Dynamics of concentrated vorticities in 2d and 3d Euler flows

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The Euler equation for an incompressible inviscid fluid in $\mathbb{R}^2$ in Vorticity-stream formulation

Find (ω, ψ) that solves the **vorticity-stream system**

$$\begin{cases} \omega_t + \nabla^{\perp}\psi \cdot \nabla\omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-1}\omega & \text{in } \mathbb{R}^2 \times (0, T) \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2 \end{cases} \qquad \text{(V)}$$

where $(a, b)^{\perp} = (b, -a)$,

$$(-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \, f(y)\, dy.$$

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- The **velocity field** (Biot-Savart law).

$$\mathbf{u}(x, t) = \nabla^\perp \psi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y-x)^\perp}{|x-y|^2} \omega(y, t)\, dy$$

- The **vorticity**, $\omega = \nabla \times \mathbf{u} = \partial_x u_2 - \partial_y u_1$.

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• **Example:** Liouville equation: $-\Delta\psi = e^{\psi} = \omega = \frac{8}{(1+|x|^2)^2}$. The Kaufmann-Scully vortex. Mass $\int_{\mathbb{R}^2} \omega = 8\pi$

We want to describe the evolution of solutions to (V) with vorticities $\omega(x,t)$ **concentrated** around a finite number of points.

$$\omega(x,t) \approx \sum_{j=1}^{k} 8\pi\kappa_j \delta(x-\xi_j(t)), \quad \Psi(x,t) \approx \sum_{j=1}^{k} 4\kappa_j \log \frac{1}{|x-\xi_j(t)|}$$

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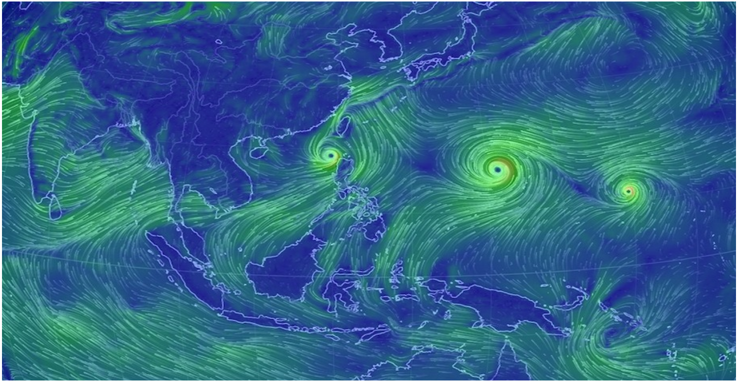

Analysis of solutions with highly concentrated vorticities:
A mathematical subject with a long history: it traces back to Helmholtz (1858), Kirchhoff (1876), Routh (1881), Lagally (1921) C.C. Lin (1941).

Formal N-vortex singular solutions (ω^s, Ψ^s) of (V):

$$\omega^s(x,t) = \sum_{j=1}^{N} 8\pi\kappa_j\, \delta(x - \xi_j(t)),$$

where $\delta(x)$ is the Dirac mass at 0, $\kappa_j \in \mathbb{R}$, $\xi_j : [0, T] \to \Omega$.
Since $\Psi^s = (-\Delta)^{-1}\omega^s$, we must have

$$\Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j \Gamma(x - \xi_j(t)), \quad \Gamma(x) = 4\log\frac{1}{|x|}.$$

Formally we compute

$$
\begin{aligned}
&\omega_t^s + \nabla^\perp \Psi^s \cdot \nabla \omega^s \\
&= -\sum_{j=1}^{N} 8\pi\kappa_j \nabla\delta(x-\xi_j)\cdot\dot{\xi}_j + \sum_{i,j=1}^{N} 8\pi\kappa_i\kappa_j \nabla^\perp\Gamma(x-\xi_i)\cdot\nabla\delta(x-\xi_j) \\
&= 8\pi\sum_{j=1}^{N}[-\kappa_j\dot{\xi}_j + \nabla_x^\perp(8\pi\sum_{i\neq j}\kappa_i\kappa_j\Gamma(x-\xi_i))]\cdot\nabla\delta(x-\xi_j).
\end{aligned}
$$

We use $\Gamma(x)$, $\delta(x)$ are "radial": $\nabla^\perp\Gamma(x-\xi_j)\cdot\nabla\delta(x-\xi_j) = 0$. Thus $(\omega^s.\Psi^s)$ is a "solution" of Problem (V) if and only if $(\xi_1,\ldots,\xi_N)$ solves the planar N-body problem

$$
\boxed{\dot{\xi}_j(t) = \sum_{i\neq j} 4\kappa_i \frac{(\xi_i(t)-\xi_j(t))^\perp}{|\xi_i(t)-\xi_j(t)|^2}, \quad j = 1,\ldots,N.} \qquad (K)
$$

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A Natural question: Are there true solutions (smooth) of (V) with vorticities highly concentrated around a finite set of points which evolve by a dynamics approximated by (K)?

We consider $\omega_{0\varepsilon}$ and $\Psi_{0\varepsilon}$ explicit ε-regularizations of the singular solution

$$\omega^s(x,t) = \sum_{j=1}^{N} 8\pi\kappa_j \delta(x - \xi_j(t)), \quad \Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j \log \frac{8}{|x - \xi_j(t)|^4},$$

using Kaufmann-Scully vortices.

$$\omega_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} W_0\left(\frac{x - \xi_j(t)}{\varepsilon}\right), \quad W_0(y) = \frac{8}{(1+|y|^2)^2}$$

$$\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \kappa_j \log \frac{8}{(|x - \xi_j(t)|^2 + \varepsilon^2)^2}$$

We have $(-\Delta)^{-1}\omega_{0\varepsilon} = \Psi_{0\varepsilon}$ and $\int_{\mathbb{R}^2} U_0 = 8\pi$. We get

$$\omega_{0\varepsilon} \rightharpoonup \sum_{j=1}^{N} \kappa_j \delta(x - \xi_i(t)), \quad \frac{1}{|\log \varepsilon|} |\nabla^{\perp} \Psi_{0\varepsilon}|^2 \rightharpoonup \sum_{j=1}^{N} 32\pi \kappa_j^2 \delta(x - \xi_j(t)).$$

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We prove that for a given colisionless solution of the system

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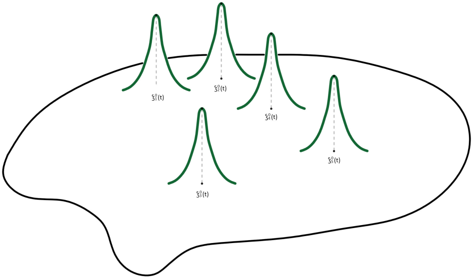

Theorem (Dávila, del Pino, Musso, Wei, ARMA 2020)

Let $\xi(t)$ *be a colisionless solution of* (K) *in* $[0, T]$. *There exists a solution* $(\omega_\varepsilon, \Psi_\varepsilon)$ *of Problem* (V) *of the form*

$$\omega_\varepsilon(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right) + \phi(x,t)$$

$$\Psi_\varepsilon(x,t) = \sum_{j=1}^{k} \kappa_j \log \frac{1}{(\varepsilon^2 + |x-\xi_j|^2)^2} + \psi(x,t)$$

where for some $0 < \sigma < 1$ *and all* $(x,t) \in \mathbb{R}^2 \times (0,T)$ *we have*

$$|\phi(x,t)| \ \le \ \varepsilon^{\sigma} \sum_{j=1}^{k} \frac{1}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right),$$

$$|\psi(x,t)| + \varepsilon |D_x \psi(x,t)| \ \le \ \varepsilon^2.$$

In particular:

$$\omega_\varepsilon \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta(x-\xi_i) \quad \frac{1}{|\log \varepsilon|} |\nabla \Psi_\varepsilon|^2 \rightharpoonup \sum_{j=1}^{k} \kappa_j^2 \delta(x-\xi_j).$$

A prior result along these lines, Marchioro and Pulvirenti (1993).

Ingredients in the construction:

• Improvement of the approximation in powers of ε using elliptic and transport equations.

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Ingredients in the construction:

• Improvement of the approximation in powers of ε using elliptic and transport equations.

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• A priori estimates to solve by a continuation (degree) argument.

Improving the approximation. Let $\Gamma_0(y) := \log \frac{8}{(1+|y|^2)^2}$,

$$\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \kappa_j \Gamma_0\left(\frac{x-\xi_j(t)}{\varepsilon}\right) - \frac{\kappa_j}{8\pi}\log 8\varepsilon^2$$

We want to solve the equation $E(\omega,\Psi)=0$, where

$$E(\omega,\Psi) := \omega_t + \nabla_x^{\perp}\Psi\cdot\nabla_x\omega, \quad -\Delta_x\Psi = \omega.$$

Near $\xi_j(t)$ write $y = \frac{x-\xi_j(t)}{\varepsilon}$. We look for a solution of the form

$$\Psi = \Psi_{0\varepsilon}(x,t) + \kappa_j\psi(y,t), \quad \omega = \frac{\kappa_j}{\varepsilon^2}U_0(y) + \frac{\kappa_j}{\varepsilon^2}\phi(y,t).$$

In terms of the y-variable we get the expression

$$\boxed{\begin{aligned} \varepsilon^4 E(\omega, \Psi) =& \varepsilon^2 \phi_t \;+\; (-\varepsilon\dot{\xi} + \nabla_y^\perp \Psi_{0\varepsilon} + \kappa_j \nabla_y^\perp \psi) \cdot \nabla_y (U_0 + \phi), \\ -\Delta_y \psi =& \phi \end{aligned}}$$

We have

$$\Psi_{0\varepsilon}(x,t) = \kappa_j \Gamma_0(y) + \varphi(x) + O(\varepsilon^2) + constant, \quad y = \frac{x - \xi_j}{\varepsilon},$$

$$\varphi(x) = \sum_{i \neq j} \kappa_i \Gamma(x - \xi_i).$$

By assumption $\dot{\xi}_j = \nabla_x^\perp \varphi(\xi_j)$, hence we get

$$-\varepsilon\dot{\xi}_j + \nabla_y^\perp \Psi_{0\varepsilon}(\xi_j + \varepsilon y) = \kappa_j \nabla^\perp (\Gamma_0 + \mathcal{R})$$

with $\mathcal{R} = O(\varepsilon^2 |y|^2)$.

$$
\begin{aligned}
\varepsilon^4 E(\omega, \Psi) =& \varepsilon^2 \phi_t \ + \ \kappa_j \nabla_y^{\perp}(\Gamma_0(y) + \mathcal{R} + \psi) \cdot \nabla_y (U_0 + \phi), , \\
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Let $f(u) = e^u$. Since $U_0 = f(\Gamma_0)$ we find

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The 0-error term:

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\varepsilon^4 E(\omega_{0\varepsilon}, \Psi_{0\varepsilon}) = \nabla^{\perp} \mathcal{R} \cdot \nabla U_0 = O(\varepsilon^2 |y|^{-4}).
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The 0-error term:

$$\varepsilon^4 E(\omega_{0\varepsilon},\Psi_{0\varepsilon}) = \nabla^{\perp}\mathcal{R}\cdot\nabla U_0 = O(\varepsilon^2|y|^{-4}).$$

We obtain a reduction in the error by solving the elliptic equation

$$-\nabla_y^{\perp}\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + \nabla^{\perp}\mathcal{R}\cdot\nabla U_0 = 0$$

After sufficiently improving the approximation we solve the problem by a continuation (degree) argument, which near each ξ_j roughly reads as

$$\boxed{\begin{aligned} \varepsilon^2\phi_t - \nabla^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + Q(\phi) + E(y,t) = 0 \\ -\Delta\psi = \phi \quad \text{ in } \mathbb{R}^2 \times [0,T] \end{aligned}}$$

with $E = O(\varepsilon^5\rho^{-3})$, $Q(\phi) = \nabla^{\perp}\psi\nabla\phi$, quadratic term.

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A basic ingredient: A priori estimates for the linear part of the equation under initial datum zero and

$\int_{B_{\delta/\varepsilon}} y\,\phi(y,t)\,dy \,=\, 0, \quad \int_{\mathbb{R}^2}\phi(y,t)\,dy \,=\, 0$

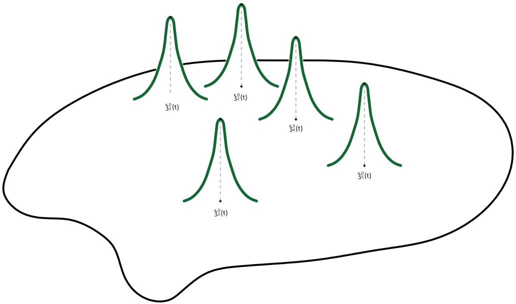
$$\|\phi(\cdot,t)U_0^{-\frac12}\|_{L^2}^2 \,\lesssim\, \varepsilon^{-2}|\log\varepsilon| \sup_{t\in[0,T]} \|EU_0^{-\frac12}\|_{L^2}^2$$

This allows a fixed point scheme to work when $E = O(\varepsilon^5\rho^{-3})$.

The **generalized surface quasigeostrophic equation (SQG)**
Let $\frac{1}{2} < s < 1$.

$$\begin{cases} \omega_t + \nabla^{\perp}\psi \cdot \nabla\omega = 0 & \text{in } \mathbb{R}^2 \times (0,T) \\ \psi = (-\Delta)^{-s}\omega & \text{in } \mathbb{R}^2 \times (0,T), \end{cases}$$

$$(-\Delta)^{-s} f(y) = c_s \int_{\mathbb{R}^2} \frac{1}{|x-y|^{2-2s}} f(y)\, dy$$



$$\dot{\xi}_j(t) = \sum_{i \neq j} \kappa_i d_s \frac{(\xi_i(t) - \xi_j(t))^\perp}{|\xi_i(t) - \xi_j(t)|^{4-2s}}, \quad j = 1, \dots, N. \qquad (K_s)$$

Theorem (M. D., Antonio Fernandez)

Let $\frac{9}{10} \leq s < 1$. For a colisionless solution $\xi(t)$ of the N-body problem (K_s) there exists a solution of (SQG) such that

$$\omega(x,t) \approx \sum_{j=1}^{N} k_j \frac{1}{\varepsilon^2} U_0\left(\frac{x - \xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{c_s}{(1 + |x|^2)^{1+s}}$$

The proof is substantially harder.

Special case: A travelling wave solution for the case of a travelling vortex pair: Ao, Dávila, del Pino, Musso, Wei (TAMS, 2021).

Partial results

-G. Cavarallo, R. Garra and C. Marchioro (2013).
-C. Geldhauser and M. Romito (2020).
-M. Rosenzweig, (2020).

Similar scheme as in Euler, but more difficult

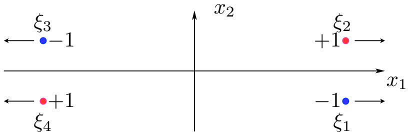
• Unlike Euler $2d$, unknown if global smooth solutions exist. Lack of regularity of the transport term is a central issue.

• Needs high accuracy of approximation (arbitrarily large order), if s is close from above to a limiting value, $s = 0.887$)

How about $t \to +\infty$?

A 4-vortex situation: (M.D., J. Davila, M. Musso, S. Parmeshwar)

There exists a solution $w(x,t)$ in Euler $2d$ which approximately looks like 2 vortex pairs travelling with constant speeds in opposite directions.



$$w(x,t) \approx \varepsilon^{-2} \sum_{j=1}^{4} U\left(\frac{x-\xi_i}{\varepsilon}\right)(-1)^i + o(1)$$

$o(1) \to 0$ as $t \to +\infty$

Nearly singular solutions for Euler in $\mathbb{R}^3$?

Open question: Solutions with concentrated vorticities near curves (filaments): *the Vortex filament conjecture* (Helmholtz, Da Rios, Levi-Civita 1858-1906-1931).

We consider the Euler equation in $\mathbb{R}^3$ in stream-vorticity formulation

$$\begin{cases} \omega_t + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \quad , \\ \mathbf{u} = \nabla \times \psi, \quad \psi(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^2} \times \omega(y,t) dy. \end{cases} \tag{V}$$

($\omega = \nabla \times \mathbf{u}$ in $\mathbb{R}^3$). We want to find solutions with vorticity concentrated on a time evolving curve (filament) $\Gamma(t)$ parametrized by arclength as $\gamma(s,t)$ in $\mathbb{R}^3$.

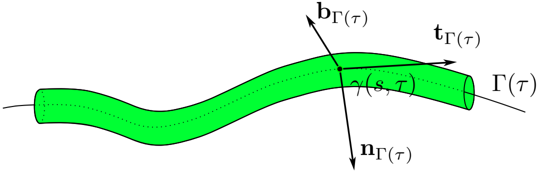
Vortex filament dynamics. (After Helmholtz and Kelvin) is a solution $\omega_\varepsilon(x,t)$ of (V) concentrated in a tube radius ε so that

$$\omega_\varepsilon(\cdot,t) \approx c\delta_{\Gamma(t)}\mathbf{t}_{\Gamma(t)} \quad \text{as } \varepsilon \to 0,$$

$\mathbf{t}_{\Gamma(t)}$ tangent vector field, $\delta_{\Gamma(t)}$ the uniform curve Dirac measure.
1904, Da Rios formal law: Letting $\tau = t|\log\varepsilon|$, $\gamma(s,\tau)$ parametrization by arclength of $\Gamma(\tau)$, κ curvature, then

$$\boxed{\gamma_\tau = \frac{c}{4\pi}(\gamma_s \times \gamma_{ss}) = \frac{c}{4\pi}\kappa\mathbf{b}_{\Gamma(\tau)}},$$

$\mathbf{b}_{\Gamma(\tau)}$ binormal vector. This is the *binormal flow of curves*.



The *vortex filament conjecture*:

Let $\Gamma(\tau)$ *be a solution curve of the binormal flow defined in* $[0, T]$ *for some* $c > 0$, $T > 0$. *For each* $\varepsilon > 0$ *there exists a smooth solution* $\omega_\varepsilon(x, t)$ *to (V) satisfying in the distributional sense,*

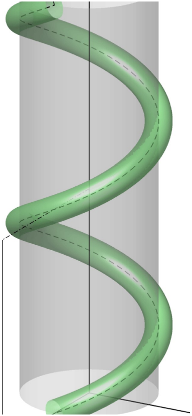
$$\omega_\varepsilon(\cdot, \frac{\tau}{|\log \varepsilon|}) \rightharpoonup c\delta_{\Gamma(\tau)}\mathbf{t}_{\Gamma(\tau)} \quad \textit{as } \varepsilon \to 0, \quad \textit{for all} \quad 0 \le \tau \le T.$$

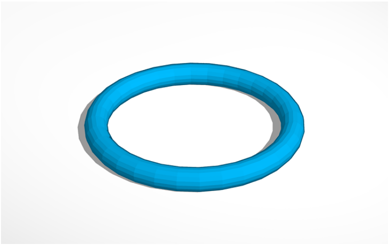
Natural: To look for a solution of the form

$$\omega_\varepsilon(x, \tau) = \frac{1}{\varepsilon^2} U_0\left(\frac{z}{\varepsilon}\right)\mathbf{t}_{\Gamma(\tau)} + o(1), \quad x = \gamma(\tau, s) + z_1\mathbf{b}_{\Gamma(\tau)} + z_2\mathbf{n}_{\Gamma(\tau)},$$

This statement is only known for special curves associated to travelling wave solutions: the thin vortex ring first found by Fraenkel, and recently a helicoidal filament.

Examples: a helix whose horizontal section rotates at a constant angular speed or a vertically translating circle are solutions of the bi-normal flow of curves.





Solutions $\vec{\omega}(x, y, z, t)$ of $3d$-Euler with Helicoidal symmetry can be obtained from a scalar function $w(x + iy, t)$ in the form

$$\vec{\omega}(x, y, z, t) = w(e^{-iz}(x + iy), t/|\log \varepsilon|) \begin{bmatrix} i(x + iy) \\ b \end{bmatrix}$$

where $w(x, \tau)$ solves

$$\begin{cases} |\log \varepsilon| w_\tau + \nabla^\perp \psi \cdot \nabla w = 0 \\ \qquad -\nabla \cdot (K \nabla \psi) = w \end{cases}$$

$$K(x, y) = \frac{1}{\kappa^2 + x^2 + y^2} \begin{pmatrix} \kappa^2 + y^2 & -xy \\ -xy & \kappa^2 + x^2 \end{pmatrix}$$

Rotating helicoidal solutions:

$$w(x + iy, \tau) = w(e^{i\alpha\tau}(x + iy)), \quad \psi((x + iy), \tau) = \psi(e^{i\alpha\tau}(x + iy)).$$

The problem reduces to the elliptic equation

$$\boxed{-\nabla \cdot (K \nabla \psi) = f(\psi - \frac{\alpha}{2} |\log \varepsilon| (x^2 + y^2)) = w \quad \text{in } \mathbb{R}^2}$$

Special case $f(u) = \varepsilon^2 e^u$. we prove:

Theorem (Dávila, del Pino, Musso, Wei, CVPDE, to appear)

There exists a solution ψ_ε *to the equation*

$$-\nabla \cdot (K \nabla \psi) = \varepsilon^2 e^{\psi + \lambda(x^2+y^2)} \quad \text{in } \mathbb{R}^2$$

such that $\varepsilon^2 e^{\psi - \frac{\alpha}{2}|\log \varepsilon|(x^2+y^2)} \rightharpoonup 8\pi \delta_{(x_0,0)}$, $x_0 > 0$, *for a suitable choice of* α.

α is precisely the number that makes the "rotating helix"

$$\gamma(s,\tau) = \begin{pmatrix} e^{i\left(\frac{s}{\sqrt{b^2+x_0^2}} - \alpha\tau\right)}(x_0 + iy_0) \\ \frac{bs}{\sqrt{b^2+x_0^2}} \end{pmatrix}$$

a solution of the binormal flow

Another known solution of the binormal flow that does not change its form in time is the vortex ring.

Axisymmetric Euler no-swirl: Cylindrical coordinates

$$\omega(r, z, t) = W(r, z, t)(-y, x).$$

After rescaling time t into $t/|\log\varepsilon|$, we get

$$\begin{cases} |\log\varepsilon| r W_t + \nabla^{\perp}(r^2\psi)\nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz}) := -\Delta_5\psi = W \\ \psi_r(0, z, t) = 0. \end{cases}$$

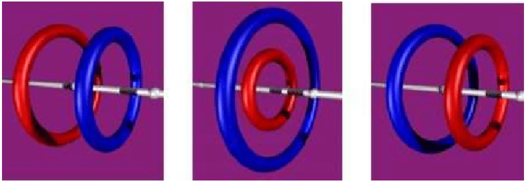
Fraenkel's exact traveling ring solutions (1970-1972):
$W = W(r, z - \alpha t)$. It solves

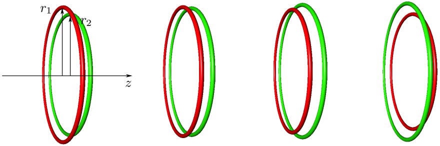
$$-\alpha r|\log\varepsilon| W_z + \nabla^{\perp}(r^2\psi)\nabla(W) = 0, \quad -\Delta_5\psi = W$$

Take $W = F(r^2(\psi - \alpha|\log\varepsilon|))$.

Leapfrogging Vortex-Rings

Helmholtz 1858: predicted the way two identical, coaxial vortex rings interact. They travel in the same direction. Due to their mutual interaction, the rear ring shrinks and accelerates, and the leading ring widens and decelerates. The rear ring then passes through the leading ring, with this process of *leapfrogging* then repeating again and again. As yet, the leapfrogging motion has not been mathematically justified in the context of the Euler equation.





Aim: Mathematically justify the leap-frogging dynamics for the 3d axisymmetric Euler flow without swirl.

$$\begin{cases} |\log \varepsilon| r W_t + \nabla^{\perp}(r^2\psi)\nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz}) = W \\ \psi_r(0, z, t) = 0 \end{cases}$$

$$W(r, z, t) = \sum_{j=1}^{2} r(a_j)^{-1} \frac{1}{\varepsilon_j^2} U_0\left(\frac{x - a_j}{\varepsilon_j}\right)$$

where $a_j = a_j(t), \varepsilon_j = \varepsilon_j(t),\ j = 1, 2$ and $\sqrt{r(a_j)}\varepsilon_j(t) = \varepsilon$,

Theorem [Dávila, del Pino, Musso, Wei, 2022] Let $a(t) = (a_1(t), \dots, a_N(t))$ be a colisionless solution of the system

$$\begin{cases} \dot{b}_i(t) = \sum_{j \neq i} \frac{(b_i - b_j)^\perp}{|b_i - b_j|^2} - \frac{r(b_i)}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ b_i(0) = b_i^0 \end{cases}$$

$$a_i(t) = \left(r_0 + \frac{r(b_i(t))}{\sqrt{|\log \varepsilon|}}, z_0 + \frac{t}{r_0} + \frac{z(b_i(t))}{\sqrt{|\log \varepsilon|}} \right)$$

in $(0, T)$. Then there exists a solution W_ε of 3D axisymmetric Euler flow (without swirl) of the form

$$\boxed{W_\varepsilon(x, t) = \sum_{j=1}^{N} \frac{1}{r(a_j)\varepsilon_j^2} U_0 \left(\frac{(r, z) - a_j}{\varepsilon_j} \right) + o(1)}$$

$\varepsilon = \sqrt{r(a_j(t))}\varepsilon_j(t)$.

Thanks for your attention