k \partial_y^m u(x, y)| \leq C \quad \text{in} \quad [\epsilon, X] \times [0, Y_1].$$

Proof. *Use the relationship*

$$\begin{aligned} \partial_y &= \sqrt{w}\partial_\psi, \\ \partial_x &= \partial_{\tilde{x}} + \partial_x\psi(x, y)\partial_\psi, \\ |\partial_x\psi| &\leq C\psi, \end{aligned}$$

we can obtain the bounds of the solution in (x, y).

Asymptotic behavior

Consider $U(x) = 1$. The steady Prandtl equation admits a family of self-similar Blasius solutions:

$$(\overline{u}, \overline{v}) = \Big(f'(\eta), \frac{1}{\sqrt{x+x_0}} \big(\eta f'(\eta) - f(\eta) \big) \Big), \quad \eta = \frac{y}{\sqrt{x+x_0}},$$

where f satisfies

$$ff'' + f''' = 0, \quad f(0) = f'(0) = 0, \quad f'(+\infty) = 1.$$

The Blasius profile $f(\zeta)$ satisfies

$$0 \le f'(\eta) \le 1 \quad \text{and} \quad f''(\eta) \ge 0 \quad \text{for } \eta \ge 0,$$
$$1 - f'(\eta) \sim \eta^{-1} e^{-\eta^2 C_1 - C_2 \eta}, \quad f''(\eta) \sim e^{-\eta^2 C_1 - C_2 \eta}.$$

Theorem.(*Serrin, Proc Roy Soc Lond 1967*)
Let u be a global Oleinik solution with $p'(x) \le 0$. Then

$$\|u(x) - \bar{u}\|_{L^\infty_y} \longrightarrow 0 \quad \text{as } x \to +\infty.$$

Iyer(*ARMA 2020*) proved the decay rates in the case when the initial data is a small perturbation of the Blasius solution.

Theorem 2.(*Wang-Zhang, arXiv 2021*)
Let u be a global Oleinik solution. Under the additional decay assumption on the data:

$$|u_0(y) - 1| \le C_4 e^{-C_5 y^2} \tag{1}$$

for some positive constants C_4, C_5, then it holds that

$$|u(x,y) - \bar{u}(x,y)| \le \frac{C}{\sqrt{x+1}} \ln(x+e) e^{-c\frac{y^2}{x+1}}.$$

Theorem 3.(*Wang-Zhang, arXiv 2021*)
Let u be a global Oleinik solution. Under the assumption (1) *and*

$$-C_6 e^{-C_7 y^2} \leq \partial_y^2 u_0(y) \leq 0$$

for some positive constants C_6, C_7, then it holds that

$$
\begin{aligned}
& -\frac{C}{x+1} e^{-c\frac{y^2}{x+1}} \leq \partial_y^2 u(x,y) \leq 0, \\
& \left|\partial_y (u(x,y) - \bar{u}(x,y))\right| \leq \frac{C}{(x+1)^{\frac{3}{4}}} \ln(x+e) e^{-c\frac{y^2}{x+1}}, \\
& |\partial_x u(x,y)| \leq \frac{C}{x+1} e^{-c\frac{y^2}{x+1}}, \\
& |\partial_{xy}^2 u(x,y)| \leq \frac{C}{(x+1)^{\frac{3}{4}}} e^{-c\frac{y^2}{x+1}}.
\end{aligned}
$$

Comparison lemma.

There exist positive constants $c < 1$ and $C > 1$ depending on w_0 such that

$$c\bar{w} \le w \le C\bar{w} \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+.$$

Proof. For any positive constant b, we have

$$\begin{aligned}&\partial_x(w - b\bar{w}) - \sqrt{w}\partial_\psi^2(w - b\bar{w})\\ &= (\sqrt{b}\sqrt{w} - b\sqrt{\bar{w}})\partial_\psi^2\bar{w} + (b - \sqrt{b})\sqrt{w}\partial_\psi^2\bar{w}.\end{aligned}$$

Thanks to $\partial_\psi^2\bar{w} < 0$ for $\psi > 0$, at a maximum point, we have

$$\partial_x(w - b\bar{w}) - \sqrt{w}\partial_\psi^2(w - b\bar{w}) < 0,$$

which is a contradiction.

Notations.

- Let

$$y(\psi; u) = \int_0^{\psi} \frac{1}{\sqrt{w}(x,\psi')} d\psi'.$$

Then we have

$$cy(\psi; \bar{u}) \leq y(\psi; u) \leq Cy(\psi; \bar{u}).$$

- Let

$$h = \frac{\psi}{\sqrt{x+1}}, \quad \zeta = \frac{y(\psi; \bar{u})}{\sqrt{x+1}}.$$

Then we have

$$h = f(\zeta), \quad h \sim \zeta \quad \text{for } h \gg 1.$$

Asymptotic behavior

We denote

$$\phi(x,\psi) = w(x,\psi) - \bar{w}(x,\psi).$$

It holds that

$$L\phi + A\phi = 0,$$

where

$$L\phi = \partial_x \phi - \sqrt{w}\partial_\psi^2 \phi,$$

$$A(x,\psi) = -\frac{\partial_\psi^2 \bar{w}}{\sqrt{\bar{w}} + \sqrt{w}} = -\frac{\partial_x \bar{w}}{\sqrt{\bar{w}}(\sqrt{\bar{w}} + \sqrt{w})}.$$

Lemma. *It holds that for any* $(x, \psi) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$|A(x,\psi)| \le \frac{C}{x+1},$$

and for any $k_0 \in (0, +\infty)$*, there exists a positive constant* λ_{k_0} *such that*

$$A(x,\psi) > \frac{\lambda_{k_0}}{x+1} \quad \text{for} \quad \zeta = \frac{y(\psi;\bar{u})}{\sqrt{x+1}} \le k_0.$$

In particular, A *is positive.*

Proof. *Use the comparison lemma and the explicit information of the Blasius solution.*

Lemma 1. *There exists a large positive constant C and a small positive constant ε such that*

$$|\phi(x,\psi)| \leq Ce^{-\varepsilon\frac{\psi^2}{x+1}} \quad \textit{in} \quad \mathbb{R}_+ \times \mathbb{R}_+.$$

Proof. *At a negative minimum point, we have*

$$(L + A)\Big(Ce^{-\varepsilon\frac{\psi^2}{x+1}} \pm \phi\Big) > 0,$$

which is a contradiction.

Asymptotic behavior: Proof of Theorem 2

Lemma 2. *There exists a positive constant C and a small positive constant λ such that*

$$|\phi(x,\psi)| < C(x+1)^{-\lambda} \quad \textit{in} \quad \mathbb{R}_+ \times \mathbb{R}_+.$$

Proof. *We introduce a barrier function*

$$g(x,\psi) = C(x+1)^{-\lambda} \begin{cases} h^{\frac{1}{2}} M^{\frac{1}{2}}, & h \le \frac{1}{M}, \\ 1, & \frac{1}{M} \le h \le h_0, \\ \frac{1}{h^{2+2\lambda}} h_0^{2+2\lambda}, & h \ge h_0. \end{cases}$$

At a negative minimum point, we have

$$(L+A)(g \pm \phi) > 0. \quad \textit{Contradiction!}$$

Asymptotic behavior: Proof of Theorem 2

Key lemma. *For any fixed $\alpha \in (0,1)$, there exist positive constants C_B, B and N large and a small positive constant $\lambda > 0$ such that*

$$|\phi(x,\psi)| \le g(x,\psi)e^{-B(x+1)^{-\frac{\lambda}{2}}} \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+,$$

where

$$g(x,\psi) = C_B \begin{cases} N^{1-\alpha}(x+1)^{-\frac{1}{2}-\frac{1-\alpha}{2}}\psi^{1-\alpha}, & h < \frac{1}{N}, \\ \frac{1}{b_0}\partial_\psi \bar{w}, & h \ge \frac{1}{N}, \end{cases}$$

with $b_0 = 2f''(\zeta_0)$ and $f(\zeta_0) = \frac{1}{N}$.

Proof. *Using Lemma 1 and 2, at a negative minimum point, we have*

$$(L+A)(ge^{-B(x+1)^{-\frac{\lambda}{2}}} \pm \phi) > 0. \quad \textit{Contradiction!}$$

Asymptotic behavior: Proof of Theorem 2

Completion of the proof.

By the key lemma, we have

$$|u(x,y) - \bar{u}(x,\bar{y})| \leq \frac{C}{\sqrt{x+1}} e^{-c\frac{y^2}{x+1}} \quad in \quad \mathbb{R}_+ \times \mathbb{R}_+.$$

By the comparison lemma, we have

$$|\bar{y} - y| \leq C + C\ln(x+1) + \frac{Cy}{\sqrt{x+1}}.$$

Thus, we obtain

$$\begin{aligned} |u(x,y) - \bar{u}(x,y)| &\leq |u(x,y) - \bar{u}(x,\bar{y})| + |\bar{u}(x,\bar{y}) - \bar{u}(x,y)| \\ &\leq \frac{C}{\sqrt{x+1}} e^{-c\frac{y^2}{x+1}} + |\partial_{\bar{y}} \bar{u}(x,\hat{y})||\bar{y} - y| \\ &\leq \frac{C}{\sqrt{x+1}} \ln(x+e) e^{-c\frac{y^2}{x+1}}. \end{aligned}$$

Asymptotic behavior: Proof of Theorem 3

Step 1. Concavity of u, i.e., $\partial_y^2 u \le 0$ if $\partial_y^2 u_0 \le 0$.

Step 2. Refined decay estimate of ϕ w.r.t. ψ:

there exist positive constants C, M and a small positive constant α such that

$$|\phi(x,\psi)| \le Cg \quad in \quad \mathbb{R}_+ \times \mathbb{R}_+,$$

where

$$g = C(x+1)^{-\alpha} \begin{cases} \frac{1}{b_1}\bar{w}, & h < \frac{1}{M}, \\ 1, & h \ge \frac{1}{M}, \end{cases}$$

with $b_1 = f'^2(\zeta_0)$ and $\zeta_0 = f^{-1}(\frac{1}{M})$.

Asymptotic behavior: Proof of Theorem 3

Step 3. There exist positive constants C_K, K and ϵ such that

$$|\partial_x \phi(x,\psi)| \leq C_K g \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+,$$

where $g = (-\partial_x \bar{w}) e^{-K(x+1)^{-\epsilon}}$.

Step 4. By Step 3 and the key lemma, we have

$$|\partial_x \phi(x,\psi)| \leq \frac{C}{x+1} e^{-c\frac{\psi^2}{x+1}}, \quad |\partial_\psi^2 \phi(x,\psi)| \leq \frac{C}{x+1} e^{-c\frac{\psi^2}{x+1}}.$$

Step 5. Decay estimates of higher order derivatives:

The idea: construction of barrier functions.

Boundary layer separation

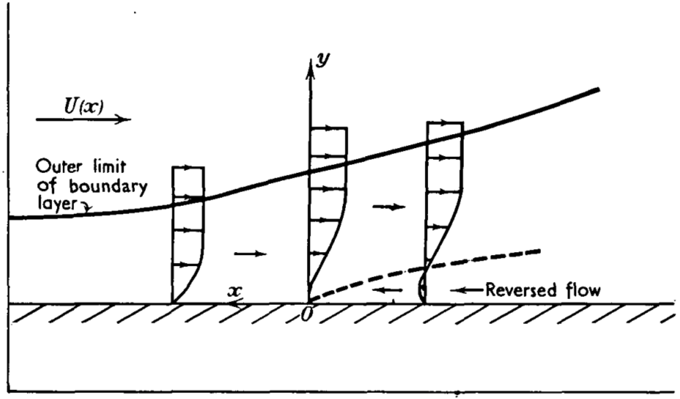


FIG. 1. Velocity profiles in the boundary layer near separation

Boundary layer separation

Theorem.(*Cafferalli and E, unpublished paper*)
If the initial data $u_0(y)$ and $p(x)$ satisfy

$$u_0(y)^2 - \frac{3}{2}u_0'(y)\int_0^y u_0(z)dz \geq 0 \quad y \geq 0, \quad p'(x) \geq c,$$

then there exists an $X^* > 0$ so that the solution can not be extended to $x > X^*$. Moreover, the sequence of u_λ defined by

$$u_\lambda(x,y) = \lambda^{-\frac{1}{2}} u(X^* - \lambda x, \lambda^{\frac{1}{4}} y)$$

is compact in $C^0(\mathbb{R}^+ \times \mathbb{R}^+)$.

Remark. *This implies in some sense that*

$$u(x,y) \sim (X^* - x)^{\frac{1}{2}} U\left(\frac{y}{(X^* - x)^{\frac{1}{4}}}\right), \quad x < X^*.$$

Boundary layer separation

Main assumptions:

- Unfavorable pressure: $p'(x) = 1$;
- The initial data $u_0(y) \in \mathcal{K}$ and for y small

$$u_0(y) = a_1 y + \frac{1}{2} y^2 + a_4 y^4 + a_7 y^7 + \cdots,$$

  where $a_1 > 0$ and $a_4 < 0$.

Theorem.(*Dalibard-Masmoudi, Publ Math IHES 2019*)
There exists $X^* > 0$ so that

$$\partial_y u(x,0) \sim \sqrt{X^* - x} \quad as \quad x \to X^*.$$

Boundary layer separation

Theorem 4.(*Shen-Wang-Zhang, Adv Math 2021*)

Let $p'(x) = 1$ *and* $U(x) = \sqrt{2(x_0 - x)}$ *for some* $x_0 > 0$. *Fix any* $\mu \in (0,1)$. *Let* u *be Oleinik solution with* $u_0 \in \mathcal{K}$ *satisfying*

$$\|\partial_y u_0\|_{L^\infty(0,+\infty)} \leq \frac{1}{2}\epsilon_0 x_0^{\frac{1}{4}}, \tag{2}$$

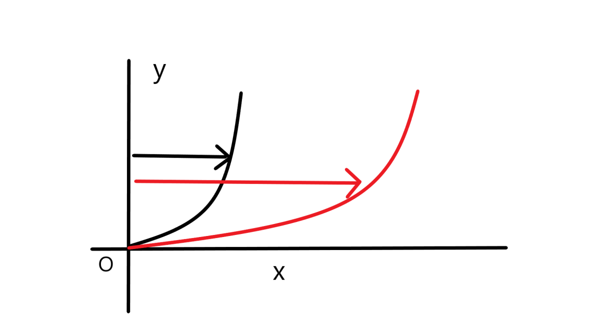
where ϵ_0 *is a positive constant depending only on* μ. *Then there exists a separation point* $x_s = X^*$ *with* $X^* < \frac{\mu}{2}x_0$.

Remark 1. *Similar result holds for* $p'(x) > 0$.

Remark 2. *The separation point has the same as the singular point.*

Remark 3.

The condition (2) can be satisfied under suitable adverse pressure gradient. Indeed, if $\|\partial_y u_0\|_{L^\infty(0,+\infty)} \le C$*, then we can take* x_0 *large so that (2) holds. On the other hand, given* $x_0 > 0$*, one can find* $u_0 \in \mathcal{K}$ *with small slope so that (2) is satisfied.*



Boundary layer separation

Goldstein's conjecture:

The separation rate should be between 1/4 and 1.

Theorem 5.(*Shen-Wang-Zhang, Adv Math 2021*)

Let u be an Oleinik solution with $X^* < x_0$, where X^* is the maximal existence time of the solution. Then there exists a positive constant C such that

$$\partial_y u(x,0) \leq C\left(X^* - x\right)^{\frac{1}{4}} \quad \text{for} \quad x \in \left(x_{near}, X^*\right)$$

for some x_{near} close enough to X^*.

Open problem 5(*Oleinik and Samokin book*):

It would be interesting to study the local structure of the solution of the Prandtl system in the vicinity of the separation point.

Theorem 6.(*Shen-Wang-Zhang, Adv Math 2021*)

Let u be Oleinik's solution with $X^* < x_0$. If u satisfies $\partial_y^2 u \le C$ in $[0, X^*) \times \mathbf{R}_+$, then for any $\bar{x} < X^* < x_0$, there exist a point $(\tilde{x}, \psi_{\tilde{x}}) \in [\bar{x}, X^*) \times \left[0, (X^* - \tilde{x})^{\frac{3}{4}}\right)$ and c, C independent of the choice of $\bar{x}$ so that

$$C(X^* - \tilde{x})^{\frac{1}{4}} \ge \partial_\psi w(\tilde{x}, \psi_{\tilde{x}}) \ge c(X^* - \tilde{x})^{\frac{1}{4}}.$$

Remark.

1. This result together with Dalibard and Masmoudi's result shows that the solution has a different separation rate when the point approaches the separation point along a different curve.

2. If u_0 *satisfies* $\partial_y^2 u_0 \le \frac{dp}{dx}(0)$ *and* $\frac{d^2p}{dx^2} \ge 0$, $X^* \le x_0$, *then it holds that*

$$\partial_y^2 u \le C \quad in \quad [0, X^*) \times \mathbf{R}_+.$$

3. **Open question:**

Is it possible to construct the solution with the separation rate such as $\frac{1}{4}, \frac{1}{2}, 1$*?*

Proof of boundary layer separation

Step 1. If $\partial_y^2 u_0 \geq -C$ and $X^* < x_0$, then

$$\partial_y^2 u(x,y) \geq -C \quad \text{in} \quad \left[0, X^*\right) \times [0, +\infty).$$

Step 2. If $X^* < x_0$, then

$$\partial_y u(x,0) \to 0 \quad \text{as} \quad x \to X^*.$$

This means that the singular point is the same as the separation point.

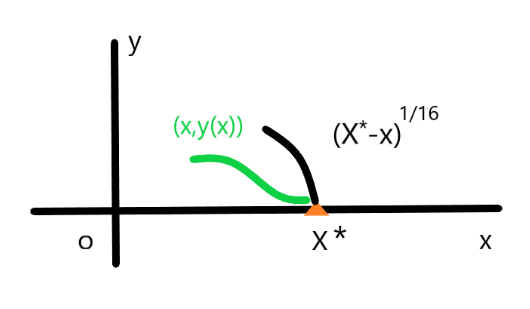
Step 3. $X^* < x_0$.

Proof of the separation rate

The idea is that we first find a special curve $(x, y(x))$ so that

$$\partial_y u(x, y(x)) \leq C\left(X^* - x\right)^{\frac{1}{4}},$$

then using one-side estimate, we can deduce the separation rate of $\partial_y u(x, 0)$:



Stability of the boundary layer solution

1. Guo and Iyer(*arXiv 2018*) prove the local stability of the Blasius solution.
2. Iyer and Masmoudi(*arXiv 2020*) prove the global stability of the Blasius solution.
3. Gao and Zhang(*arXiv 2021*) prove the local stability of the Oleinik's solution.
4. **Open question**:

   Global stability of the Oleinik solution with $p'(x) = 0$?

Thanks for your attention!